LARGE-TIME BEHAVIOR OF COMPRESSIBLE POLYTROPIC FLUIDS AND NONLINEAR SCHRÖDINGER EQUATION

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ABSTRACT. In this paper we analyze the large-time behavior of weak solutions to polytropic fluid models possibly including quantum and viscous effects. Formal a priori estimates show that the density of solutions to these systems should disperse with time. Scaling appropriately the system, we prove that, under a reasonable assumption on the decay of energy, the density of weak solutions converges in large times to an unknown profile. In contrast with the isothermal case, we also show that there exists a large variety of asymptotic profiles. We complement the study by providing existence of global-in-time weak solutions satisfying the required decay of energy. As a byproduct of our method, we also obtain results concerning the large-time behavior of solutions to nonlinear Schrödinger equation, allowing the presence of a semi-classical parameter as well as long range nonlinearities.

CONTENTS

1. INTRODUCTION

We consider isentropic compressible fluid models describing the evolution of the density and velocity field of a fluid on the whole space $\mathbb{R}^d$, given by

$$(1.1) \qquad \partial_t \rho + \operatorname{div}(\rho u) = 0,$$

$$(1.2) \qquad \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla P(\rho) = \operatorname{div}\left(\frac{\varepsilon^2}{2}\mathbb{K}[\rho] + \nu\sqrt{\rho}\,\mathbb{S}[\rho, u]\right).$$

Here $t \in \mathbb{R}_+$ is the time variable and $x \in \mathbb{R}^d$ is the spatial variable, $\rho = \rho(t, x) : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}_+$ is the density of the fluid, $u = u(t, x) : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$ is the velocity field, and $P(\rho)$ is

the pressure which depends on the density. The terms appearing on the right-hand side of (1.2) account for quantum correction and diffusion term, respectively. More precisely, we consider constants $\varepsilon \geqslant 0$ and $\nu \geqslant 0$ and

$$(1.3) \qquad \frac{1}{2}\mathbb{K}[\rho] = \frac{1}{4}\rho\nabla^2 \ln\rho = \frac{1}{2}(\sqrt{\rho}\nabla^2\sqrt{\rho} - \nabla\sqrt{\rho} \otimes \nabla\sqrt{\rho}),$$

$$(1.4) \qquad \mathbb{S}[\rho, u] = \sqrt{\rho}\mathbf{D}u,$$

where $\mathbf{D}u = \frac{1}{2}(\nabla u + \nabla u^\top)$ denotes the symmetric part of the gradient. We also denote below $\mathbf{A}u = \frac{1}{2}(\nabla u - \nabla u^\top)$ the skew-symmetric one. We emphasize that we have split the term $\rho\mathbf{D}u = \sqrt{\rho}\sqrt{\rho}\mathbf{D}u$ in preparation for the following computations. We emphasize also that we allow the parameters ε and ν to vanish separately or simultaneously. We will classically refer to the equations obtained according to these cases as:

- Euler: $\varepsilon = \nu = 0$.
- Euler-Korteweg: $\varepsilon > 0 = \nu$.
- Navier-Stokes (with degenerate viscosity): $\nu > 0 = \varepsilon$.
- Quantum Navier-Stokes (with degenerate viscosity): $\nu > 0$ and $\varepsilon > 0$.

As for the pressure, we shall consider hereafter laws corresponding to polytropic fluids

$$(1.5) \qquad P(\rho) = \rho^\gamma \quad \text{with} \quad \gamma > 1.$$

We refer the reader to [9, 36, 1] for more details on the modeling. In previous studies [10, 9], we focused on the particular case $\gamma = 1$, which corresponds to isothermal fluids. Our main motivation herein is to show that some tools of these former analyses extend to the polytropic case and yield relevant information on the large-time behavior of the density ρ.

1.1. **Main results.** System (1.1)–(1.2) has, at least formally, two fundamental properties: the conservation of mass

$$(1.6) \qquad \int_{\mathbb{R}^d} \rho(t, x)\, \mathrm{d}x = \int_{\mathbb{R}^d} \rho_0(x)\, \mathrm{d}x \quad \forall t \geqslant 0,$$

and the energy identity

$$(1.7) \qquad E[\rho, u](t) + \nu \int_0^t D[\rho, u](s)\, \mathrm{d}s = E_0 \quad \forall t \geqslant 0,$$

where the energy $E = E[\rho, u]$ is defined by

$$(1.8) \qquad E[\rho, u] := \frac{1}{2}\int_{\mathbb{R}^d} (\rho|u|^2 + \varepsilon^2 |\nabla\sqrt{\rho}|^2)\, \mathrm{d}x + \frac{1}{\gamma - 1}\int_{\mathbb{R}^d} \rho^\gamma\, \mathrm{d}x,$$

and its dissipation by

$$(1.9) \qquad D[\rho, u] := \nu \int_{\mathbb{R}^d} \rho|\mathbf{D}u|^2\, \mathrm{d}x.$$

For sufficiently localized solutions (ρ, u) of (1.1)–(1.2) we have also at hand alternative functionals which are well-suited for the study of large-time behavior. First, we observe that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d} \rho|x|^2 = -\int_{\mathbb{R}^d} \mathrm{div}(\rho u)|x|^2 = 2\int_{\mathbb{R}^d} \rho u \cdot x,$$

and Cauchy-Schwarz inequality yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d} \rho|x|^2 \leqslant 2\left(\int_{\mathbb{R}^d} \rho|x|^2\right)^{1/2}\left(\int_{\mathbb{R}^d} \rho|u|^2\right)^{1/2}.$$

In view of the above energy identity, we infer, for all $t \geqslant 0$,

$$(1.10) \qquad \int_{\mathbb{R}^d} \rho|x|^2(t) \leqslant C(E_0)(1 + t^2),$$

for some constant $C(E_0) > 0$ depending on the energy E_0 of the initial data. Second, defining the functional

$$B[\rho, u] := \frac{1}{2} \int_{\mathbb{R}^d} \left(\rho \left| u - \frac{x}{t} \right|^2 + \varepsilon^2 |\nabla \sqrt{\rho}|^2 \right) + \frac{1}{\gamma - 1} \int_{\mathbb{R}^d} \rho^\gamma,$$

we may adapt computations of [32, Section 4] (see also [37]) to obtain that, for some constant $C > 0$:

$$(1.11) \qquad B[\rho, u](t) \leqslant \frac{C(E_0)}{(1+t)^{\min(2, d(\gamma-1))}} + \frac{C\nu}{1+t}, \qquad \forall t > 0.$$

For completeness we provide an exhaustive proof of this estimate in Appendix A. From (1.11) we infer that the L^γ-norm of ρ has to decay to zero with time. Since the total mass is conserved in view of (1.6), we expect the density of solutions to (1.1)-(1.2) to disperse. To give relevant information on the asymptotic state of the density, we consider the (L^1-unitary in space) rescaling

$$R(t, x) = \tau^d(t) \rho(t, \tau(t) x), \quad \forall t > 0,$$

for some well-chosen scaling-parameter family $t \mapsto \tau(t)$. In order to compel with the energy bounds (1.10)-(1.11), one notices that a natural choice is

$$(1.12) \qquad \tau(t) \sim t.$$

The precise choice of τ will be motivated by the fact that our approach relies on compactness properties requiring a priori estimates of the form (1.11). Typically, in the case $\nu = 0$, we use different choices whether γ is smaller or larger than $1 + 2/d$.

In the case of isothermal models $\gamma = 1$, we showed that, by choosing the scaling τ as solution to an appropriate differential equation (which can be inferred *via* a parallel with nonlinear Schrödinger equations), we may not only analyze the asymptotics of solutions to (1.1)-(1.2) but also identify new energy estimates crucial to the construction of a Cauchy theory for this system (see [10, 9]). Such ideas were already hinted in [32]. Following these previous approaches, we propose to introduce the scaling function τ as follows. We fix $\alpha > 0$ and compute $\tau : \mathbb{R}_+ \to \mathbb{R}_+$ as the solution of

$$(1.13) \qquad \ddot{\tau} = \frac{\alpha}{2\tau^{1+\alpha}}, \quad \tau(0) = 1, \ \dot{\tau}(0) = 0.$$

Its large time behavior turns out to be independent of $\alpha > 0$:

Lemma 1.1. *Let $\alpha > 0$. The ordinary differential equation* (1.13) *has a unique, global, smooth solution $\tau \in C^\infty(\mathbb{R}; \mathbb{R}_+)$. In addition, its large time behavior is given by*

$$\dot{\tau}(t) \underset{t \to \infty}{\longrightarrow} 1, \quad hence \ \tau(t) \underset{t \to \infty}{\sim} t.$$

The proof is postponed to Appendix B. Define the new unknowns (R, U) by

$$(1.14) \qquad \rho(t, x) = \frac{1}{\tau(t)^d} R\left(t, \frac{x}{\tau(t)}\right), \quad u(t, x) = \frac{1}{\tau(t)} U\left(t, \frac{x}{\tau(t)}\right) + \frac{\dot{\tau}(t)}{\tau(t)} x.$$

This change of unknowns does not affect the initial data:

$$R(0, x) = \rho(0, x), \quad U(0, x) = u(0, x).$$

In terms of $(R, U) = (R(t, y), U(t, y))$, system (1.1)–(1.2) then becomes,

$$(1.15) \qquad \partial_t R + \frac{1}{\tau^2} \operatorname{div}(RU) = 0$$

$$(1.16) \qquad \partial_t(RU) + \frac{1}{\tau^2} \operatorname{div}(RU \otimes U) + \frac{\alpha}{2\tau^\alpha} yR + \frac{1}{\tau^{d(\gamma-1)}} \nabla R^\gamma$$

$$= \frac{1}{\tau^2} \operatorname{div}\left(\frac{\varepsilon^2}{2} \mathbb{K}[R] + \nu \sqrt{R} \mathbb{S}[R, U] \right) + \frac{\nu \dot{\tau}}{\tau} \nabla R,$$

with $\mathbb{K}$ and $\mathbb{S}$ defined as previously. Since τ depends on some parameter α, so do the new unknowns R and U, and the system (1.15)-(1.16). However, since the large-time behavior

of τ does not depend on $\alpha > 0$ (see Lemma 1.1), the different changes of unknowns describe the same quantities asymptotically. The precise value of α is motivated by the a priori estimates we obtain for (R, U). Yet, the analysis could be performed for various α's, and since the large-time behavior of (R, U) should be independent of α, one expects that α-dependent terms in (1.15)-(1.16) will be subdominant in large times.

The analogues of (1.6), (1.8) and (1.9) in terms of the new unknown (R, U) are the following. First, for fixed t, the map $\rho(t, \cdot) \mapsto R(t, \cdot)$ preserves the L^1-norm, hence (1.6) becomes

$$(1.17) \qquad \int_{\mathbb{R}^d} R(t, y) \mathrm{d}y = \int_{\mathbb{R}^d} \rho_0(x) \mathrm{d}x.$$

Define the *pseudo-energy* $\mathcal{E}[R, U]$ by

$$(1.18) \qquad \begin{aligned} \mathcal{E}[R, U] &= \frac{1}{2\tau^2} \int_{\mathbb{R}^d} \left(R|U|^2 + \varepsilon^2 |\nabla \sqrt{R}|^2 \right) \mathrm{d}y + \frac{\alpha}{4\tau^\alpha} \int_{\mathbb{R}^d} |y|^2 R \, \mathrm{d}y \\ &\quad + \frac{1}{(\gamma - 1)\tau^{d(\gamma-1)}} \int_{\mathbb{R}^d} R^\gamma \, \mathrm{d}y, \end{aligned}$$

as well as its nonnegative dissipation $\mathcal{D}$ by

(1.19)
$$\begin{aligned} \mathcal{D}[R, U] &= \frac{\dot{\tau}}{\tau} \left\{ \frac{1}{\tau^2} \int_{\mathbb{R}^d} \left(R|U|^2 + \varepsilon^2 |\nabla \sqrt{R}|^2 \right) \mathrm{d}y + \frac{\alpha^2}{4\tau^\alpha} \int_{\mathbb{R}^d} |y|^2 R \, \mathrm{d}y + \frac{d}{\tau^{d(\gamma-1)}} \int_{\mathbb{R}^d} R^\gamma \, \mathrm{d}y \right\} \\ &\quad + \frac{\nu}{\tau^4} \int_{\mathbb{R}^d} R|\mathbf{D}U|^2. \end{aligned}$$

We obtain, at least formally, the following pseudo-energy identity

$$(1.20) \qquad \frac{\mathrm{d}}{\mathrm{d}t} \mathcal{E}[R, U] + \mathcal{D}[R, U] = -\nu \frac{\dot{\tau}}{\tau^3} \int_{\mathbb{R}^d} R \operatorname{div} U \, \mathrm{d}y.$$

We remark that $\mathcal{E}$ does not correspond to the energy E written in the (R, U) variables. These formal identities imply that densities $(R(t, \cdot))_{t>0}$ are positive with finite mass and second order moment. An appropriate functional space to tackle the large-time behavior is then the set of positive measures on $\mathbb{R}^d$. Up to a scaling argument – which may only change the amplitude of pressure law and Korteweg terms – we restrict to the case where $(R(t, \cdot))_{t>0}$ is a family of probability measures.

Notations. We use classical notations $C_c^\infty(\mathbb{R}^d)$, $\mathcal{S}(\mathbb{R}^d)$ for smooth functions with compact support and Schwartz space. Notations $L^p(\mathbb{R}^d)$ (resp. $H^s(\mathbb{R}^d)$, $W^{m,p}(\mathbb{R}^d)$) refer to Lebesgue (resp. Sobolev spaces). We shall make repeated use of Bochner spaces $L^p(0, \infty; L^q(\mathbb{R}^d))$, of $L^p(0, \infty; H^s(\mathbb{R}^d))$, and their local-in-time variants. In the space $L^\infty_{\mathrm{loc}}(0, \infty; L^p(\mathbb{R}^d))$ we denote $C([0, \infty); L^p(\mathbb{R}^d) - w)$ the subspace of continuous functions when endowing $L^p(\mathbb{R}^d)$ with its weak topology. The space $\mathcal{D}'(\Omega)$ is made of distributions on the open set Ω (not to be confused with $\mathcal{D}$). We denote by $\mathbb{P}(\mathbb{R}^d)$ the set of probability measures on $\mathbb{R}^d$. More generally, for $j \in \mathbb{N}$, $\mathbb{P}_j(\mathbb{R}^d)$ denotes the space of probability measures on $\mathbb{R}^d$ with finite j-th moment.

1.1.1. *Rigidity result.* Our main contribution consists in analyzing large-time properties of potential weak solutions to (1.15)-(1.16). Building up on our previous construction in [9] we consider weak solutions which read $(\sqrt{R}, \sqrt{R}U)$ and that enjoy the following properties:

(H1) $\sqrt{R} \in L^\infty(0, \infty; L^2(\mathbb{R}^d)) \cap L^\infty_{\mathrm{loc}}(0, \infty; L^{2\gamma}(\mathbb{R}^d))$, $(\varepsilon + \nu)\sqrt{R} \in L^\infty_{\mathrm{loc}}(0, \infty; H^1(\mathbb{R}^d))$, with $R(t, \cdot) \in \mathbb{P}_2(\mathbb{R}^d)$ for a.e. $t > 0$,

(H2) $\sqrt{R}U \in L^\infty_{\mathrm{loc}}(0, \infty; L^2(\mathbb{R}^d))$,

(H3) There exists $\mathbb{T} \in L^2_{\text{loc}}(0, \infty; L^2(\mathbb{R}^d))$, such that

$$\begin{cases} \partial_t R + \dfrac{1}{\tau^2} \operatorname{div}(\sqrt{R}\sqrt{R}U) = 0 \\[2mm] \partial_t(\sqrt{R}\sqrt{R}U) + \dfrac{1}{\tau^2} \operatorname{div}(\sqrt{R}U \otimes \sqrt{R}U) + \dfrac{\alpha}{2\tau^\alpha} yR + \dfrac{1}{\tau^{d(\gamma-1)}} \nabla R^\gamma \\[2mm] \hspace{4cm} = \dfrac{1}{\tau^2} \operatorname{div}\left(\dfrac{\varepsilon^2}{2}\mathbb{K} + \nu\sqrt{R}\mathbb{T}^s \right) + \dfrac{\nu\dot{\tau}}{\tau} \nabla R, \end{cases}$$

holds in $\mathcal{D}'((0, \infty) \times \mathbb{R}^d)$ with the compatibility conditions, when these terms are present:

$$\mathbb{K} = \sqrt{R}\nabla^2\sqrt{R} - \nabla\sqrt{R} \otimes \nabla\sqrt{R}, \qquad \sqrt{R}\mathbb{T} = \nabla(\sqrt{R}\sqrt{R}U) - 2\sqrt{R}U \otimes \nabla\sqrt{R}.$$

For legibility, we have written equations in terms of $R = \left(\sqrt{R}\right)^2$ in this definition whereas, since $\sqrt{R}$ is the involved unknown, these quantities must be computed in terms of $\sqrt{R}$. Similarly, U is not an appropriate unknown in our framework. So, we do not write the quantity $\sqrt{R}\nabla U$ but the symbol $\mathbb{T}$ which plays its role (expressed in terms of $(\sqrt{R}, \sqrt{R}U)$), hence our second compatibility condition in (H3). The exponent s denotes here the symmetric part of $\mathbb{T}$. Such assumptions are also inspired by the definition of weak solution in [9, Definition 1.1] (isothermal case), as in [1, Definition 2.1]), with further momenta requirements for the density (see also [27] in the case of the torus).

To complete the set of assumptions, it is mandatory to enforce in one way or another the decay properties inherited from (1.20) (as it is classical for weak solutions to dissipative systems). By abuse of notations, we keep the symbols $\mathcal{E}$ for energy and $\mathcal{D}$ for its dissipation, though they will be computed in terms of $\sqrt{R}, \sqrt{R}U$ and $\mathbb{T}$ (and not R and U which are not the good unknowns in this weak-solution framework). Our last requirement builds on the following formal analysis. First we bound the right-hand side in (1.20) as follows:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E} + \mathcal{D} \leqslant \nu\frac{\dot{\tau}}{\tau^3}\left(\int_{\mathbb{R}^d} R\,\mathrm{d}y\right)^{1/2}\left(\int_{\mathbb{R}^d} |\mathbb{T}^s|^2\,\mathrm{d}y\right)^{1/2}$$

$$\leqslant 2\nu\frac{(\dot{\tau})^2}{\tau^2}\int_{\mathbb{R}^d} R\,\mathrm{d}y + \frac{\nu}{2\tau^4}\int_{\mathbb{R}^d} |\mathbb{T}^s|^2\,\mathrm{d}y,$$

which implies

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E} + \frac{1}{2}\mathcal{D} \leqslant 2\nu\frac{(\dot{\tau})^2}{\tau^2}.$$

Remarking that $\int_0^\infty \frac{(\dot{\tau}(t))^2}{\tau(t)^2}\mathrm{d}t < \infty$ (see Lemma 1.1), we already obtain that

$$(1.21) \qquad \sup_{t \geqslant 0}\mathcal{E}(t) + \int_0^\infty \mathcal{D}(t)\,\mathrm{d}t \leqslant C(\mathcal{E}_0),$$

where $C(\mathcal{E}_0) > 0$ is a constant depending on the pseudo-energy of the initial data $\mathcal{E}_0$. This yields at first that $\mathcal{D}$ is in $L^1(0, \infty)$. Observing from (1.19) that if $\alpha \leqslant \min(2, (d(\gamma - 1))$,

$$\mathcal{D}[R, U] \geqslant \alpha\frac{\dot{\tau}}{\tau}\mathcal{E}[R, U],$$

we therefore deduce the following differential inequality

$$(1.22) \qquad \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E} \leqslant -\frac{\dot{\tau}}{\tau}\alpha\mathcal{E} + 2\nu\frac{(\dot{\tau})^2}{\tau^2},$$

which entails after integration that, for all $t \geqslant 0$ (the outcome is slightly different whether $\alpha \neq 1$ or $\alpha = 1$),

$$(1.23) \qquad \mathcal{E}(t) \leqslant C_0\left(\frac{1}{1 + t^\alpha} + \frac{\nu}{1 + t}(\mathbf{1}_{\alpha \neq 1} + \ln(1 + t)\,\mathbf{1}_{\alpha = 1})\right).$$

We note that, when $\nu > 0$ and $\alpha > 1$, these computations only yield a bound on the growth of the second order moment of $R(t)$. This can be improved thanks to the following remark. By multiplying the continuity equation with $|y|^2$ we have formally that:

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^d} |y|^2 R \, \mathrm{d}y = \frac{2}{\tau^2} \int_{\mathbb{R}^d} R y \cdot U \, \mathrm{d}y,$$

which implies

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^d} |y|^2 R \, \mathrm{d}y \leqslant \frac{2}{\tau^2} \left(\int_{\mathbb{R}^d} R|y|^2 \, \mathrm{d}y \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^d} R|U|^2 \, \mathrm{d}y \right)^{\frac{1}{2}},$$

and thus

$$(1.24) \qquad \frac{\mathrm{d}}{\mathrm{d}t} \left(\int_{\mathbb{R}^d} |y|^2 R \, \mathrm{d}y \right)^{\frac{1}{2}} \leqslant \frac{1}{\tau} \left(\frac{1}{\tau^2} \int_{\mathbb{R}^d} R|U|^2 \, \mathrm{d}y \right)^{\frac{1}{2}} \leqslant \frac{C}{\tau} \sqrt{\mathcal{E}}.$$

The combined decay of $\mathcal{E}$ and growth of τ entail finally that the second order moment of R remains bounded whatever the value of α.

Eventually, these formal considerations lead us to the following last assumption:

(H4) Set $\alpha = \min(2, d(\gamma - 1))$. Introducing $\mathcal{E}, \mathcal{D}$ as defined previously (see (1.18) and (1.19)), there exists a constant $C_0 > 0$ such that:

$$(1.25) \qquad \mathcal{E}(t) \leqslant C_0 \left(\frac{1}{(1+t)^\alpha} + \frac{\nu}{(1+t)} (\mathbf{1}_{\alpha \neq 1} + \ln(1+t) \, \mathbf{1}_{\alpha=1}) \right), \qquad \forall \, t > 0,$$

$$(1.26) \qquad \sup_{t>0} \left(\int_{\mathbb{R}^d} |y|^2 R(t,y) \mathrm{d}y \right) + \int_0^\infty \mathcal{D}(t) \, \mathrm{d}t \leqslant C_0.$$

More details on the derivation of (1.25)-(1.26) are given in Section 4. With these assumptions, our main result yields a description of the large-time behavior of the density $R(t) = [\sqrt{R}(t)]^2$. This is the content of the following theorem:

Theorem 1.2. *Assume that $(\sqrt{R}, \sqrt{R}U)$ is a global weak solution to (1.15)–(1.16) such that (H1)–(H4) hold true. There exists $R_\infty \in \mathbb{P}_2(\mathbb{R}^d)$ such that*

$$R(t, \cdot) \rightharpoonup R_\infty \quad \text{in } \mathbb{P}(\mathbb{R}^d).$$

We have in addition $R_\infty \in L^1(\mathbb{R}^d)$ (at least) in the following cases:

- $\varepsilon = \nu = 0$ *and* $1 < \gamma \leqslant 1 + 2/d$,
- $\varepsilon > 0$, $\nu = 0$ *and* $\gamma > 1$,
- $\varepsilon \geqslant 0$, $\nu > 0$ *and* $1 < \gamma \leqslant 1 + 1/d$.

Remark 1.3. In [3], the authors consider the Korteweg case $\varepsilon > 0 = \nu$ in dimension $d = 1$, and show dispersive estimates of another nature, in the sense that they involve a strong topology: under (H1), (H2) and (H3) (the assumptions actually involve (ρ, u) instead of (R, U), which means in particular that one should consider $\tau \equiv 1$ in (H3)), the authors prove that $\rho \in L^{\gamma+1}(\mathbb{R}_t \times \mathbb{R}_x^d)$ and $\partial_x \rho \in L^2(\mathbb{R}_t \times \mathbb{R}_x^d)$. The proof relies on an adaptation of Morawetz estimates, which are a classical tool in the study of nonlinear Schrödinger equations; see e.g. [21] for a presentation which clearly uses the link between the nonlinear Schrödinger equation and hydrodynamical equations.

We obtain in the course of the proof an explicit polynomial rate of convergence from $R(t, \cdot)$ to R_∞. Reconstructing the solution (ρ, u) from (R, U) via the formulas (1.14), we infer

$$\lim_{t \to \infty} \tau^d(t) \rho(t, \tau(t) \cdot) = R_\infty \text{ in } \mathbb{P}(\mathbb{R}^d).$$

We emphasize that contrary to the isothermal case $\gamma = 1$, where, as proven in [10], the only possible R_∞ is given by

$$R_\infty(y) = \frac{\|\rho_0\|_{L^1(\mathbb{R}^d)}}{\pi^{d/2}} e^{-|y|^2},$$

in the polytropic case $\gamma > 1$, the range of the map $\rho_0 \mapsto R_\infty$ is very broad. In the case of the Euler equation, we have, as established in [10] by adapting the approach from [32]:

Proposition 1.4. *Let $\varepsilon = \nu = 0$, $1 < \gamma \leqslant 1 + 2/d$ and $s > d/2 + 1$. There exists $\eta > 0$ such that if $0 \leqslant a_\infty \in H^s(\mathbb{R}^d)$ is such that $\|a_\infty\|_{H^s(\mathbb{R}^d)} \leqslant \eta$, then there exists a solution to (1.1)-(1.2) which is global in time, with*

$$\left\| \rho(t,x) - \frac{1}{t^d} R_\infty \left(\frac{x}{t} \right) \right\|_{L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)} \xrightarrow[t\to\infty]{} 0, \quad R_\infty := a_\infty^{\frac{2}{\gamma-1}}.$$

In the case of the Euler-Korteweg system, we will prove (for a different range of γ):

Proposition 1.5. *Let $\varepsilon > 0 = \nu$,*

$$\gamma > 3 \text{ if } d = 1, \quad 1 + \frac{4}{d+2} < \gamma < 1 + \frac{4}{(d-2)_+} \text{ if } d \geqslant 2.$$

For any $a_\infty \in \mathcal{S}(\mathbb{R}^d)$, there exists a solution to (1.1)-(1.2) such that

$$\left\| \rho(t,x) - \frac{1}{t^d} R_\infty \left(\frac{x}{t} \right) \right\|_{L^1(\mathbb{R}^d)} \xrightarrow[t\to\infty]{} 0, \quad R_\infty := |a_\infty|^2.$$

This result is a direct consequence of scattering theory for nonlinear Schrödinger equations, as discussed more precisely in Section 3.3.

1.1.2. *Existence results.* The second natural contribution consists in making sure that the assumptions of Theorem 1.2 are not empty.

Theorem 1.6. *In the three following cases, initial data (ρ_0, u_0) yield at least one global weak solution $(\sqrt{R}, \sqrt{R}U)$ to (1.15)-(1.16) satisfying the assumptions of Theorem 1.2:*

(i) **Euler equations**. *Assume $\varepsilon = \nu = 0$. Let $\gamma > 1$, $s > d/2 + 1$ and $r_0 \in H^s(\mathbb{R}^d)$ such that $r_0 \geqslant 0$ is compactly supported with $\|r_0\|_{H^s(\mathbb{R}^d)}$ sufficiently small. Then, assume $\rho_0(x) = r_0(x)^{\frac{2}{\gamma-1}}$, and u_0 satisfies $\nabla^2 u_0 \in H^{s-1}(\mathbb{R}^d)$, $\nabla u_0 \in L^\infty(\mathbb{R}^d)$, and there exists $\delta > 0$ such that for all $x \in \mathbb{R}^d$, $\text{dist}(\text{Sp}(\nabla u_0(x)), \mathbb{R}_- \geqslant \delta$, where we denote by $\text{Sp}(M)$ the spectrum of a matrix M.*

(ii) **Euler-Korteweg equations**. *Assume $\varepsilon > 0$, $\nu = 0$ and $1 < \gamma < 1 + \frac{4}{(d-2)_+}$, and there exists*

$$\psi_0 \in \Sigma := \{ f \in H^1(\mathbb{R}^d), \quad x \mapsto xf(x) \in L^2(\mathbb{R}^d) \},$$

such that $\rho_0 = |\psi_0|^2$, $\rho_0 u_0 = \varepsilon \text{Im}(\bar{\psi}_0 \nabla \psi_0)$.

(iii) **Navier-Stokes equations**. *Assume $d \leqslant 3$, $\gamma > 1$, $\nu > 0$ and $\varepsilon \geqslant 0$. Let (ρ_0, u_0) satisfy:*

$$(1 + |x| + |u_0|)\sqrt{\rho_0} \in L^2(\mathbb{R}^d), \quad \rho_0 \in L^\gamma(\mathbb{R}^d), \quad \sqrt{\rho_0} \in H^1(\mathbb{R}^d).$$

We remind that the change of unknown from small-letter to capital-letter unknowns does not affect initial data. In particular, depending on the case, we may prefer to solve the small-letter system (1.1)-(1.2) and then apply the change of unknown to yield weak solutions satisfying (H1)-(H4) or directly work on the scaled system (1.15)-(1.16) with the capital-letter unknowns. More details are given in Section 4.

1.2. **Nonlinear Schrödinger equation.** It is well-known (see e.g. [2, 11]) that the Euler-Korteweg equation is intimately related to the nonlinear Schrödinger equation (NLS)

$$(1.27) \qquad i\varepsilon \partial_t \psi^\varepsilon + \frac{\varepsilon^2}{2} \Delta \psi^\varepsilon = \lambda |\psi^\varepsilon|^{2\sigma} \psi^\varepsilon, \quad \psi^\varepsilon_{|t=0} = \psi_0^\varepsilon \in H^1(\mathbb{R}^d),$$

through the Madelung transform,

$$(1.28) \qquad \rho = |\psi^\varepsilon|^2, \quad \rho u = \varepsilon \text{Im} \left(\bar{\psi}^\varepsilon \nabla \psi^\varepsilon \right), \quad \text{with} \quad \lambda = \frac{\gamma}{\gamma-1}, \quad \sigma = \frac{\gamma-1}{2}.$$

We emphasize the dependence of ψ^ε upon ε through the notation, for the limit $\varepsilon \to 0$ corresponds to the semi-classical limit, and will be discussed in the present paper. The Cauchy problem (1.27) is easier than its fluid mechanical counterpart: if $\lambda > 0$ and $0 < \sigma < \frac{2}{(d-2)_+}$ (defocusing, energy-subcritical nonlinearity), then (1.27) has a unique solution

$$\psi^\varepsilon \in C(\mathbb{R}; H^1(\mathbb{R}^d)) \cap L^{\frac{4\sigma+4}{d\sigma}}_{\mathrm{loc}}(\mathbb{R}; L^{2\sigma+2}(\mathbb{R}^d)).$$

See e.g. [12]. If in addition $x \mapsto x\psi_0^\varepsilon \in L^2(\mathbb{R}^d)$, then this integrability property is propagated by the flow. The analogue in the context of nonlinear Schrödinger equations of the evolution of $B[\rho, u]$ in the fluid mechanical case was discovered by Ginibre and Velo [17] (and thus actually before its counterpart in fluid mechanics), and goes under the name of *pseudo-conformal conservation law.*

Theorem 1.7. *Let $d \geqslant 1$, $\varepsilon, \lambda > 0$, $0 < \sigma < \frac{2}{(d-2)_+}$, and*

$$\psi_0^\varepsilon \in \Sigma := \{f \in H^1(\mathbb{R}^d), \quad x \mapsto xf(x) \in L^2(\mathbb{R}^d)\}.$$

Rescale the function ψ^ε to Ψ^ε via

$$(1.29) \qquad \psi^\varepsilon(t, x) = \frac{1}{\tau(t)^{d/2}} \Psi^\varepsilon\left(t, \frac{x}{\tau(t)}\right) e^{i\frac{\dot{\tau}(t)}{\tau(t)}\frac{|x|^2}{2\varepsilon}} \|\psi_0^\varepsilon\|_{L^2(\mathbb{R}^d)},$$

where $\tau(t)$ is a scaling like before (in particular, $\tau(t) \sim t$ as $t \to \infty$). There exists $R_\infty^\varepsilon \in \mathbb{P}_2(\mathbb{R}^d)$ such that

$$|\Psi^\varepsilon(t, \cdot)|^2 \rightharpoonup R_\infty^\varepsilon \quad in \ \mathbb{P}(\mathbb{R}^d).$$

More details are given in Section 3. At this stage, we emphasize the fact that σ is arbitrarily small. In particular, for $0 < \sigma \leqslant 1/d$, the nonlinearity is long range, in the sense that no standard scattering result is possible: fix $\varepsilon > 0$, and assume that there exists $\psi_+^\varepsilon \in L^2(\mathbb{R}^d)$ such that

$$(1.30) \qquad \left\|\psi^\varepsilon(t) - e^{i\varepsilon\frac{t}{2}\Delta}\psi_+^\varepsilon\right\|_{L^2(\mathbb{R}^d)} \underset{t\to\infty}{\longrightarrow} 0,$$

then necessarily $\psi^\varepsilon \equiv 0$, from [4]. On the other hand, it is a common belief that long range effects affect only the behavior of the phase, at leading order, meaning that the dispersion is the same as in the linear case. Indeed, for $\sigma > 1/d$, under the assumptions of Theorem 1.7, there exists ψ_+^ε (with in particular $\|\psi_+^\varepsilon\|_{L^2(\mathbb{R}^d)} = \|\psi_0^\varepsilon\|_{L^2(\mathbb{R}^d)}$) such that (1.30) holds ([34]). Recall that in $L^2(\mathbb{R}^d)$ (see e.g. [33]),

$$e^{i\varepsilon\frac{t}{2}\Delta} f(x) \underset{t\to\infty}{\sim} \frac{1}{(\varepsilon t)^{d/2}} \hat{f}\left(\frac{x}{\varepsilon t}\right) e^{i\frac{|x|^2}{2\varepsilon t}}.$$

Therefore, for $\varepsilon > 0$ fixed, Theorem 1.7 shows that long range effects do not alter the standard dispersion.

1.3. **Outline of the paper.** In brief, the paper splits into 3 sections and 2 appendices. In **Section 2** we provide a proof of **Theorem 1.2**. The next section is devoted to the analysis of nonlinear Schrödinger equations to provide the examples of **Proposition 1.5**. We complement the analysis in **Section 4** with the proof of the existence result **Theorem 1.6**. The two appendices are devoted to the formal computation of decay estimate (1.11), and to the properties of the scaling parameter families $(\tau(t))_{t>0}$, respectively.

2. Proof of Theorem 1.2

We consider non-negative parameters ε, ν, and assume that $(\sqrt{R}, \sqrt{R}U)$ is a global weak solution to (1.15)-(1.16) in the sense of (H1)-(H3), enjoying the decay properties (H4).

As a preliminary, we note from (H4) that

$$\left(\int_{\mathbb{R}^d} |y|^2 R(t, y)\mathrm{d}y\right)_{t>0} \quad \text{is bounded.}$$

So the family of probability densities $(R(t, \cdot))_{t>0}$ is tight and precompact in $\mathbb{P}(\mathbb{R}^d)$. Remark that this already implies that there is some sequence of times $(t_n)_{n \geqslant 0}$ with $t_n \to \infty$ as $n \to \infty$, such that $(R(t_n, \cdot))_{n \geqslant 0}$ converges weakly in $\mathbb{P}(\mathbb{R}^d)$ to some probability measure R_∞. Unlike in the isothermal case, we have not been able to identify a limiting equation for R_∞, which could make it possible to infer uniqueness of the accumulation point ($R(t, \cdot)$ might keep oscillating as $t \to \infty$). However, given the uniform bound on $(R(t, \cdot))$ in $\mathbb{P}_2(\mathbb{R}^d)$, our proof reduces to obtaining convergence in some sufficiently large dual space. To this end, we will make repeated use of the following lemma:

Lemma 2.1. *Let $T > 0$, $m \in \mathbb{N}$ and $(p, q) \in (1, \infty)$. Assume that $X \in L^\infty(0, T; L^p(\mathbb{R}^d))$ satisfies $\partial_t X \in L^1(0, T; W^{-m,q}(\mathbb{R}^d))$. Then there holds:*
- $X \in C([0, T]; L^p(\mathbb{R}^d) - w)$
- *for arbitrary $\varphi \in C_c^\infty(\mathbb{R}^d)$ there holds:*

$$\left[\int_{\mathbb{R}^d} X(\cdot, y) \varphi(y) \mathrm{d}y \right]_{t_1}^{t_2} = \left\langle \partial_t X, (t, y) \mapsto \varphi(y) \mathbf{1}_{[t_1,t_2]}(t) \right\rangle \quad \forall 0 \leqslant t_1 < t_2 \leqslant T.$$

This lemma is part of the folklore and is stated without proof. Formally, it is tempting to invoke (1.15) and use Cauchy-Schwarz inequality to obtain

$$\int_{\mathbb{R}^d} R|U| \mathrm{d}y \leqslant \left(\int_{\mathbb{R}^d} R \mathrm{d}y \right)^{1/2} \left(\int_{\mathbb{R}^d} R|U|^2 \mathrm{d}y \right)^{1/2} \lesssim \tau.$$

Then one may want to write, in view of (1.15),

$$\|\partial_t R\|_{W^{-1,1}(\mathbb{R}^d)} = \frac{1}{\tau^2} \| \operatorname{div}(RU) \|_{W^{-1,1}(\mathbb{R}^d)} \leqslant \frac{1}{\tau^2} \|RU\|_{L^1(\mathbb{R}^d)} \lesssim \frac{1}{\tau}.$$

We see that we barely miss integrability on the right hand side, due to a logarithmic divergence. Also, this estimate implicitly relies on duality properties of $W^{-1,1}$, which is a delicate matter. To overcome these issues, we estimate $\partial_t R$ at a lower regularity level in order to obtain integrability in time, and we consider estimates related to L^p spaces with $1 < p < \infty$ (for reflexivity), and $p > d$ so we can use Sobolev embeddings $W^{s,p}(\mathbb{R}^d) \subset W^{s-1,\infty}(\mathbb{R}^d)$. This again reduces the level of regularity at which we estimate $\partial_t R$. More precisely, we estimate $\|\partial_t R\|_{W^{-4,p'}}$ for $d < p < \infty$, in **Proposition 2.3** below.

The core of the proof is then two successive applications of **Lemma 2.1**. First, we obtain:

Proposition 2.2. *Let $\gamma_* = 2\gamma/(\gamma+1)$. There holds $RU = \sqrt{R}\sqrt{R}U \in C([0, \infty); L^{\gamma_*}(\mathbb{R}^d) - w)$ and, given $p > d$, there exists $K_p > 0$ depending on C_0 in (H4) and $\alpha, p, \varepsilon, \nu$ for which:*

$$\left| \int_{\mathbb{R}^d} RU \cdot w \right| \leqslant K_p \left((1+t)^{(1-\alpha)_+} + \ln(1+t) \mathbf{1}_{\alpha=1} + \mathbf{1}_{\nu>0} (1+t)^{1/2} \right) \|w\|_{W^{3,p}(\mathbb{R}^d)},$$

for all $w \in [C_c^\infty(\mathbb{R}^d)]^d$.

Proof. Since we have $RU \in L_{\text{loc}}^\infty(0, \infty); L^{\gamma_*}(\mathbb{R}^d))$, a direct application of **Lemma 2.1** yields our result if we prove, for any $t > 0$, that:

$$|\langle \partial_t(RU), w \rangle| \leqslant K_p \left((1+t)^{(1-\alpha)_+} + \ln(1+t) \mathbf{1}_{\alpha=1} + \mathbf{1}_{\nu>0} (1+t)^{1/2} \right) \sup_{s \in (0,t)} \|w(s, \cdot)\|_{W^{3,p}(\mathbb{R}^d)}$$

for arbitrary $w \in C_c^\infty((0, t) \times \mathbb{R}^d)^d$. To this respect, we will make repeated use without mention of the property, stemming from (1.23):

$$\int_0^t \mathcal{E}(s) \, \mathrm{d}s \leqslant C_0 \int_0^t \left[\frac{1}{(1+s)^\alpha} + \frac{\nu}{(1+s)} \left(\mathbf{1}_{\alpha \neq 1} + \ln(1+s) \mathbf{1}_{\alpha=1} \right) \right] \mathrm{d}s$$

$$\leqslant K_p \left((1+t)^{(1-\alpha)_+} + \ln(1+t) \mathbf{1}_{\alpha=1} + \mathbf{1}_{\nu>0} \sqrt{1+t} \right).$$

So, given $t > 0$ and $w \in C_c^\infty((0, t) \times \mathbb{R}^d)^d$, we apply (1.16) to split:

$$\langle \partial_t(RU), w \rangle = \sum_{k=1}^{5} \langle L_i, w \rangle,$$

where:

$$\langle L_1, w \rangle = -\int_0^t \frac{1}{\tau^2} \int_{\mathbb{R}^d} (\frac{\varepsilon^2}{2}\mathbb{K} + \nu\sqrt{R}\mathbb{S}) : \nabla w,$$

$$\langle L_2, w \rangle = -\int_0^t \frac{\nu\dot\tau}{\tau} \int_{\mathbb{R}^d} R\,\mathrm{div}\,w,$$

$$\langle L_3, w \rangle = \int_0^t \frac{1}{\tau^2} \int_{\mathbb{R}^d} \sqrt{R}U \otimes \sqrt{R}U : \nabla w,$$

$$\langle L_4, w \rangle = -\int_0^t \frac{\alpha}{2\tau^\alpha} \int_{\mathbb{R}^d} Ry \cdot w,$$

$$\langle L_5, w \rangle = \int_0^t \frac{1}{\tau^{d(\gamma-1)}} \int_{\mathbb{R}^d} R^\gamma \cdot \mathrm{div}\,w.$$

We now estimate these five terms independently.

Concerning L_1, we split $L_1 = L_1[\mathbb{K}] + L_1[\mathbb{S}]$ with obvious notations. First we bound:

$$|\langle L_1[\mathbb{K}], w \rangle| = \left| \int_0^t \frac{\varepsilon^2}{2\tau^2} \int_{\mathbb{R}^d} \left(\sqrt{R}\nabla^2\sqrt{R} - \nabla\sqrt{R} \otimes \nabla\sqrt{R} \right) : \nabla w \right|$$

$$\lesssim \int_0^t \frac{1}{\tau^2} \int_{\mathbb{R}^d} \left(|\varepsilon\nabla\sqrt{R}|^2|\nabla w| + \varepsilon\sqrt{R}|\varepsilon\nabla\sqrt{R}||\nabla^2 w| \right)$$

$$\lesssim \int_0^t \left(\frac{\varepsilon^2}{\tau^2} \int_{\mathbb{R}^d} |\nabla\sqrt{R}|^2 + \frac{\varepsilon^2}{\tau^2} \int_{\mathbb{R}^d} R \right) \sup_{[0,t]} \|w\|_{W^{2,\infty}(\mathbb{R}^d)}.$$

We remind here that R is a probability measure and the definition (1.18) of $\mathcal{E}$. This entails by Sobolev embedding that:

$$|\langle L_1[\mathbb{K}], w \rangle| \leqslant C_p \left(\int_0^t \mathcal{E} + \frac{\varepsilon^2}{\tau^2} \right) \sup_{[0,t]} \|w\|_{W^{3,p}(\mathbb{R}^d)}$$

$$\leqslant K_p \left((1+t)^{(1-\alpha)_+} + \ln(1+t)\,\mathbf{1}_{\alpha=1} + \mathbf{1}_{\nu>0}\sqrt{1+t} \right) \sup_{[0,t]} \|w\|_{W^{3,p}(\mathbb{R}^d)},$$

where we used that τ^{-1} decays like $1/(1+t)$ to integrate $1/\tau^2$. Similarly, we apply the control induced by $\mathcal{D}$ to bound:

$$|\langle L_1[\mathbb{S}], w \rangle| \leqslant \int_0^t \frac{\nu}{\tau^2} \int_{\mathbb{R}^d} \sqrt{R}\mathbb{S} : \nabla w$$

$$\leqslant \sqrt{\nu} \int_0^t \left(\frac{\nu}{\tau^4} \int_{\mathbb{R}^d} |\mathbb{T}^s|^2 \right)^{1/2} \left(\int_{\mathbb{R}^d} R \right)^{1/2} \sup_{[0,t]} \|\nabla w\|_{L^\infty(\mathbb{R}^d)}$$

$$\leqslant C_p\sqrt{\nu t} \left(\int_0^\infty \mathcal{D} \right)^{\frac{1}{2}} \sup_{[0,t]} \|w\|_{W^{2,p}(\mathbb{R}^d)}.$$

Combining the previous two estimates yields finally:

$$(2.1) \qquad |\langle L_1, w \rangle| \leqslant K_p \left((1+t)^{(1-\alpha)_+} + \ln(1+t)\,\mathbf{1}_{\alpha=1} + \mathbf{1}_{\nu>0}\sqrt{1+t} \right) \sup_{[0,t]} \|w\|_{W^{3,p}(\mathbb{R}^d)}.$$

To handle L_2, we use that R has constant mass and the growth of τ at infinity:

$$|\langle L_2, w \rangle| \leqslant \left| \int_0^t \frac{\nu\dot\tau}{\tau} \int_{\mathbb{R}^d} R\,\mathrm{div}\,w \right| \leqslant C\nu \int_0^t \frac{1}{\tau} \sup_{[0,t]} \|\mathrm{div}\,w\|_{L^\infty(\mathbb{R}^d)} \leqslant K_p\nu\ln(1+t) \sup_{[0,t]} \|w\|_{W^{2,p}}.$$

We proceed with L_3. First, we make controlled quantities appear via Hölder inequality:

$$\begin{aligned}
|\langle L_3, w \rangle| &\leqslant \left| \int_0^t \frac{1}{\tau^2} \int_{\mathbb{R}^d} \sqrt{R}U \otimes \sqrt{R}U : \nabla w \right| \\
&\leqslant \int_0^t \frac{1}{\tau^2} \int_{\mathbb{R}^d} |\sqrt{R}U|^2 \sup_{[0,t]} \|\nabla w\|_{L^\infty(\mathbb{R}^d)} \\
&\leqslant \left(\int_0^t \mathcal{E} \, \mathrm{d}s \right) \sup_{[0,t]} \|\nabla w\|_{L^\infty(\mathbb{R}^d)} \\
&\leqslant K_p \left((1+t)^{(1-\alpha)_+} + \ln(1+t)\, \mathbf{1}_{\alpha=1} + \mathbf{1}_{\nu>0} \sqrt{1+t} \right) \sup_{[0,t]} \|w\|_{W^{2,p}(\mathbb{R}^d)}.
\end{aligned}$$

Concerning L_4, we have

$$\begin{aligned}
|\langle L_4, w \rangle| &\leqslant \frac{1}{2} \int_0^t \frac{\alpha}{\tau^\alpha} \left(\int_{\mathbb{R}^d} R \right)^{1/2} \left(\int_{\mathbb{R}^d} R|y|^2 \right)^{1/2} \sup_{[0,t]} \|w\|_{L^\infty(\mathbb{R}^d)} \\
&\leqslant \frac{1}{2} \left(\int_0^t \frac{1}{\tau^{\alpha/2}} \sqrt{\mathcal{E}} \, \mathrm{d}s \right) \sup_{[0,t]} \|w\|_{W^{1,p}(\mathbb{R}^d)} \\
&\leqslant K_p \left(\int_0^t \left(\frac{1}{(1+s)^\alpha} + \frac{\nu(1 + \ln(1+s))}{(1+s)^{(1+\alpha)/2}} \right) \mathrm{d}s \right) \sup_{[0,t]} \|w\|_{W^{1,p}(\mathbb{R}^d)} \\
&\leqslant K_p \left((1+t)^{(1-\alpha)_+} + \ln(1+t)\mathbf{1}_{\alpha=1} + \mathbf{1}_{\nu>0} \sqrt{1+t} \right) \sup_{[0,t]} \|w\|_{W^{1,p}(\mathbb{R}^d)}.
\end{aligned}$$

Finally, for L_5, we obtain directly that:

$$\begin{aligned}
|\langle L_5, w \rangle| &\leqslant (\gamma - 1) \left(\int_0^t \mathcal{E} \, \mathrm{d}s \right) \sup_{[0,t]} \|\operatorname{div} w\|_{L^\infty} \\
&\leqslant K_p \left((1+t)^{(1-\alpha)_+} + \ln(1+t)\, \mathbf{1}_{\alpha=1} + \mathbf{1}_{\nu>0} \sqrt{1+t} \right) \sup_{[0,t]} \|w\|_{W^{2,p}(\mathbb{R}^d)}.
\end{aligned}$$

This completes the proof. $\qquad\square$

We apply now this control of RU in order to handle $\partial_t R$. We have:

Proposition 2.3. *For arbitrary $\phi \in C_c^\infty(\mathbb{R}^d)$ the function*

$$R_\phi : t \mapsto \int_{\mathbb{R}^d} R(t,y)\phi(y)\mathrm{d}y$$

enjoys the properties:

i) $R_\phi \in C([0,\infty))$,

ii) R_ϕ *converges to some limit R_ϕ^∞ as $t \to \infty$, satisfying:*

$$|R_\phi^\infty| \leqslant C_{p,\infty} \|\phi\|_{W^{4,p}(\mathbb{R}^d)},$$

for a constant $C_{p,\infty}$ depending on $p > d$, but independent of ϕ.

This latter result shows the convergence of $(R(t,\cdot))_{t>0}$ through the mapping $\phi \mapsto R_\phi^\infty$ in $W^{-4,p'}(\mathbb{R}^d)$. This mapping is bound to be a probability measure thanks to the tightness of $(R(t,\cdot))_{t>0}$. Hence, the proof of this proposition ends up this part.

Proof. Similarly to the previous proof, we have here that $R \in L_{\mathrm{loc}}^\infty(0,\infty; L^\gamma(\mathbb{R}^d))$ and, thanks to Equation (1.15) with (H2), there holds: $\partial_t R \in L_{\mathrm{loc}}^1(0,\infty; W^{-1,\gamma_*'}(\mathbb{R}^d))$ (where γ_*' is the conjugate exponent of γ_*). Applying **Lemma 2.1**, we have then that, for arbitrary $\phi \in C_c^\infty(\mathbb{R}^d)$ and $t_1 < t_2$ there holds $R_\phi \in C([0,\infty))$ and

$$R_\phi(t_2) - R_\phi(t_1) = \int_{t_1}^{t_2} \frac{1}{\tau^2} RU \cdot \nabla \phi.$$

In this equality, we apply the bound of **Proposition 2.2** with $p > d$. This yields:

$$|R_\phi(t_2) - R_\phi(t_1)|$$
$$\leqslant K_p \left(\int_{t_1}^{t_2} \frac{(1+t)^{(1-\alpha)_+} + \ln(1+t)\mathbf{1}_{\alpha=1} + \mathbf{1}_{\nu>0}(1+t)^{1/2}}{\tau(t)^2} \, \mathrm{d}t \right) \|\nabla\phi\|_{W^{3,p}(\mathbb{R}^d)}.$$

Since $\tau \sim t$ for large t, we obtain that

$$t \mapsto \frac{(1+t)^{(1-\alpha)_+} + \ln(1+t)\mathbf{1}_{\alpha=1} + \mathbf{1}_{\nu>0}(1+t)^{1/2}}{\tau(t)^2} \in L^1([0,\infty)).$$

By a standard domination argument, we infer the conclusions of our proposition: R_ϕ admits a limit R_ϕ^∞ when $t \to \infty$, and

$$|R_\phi^\infty| \leqslant \left| \int_{\mathbb{R}^d} R(0,\cdot)\phi \right|$$
$$+ K_p \left(\int_0^\infty \frac{(1+t)^{(1-\alpha)_+} + \ln(1+t)\mathbf{1}_{\alpha=1} + \mathbf{1}_{\nu>0}(1+t)^{1/2}}{\tau(t)^2} \, \mathrm{d}t \right) \|\phi\|_{W^{4,p}(\mathbb{R}^d)}$$
$$\leqslant C_{p,\infty} \|\phi\|_{W^{4,p}(\mathbb{R}^d)}.$$

$\square$

As a straightforward corollary to the above computations, we also have the following convergence result for any $p > d$:

$$\|R(t,\cdot) - R_\infty\|_{W^{-4,p'}(\mathbb{R}^d)} \leqslant K_p \left(\frac{1}{(1+t)^{\min(\alpha,1)}} + \frac{\ln(1+t)}{(1+t)}\mathbf{1}_{\alpha=1} + \frac{\mathbf{1}_{\nu>0}}{\sqrt{1+t}} \right), \quad \forall t > 0.$$

Furthermore, for sufficiently small γ, we can also state more properties of the asymptotic R_∞. Indeed, from (H4) we infer that:

$$\int_{\mathbb{R}^d} R^\gamma(t,\cdot) \leqslant C_0 \tau^{d(\gamma-1)} \left(\frac{1}{(1+t)^\alpha} + \frac{\nu}{(1+t)} \left(\mathbf{1}_{\alpha\neq1} + \ln(1+t)\mathbf{1}_{\alpha=1} \right) \right) \quad \forall t > 0.$$

Consequently, when $\nu > 0$, if $d(\gamma-1) \leqslant 1$ (i.e. $\gamma < 1 + 1/d$ and $\alpha = d(\gamma-1)$) we obtain that $(R(t,\cdot))_{t>0}$ is bounded in $L^\gamma(\mathbb{R}^d)$. While, when $\nu = 0$, the same holds true for $d(\gamma-1) \leqslant 2$ i.e. $\gamma \leqslant 1 + 2/d$. In both cases, the uniform L^γ-bound ensures that the asymptotic profile R_∞ is not only a probability measure but also an L^1-function. Finally, when $\varepsilon > 0$, $\nu = 0$ and $\gamma \geqslant 1 + 2/d$, we have $\alpha = 2$ and from (H4) we obtain

$$\frac{\varepsilon^2}{2} \int_{\mathbb{R}^d} |\nabla\sqrt{R}(t,\cdot)|^2 \leqslant C_0 \frac{\tau^2(t)}{(1+t)^2} \quad \forall t > 0,$$

which implies that $(\nabla\sqrt{R}(t,\cdot))_{t>0}$ is bounded in $L^2(\mathbb{R}^d)$ and thus that $R_\infty \in L^1(\mathbb{R}^d)$ also in this case.

3. NONLINEAR SCHRÖDINGER EQUATION

3.1. **A priori estimates.** For τ solution to (1.13), and ψ^ε solution to (1.27) with $\psi_0^\varepsilon \in \Sigma$ as defined in Theorem 1.7, Ψ^ε given by (1.29) solves

$$(3.1) \quad i\varepsilon\partial_t\Psi^\varepsilon + \frac{\varepsilon^2}{2\tau(t)^2}\Delta\Psi^\varepsilon = \frac{\alpha}{\tau(t)^{2\alpha}}\frac{|y|^2}{2}\Psi^\varepsilon + \frac{\mu^\varepsilon}{\tau(t)^{d\sigma}}|\Psi^\varepsilon|^{2\sigma}\Psi^\varepsilon, \quad \Psi^\varepsilon_{|t=0} = \frac{\psi_0^\varepsilon}{\|\psi_0^\varepsilon\|_{L^2(\mathbb{R}^d)}},$$

where

$$\mu^\varepsilon = \lambda\|\psi_0^\varepsilon\|_{L^2(\mathbb{R}^d)}^{2\sigma}.$$

The pseudo-energy for Ψ^ε is

$$\mathcal{E}^\varepsilon(\Psi^\varepsilon) = \frac{\varepsilon^2}{2\tau(t)^2}\|\nabla\Psi^\varepsilon(t)\|_{L^2(\mathbb{R}^d)}^2 + \frac{\alpha}{2\tau(t)^{2\alpha}}\int_{\mathbb{R}^d}|y|^2|\Psi^\varepsilon(t,y)|^2\mathrm{d}y$$
$$(3.2)$$
$$+ \frac{\mu^\varepsilon}{(\sigma+1)\tau(t)^{d\sigma}}\int_{\mathbb{R}^d}|\Psi^\varepsilon(t,y)|^{2\sigma+2}\mathrm{d}y,$$

and satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}^\varepsilon(\Psi^\varepsilon) + \mathcal{D}^\varepsilon(\Psi^\varepsilon) = 0,$$

where the dissipation is given by

(3.3)
$$\mathcal{D}^\varepsilon(\Psi^\varepsilon) = \frac{\dot{\tau}}{\tau}\left(\frac{\varepsilon^2}{\tau(t)^2}\|\nabla\Psi^\varepsilon(t)\|_{L^2(\mathbb{R}^d)}^2 + \frac{\alpha^2}{\tau(t)^{2\alpha}}\int_{\mathbb{R}^d}|y|^2|\Psi^\varepsilon(t,y)|^2\mathrm{d}y \right.$$
$$\left. + \frac{d\sigma\mu^\varepsilon}{(\sigma+1)\tau(t)^{d\sigma}}\int_{\mathbb{R}^d}|\Psi^\varepsilon(t,y)|^{2\sigma+2}\mathrm{d}y\right).$$

In the case of the nonlinear Schrödinger equation, justifying the above identity is standard at the level of regularity that we consider, and we refer to [12] for details. We infer:

Proposition 3.1. *Let* $d \geqslant 1$, $\varepsilon, \lambda > 0$, $0 < \sigma < \frac{2}{(d-2)_+}$, *and* $\psi_0^\varepsilon \in \Sigma$. *Then for* τ *solution to* (1.13) *with* $\alpha = \min\left(\frac{d\sigma}{2}, 1\right)$, *the function* Ψ^ε *defined by* (1.29) *satisfies*

$$\mathcal{E}^\varepsilon(\Psi^\varepsilon(t)) \leqslant \frac{\mathcal{E}^\varepsilon(\Psi^\varepsilon(0))}{\tau(t)^{\min(2,d\sigma)}}, \quad \forall t \geqslant 0, \quad \int_0^\infty \mathcal{D}^\varepsilon(\Psi^\varepsilon(t))\mathrm{d}t < \infty,$$

where $\mathcal{E}^\varepsilon$ *is given by* (3.2) *and* $\mathcal{D}^\varepsilon$ *is given by* (3.3).

The above proposition provides the same a priori estimates as we have used in the case of the Euler-Korteweg system. More precisely, for (ρ, u) related to ψ^ε thanks to Madelung transform like in (1.28), we note that Madelung transform for Ψ^ε provides

$$R^\varepsilon = |\Psi^\varepsilon|^2, \quad R^\varepsilon U^\varepsilon = \varepsilon \operatorname{Im}\left(\bar{\Psi}^\varepsilon \nabla \Psi^\varepsilon\right),$$

and thus, (ρ, u) and $(R^\varepsilon, U^\varepsilon)$ are related through

$$\rho(t,x) = \frac{1}{\tau(t)^d}R^\varepsilon\left(t, \frac{x}{\tau(t)}\right), \quad u(t,x) = \frac{1}{\tau(t)}U^\varepsilon\left(t, \frac{x}{\tau(t)}\right) + \frac{\dot{\tau}(t)}{\tau(t)}x,$$

which is exactly (1.14). Theorem 1.7 then appears as a direct consequence of Theorem 1.2 in the Euler-Korteweg case.

3.2. **Interpretation.** We now comment on some consequences of Proposition 3.1.

3.2.1. *Long range scattering.* Suppose $\varepsilon = 1$. In the case $\sigma \geqslant 2/d$, a complete scattering theory is available for (1.27), in the sense that given $\psi_0 \in \Sigma$, there exists $\psi_+ \in \Sigma$ such that

$$\left\|e^{-i\frac{t}{2}\Delta}\psi(t) - \psi_+\right\|_\Sigma \underset{t\to\infty}{\longrightarrow} 0, \quad \|f\|_\Sigma^2 := \|f\|_{L^2(\mathbb{R}^d)}^2 + \|\nabla f\|_{L^2(\mathbb{R}^d)}^2 + \|xf\|_{L^2(\mathbb{R}^d)}^2.$$

As a matter of fact, the same is true under the weaker assumption $\sigma \geqslant \sigma_0(d)$ for some $1/d < \sigma_0(d) < 2/d$; see e.g. [12]. A weaker convergence (in $L^2(\mathbb{R}^d)$ instead of Σ, with $\psi_+ \in L^2(\mathbb{R}^d)$, [34], and even $\psi_+ \in H^1(\mathbb{R}^d)$, [16]) holds for $\sigma > 1/d$. For $0 < \sigma \leqslant 1/d$, long range effects are present, as evoked in the introduction. In the case of (1.27), the long range effects are understood only in the critical case $\sigma = 1/d$: see [24] and references therein. See also [30] and references therein for the existence of wave operators (Cauchy problem with prescribed behavior at $t = \infty$ instead of $t = 0$) in the case $\sigma = 1/d$. It seems that so far, the long range scattering has not been studied for (1.27) in the case $0 < \sigma < 1/d$. The lack of regularity of the nonlinearity is an important technical difficulty, which was bypassed in the analogous case of (generalized) Hartree nonlinearities, see [18, 19, 20] and references therein.

In the case $0 < \sigma \leqslant 1/d$, Proposition 3.1 yields, for $\varepsilon = 1$,

$$\int_{\mathbb{R}^d}|y|^2|\Psi(t,y)|^2\mathrm{d}y + \|\Psi(t)\|_{L^{2\sigma+2}(\mathbb{R}^d)}^{2\sigma+2} \lesssim 1, \quad \forall t \geqslant 0,$$

and Theorem 1.7 shows the convergence of $|\Psi(t,\cdot)|^2$ in the limit $t \to \infty$, indicating that for the full range $0 < \sigma \leqslant 1/d$, long range effects do not affect the dispersive behavior, and present at leading order only in a phase modification.

3.2.2. *Semi-classical limit.* Consider the limit $\varepsilon \to 0$ in (1.27), for initial data under a WKB form,

$$\psi_0^\varepsilon(x) = a_0(x)e^{i\phi_0(x)/\varepsilon},$$

with a_0 and ϕ_0 smooth and independent of ε, ϕ_0 being real-valued. In particular, the L^2-norm of ψ_0^ε is independent of ε. Proposition 3.1 then yields, in the case $0 < \sigma \leqslant 1/d$,

$$(3.4) \qquad \int_{\mathbb{R}^d} |y|^2 |\Psi^\varepsilon(t,y)|^2 \mathrm{d}y + \|\Psi^\varepsilon(t)\|_{L^{2\sigma+2}(\mathbb{R}^d)}^{2\sigma+2} \leqslant C_0, \quad \forall t \geqslant 0,$$

for some $C_0 > 0$ independent of ε. This estimate indicates dispersive properties which are uniform in ε, a phenomenon which cannot hold in the linear case

$$i\varepsilon\partial_t\psi_{\mathrm{lin}}^\varepsilon + \frac{\varepsilon^2}{2}\Delta\psi_{\mathrm{lin}}^\varepsilon = V(x)\psi_{\mathrm{lin}}^\varepsilon, \quad \psi_0^\varepsilon(x) = a_0(x)e^{i\phi_0(x)/\varepsilon},$$

where the formation of caustics is incompatible with (3.4). Indeed in the linear case, the rapid oscillation are described, initially, by a Hamilton-Jacobi equation, whose solution may become singular in finite time, precisely on the caustic set: this geometrical phenomenon coincides with the amplification of the order of magnitude of $\psi_{\mathrm{lin}}^\varepsilon$ in the limit $\varepsilon \to 0$. While the L^2-norm of $\psi_{\mathrm{lin}}^\varepsilon$ is independent of (time and) ε, its $L^{2\sigma+2}$-norm blows up like some negative power of ε. The precise value of this power depends on the geometry of the caustic (see e.g. [13, 14, 25]), and is for instance equal to $-\frac{d\sigma}{2\sigma+2}$ in the case of a focal point (see e.g. [8]). In the case of (1.27), the Hamilton-Jacobi equation is replaced by a compressible Euler equation ((1.1)-(1.2) with $\varepsilon = \nu = 0$, where λ, σ and γ are related like in (1.28)), whose solution may develop singularities in finite time, from [29]. However, there is no amplification of Ψ^ε, at least in $L^2 \cap L^{2\sigma+2}$. This suggests that the notion of caustic must be adapted in this case, for the geometrical phenomenon and the analytical phenomenon, which coincide in the linear case, no longer do: the nonlinearity in (1.27) prevents the amplification phenomenon.

3.3. **Proof of Proposition 1.5.** Proposition 1.5 is actually valid for more general profiles, $a_\infty \in \Sigma$. Let $\varepsilon > 0 = \nu$, γ like in Proposition 1.5, and $a_\infty \in \Sigma$. Define ψ_+^ε by

$$a_\infty(x) = \frac{1}{\varepsilon^{d/2}}\hat{\psi}_+^\varepsilon\left(\frac{x}{\varepsilon}\right), \quad \text{where} \quad \hat{f}(\xi) = \frac{1}{(2\pi)^{d/2}}\int_{\mathbb{R}^d} e^{-ix\cdot\xi}f(x)\mathrm{d}x, \quad f \in \mathcal{S}(\mathbb{R}^d).$$

Since $\Sigma = H^1 \cap \mathcal{F}(H^1)$, $\psi_+^\varepsilon \in \Sigma$. Standard scattering theory for NLS (see e.g. [12, 16]) implies that there exists a unique solution $\psi^\varepsilon \in C(\mathbb{R}_+; \Sigma) \cap L^{\frac{4\sigma+4}{d\sigma}}(\mathbb{R}_+; L^{2\sigma+2}(\mathbb{R}^d))$ to (1.27), with

$$\lambda = \frac{\gamma}{\gamma-1} > 0, \quad \sigma = \frac{\gamma-1}{2},$$

such that

$$\|e^{-i\varepsilon\frac{t}{2}\Delta}\psi^\varepsilon(t) - \psi_+\|_\Sigma \underset{t\to\infty}{\longrightarrow} 0.$$

Since $e^{i\varepsilon\frac{t}{2}\Delta}$ is unitary on $L^2(\mathbb{R}^d)$, this implies

$$\|\psi^\varepsilon(t) - e^{i\varepsilon\frac{t}{2}\Delta}\psi_+\|_{L^2(\mathbb{R}^d)} \underset{t\to\infty}{\longrightarrow} 0.$$

On the other hand (see e.g. [33]),

$$\|e^{i\varepsilon\frac{t}{2}\Delta}\psi_+^\varepsilon - A(\psi_+^\varepsilon)(t)\|_{L^2(\mathbb{R}^d)} \underset{t\to\infty}{\longrightarrow} 0, \quad \text{where} \quad A(\psi_+)(t,x) = \frac{1}{(i\varepsilon t)^{d/2}}\hat{\psi}_+^\varepsilon\left(\frac{x}{\varepsilon t}\right)e^{i\frac{|x|^2}{2\varepsilon t}}$$

$$= \frac{1}{(it)^{d/2}}a_\infty\left(\frac{x}{t}\right)e^{i\frac{|x|^2}{2\varepsilon t}}.$$

We infer, from Cauchy–Schwarz and triangle inequalities,

$$
\begin{aligned}
\left\| |\psi^\varepsilon(t)|^2 - \frac{1}{t^d}\left|a_\infty\left(\frac{x}{t}\right)\right|^2 \right\|_{L^1(\mathbb{R}^d)} &= \left\| |\psi^\varepsilon(t)|^2 - |A(\psi_+^\varepsilon)(t)|^2 \right\|_{L^1(\mathbb{R}^d)} \\
&\leqslant \left(\|\psi^\varepsilon(t)\|_{L^2(\mathbb{R}^d)} + \|A(\psi_+^\varepsilon)(t)\|_{L^2(\mathbb{R}^d)} \right) \|\psi^\varepsilon(t) - A(\psi_+^\varepsilon)(t)\|_{L^2(\mathbb{R}^d)} \\
&\leqslant 2\|\psi_+^\varepsilon\|_{L^2(\mathbb{R}^d)} \|\psi^\varepsilon(t) - A(\psi_+^\varepsilon)(t)\|_{L^2(\mathbb{R}^d)} \underset{t\to\infty}{\longrightarrow} 0,
\end{aligned}
$$

hence Proposition 1.5 by defining (ρ, u) by Madelung transform (1.28), so it solves the Euler-Korteweg system.

4. Proof of Theorem 1.6

We end the paper with the proof of **Theorem 1.6**. This section is split into three subsections corresponding to the three different cases in **Theorem 1.6**.

4.1. **Euler.** The first case of Theorem 1.6 is simply a reformulation of the main result from [23]. The assumption made on u_0 ensures that the (multidimensional) Burgers equation

$$
\partial_t \bar{u} + \bar{u} \cdot \nabla \bar{u} = 0, \quad \bar{u}_{|t=0} = u_0,
$$

has a unique, global solution for $t \geqslant 0$. A typical example is $u_0(x) = x$. Then [23, Theorem 1] asserts that the Euler equation (1.1)-(1.2) has a global smooth solution such that

$$
\rho^{\frac{\gamma-1}{2}}, u - \bar{u} \in C^j([0,\infty[; H^{s-j}(\mathbb{R}^d)), \quad j = 0, 1.
$$

At this level of regularity, the unknowns (R, U) obtained from (ρ, u) through the change of unknown (1.14) satisfy a fortiori (H1)–(H3). Moreover, when $\alpha = \min(2, d(\gamma-1))$ all the formal manipulations leading to (1.21)-(1.23) and (1.24) are rigorously justified, and so (H4) is satisfied too.

Remark 4.1. In [32], the assumption on γ is restricted to $1 < \gamma \leqslant 1 + 2/d$, ρ_0 need not be compactly supported, and the assumption on u_0 reads $v_0 \in H^s(\mathbb{R}^d)$ with $\|v_0\|_{H^s(\mathbb{R})} \ll 1$, where $v_0(x) = u_0(x) - x$. The conclusion is then the same as above, with $\bar{u}$ replaced by

$$
\bar{u}(t, x) = \frac{x}{t+1},
$$

which is a particular solution of the Burgers equation. Therefore, the assumption on ρ_0 is slightly weaker, but the assumption on u_0 appears to be a particular case of the framework considered in [23].

4.2. **Euler-Korteweg.** The second case of Theorem 1.6 is a consequence of Madelung transform (1.28) and of the identities presented in Section 3. Indeed the assumption $1 < \gamma < 1 + \frac{4}{(d-2)_+}$ from Theorem 1.6 corresponds to $0 < \sigma < \frac{2}{(d-2)_+}$ in (1.27), with $\lambda > 0$ (referred to as defocusing case). Since we assume $\psi_0 \in \Sigma$, standard Cauchy theory for (1.27) (see e.g. [12]) yields the existence of a unique solution

$$
\psi^\varepsilon \in C(\mathbb{R}; \Sigma) \cap L_{\mathrm{loc}}^{\frac{4\sigma+4}{d\sigma}}(\mathbb{R}; L^{2\sigma+2}(\mathbb{R}^d)).
$$

In particular, we also have

$$
\Psi^\varepsilon \in C(\mathbb{R}; \Sigma) \cap L_{\mathrm{loc}}^{\frac{4\sigma+4}{d\sigma}}(\mathbb{R}; L^{2\sigma+2}(\mathbb{R}^d)).
$$

As noticed in Section 3, $R = |\Psi^\varepsilon|^2$ and $RU = \varepsilon \operatorname{Im}(\bar{\Psi}^\varepsilon \nabla \Psi^\varepsilon)$, and (H1)–(H3) are satisfied. In addition (see [2] or [11]),

$$
\varepsilon^2 |\nabla \Psi^\varepsilon|^2 = |\varepsilon \nabla \sqrt{R}|^2 + R|U|^2.
$$

Therefore (3.2) corresponds exactly to (1.18), and **Proposition 3.1** shows that (H4) is satisfied.

4.3. **Navier-Stokes.** We proceed with case (iii) of **Theorem 1.6** and consider the Navier-Stokes system with capital-letter unknowns (1.15)-(1.16) in dimension $d \leqslant 3$, where $\nu > 0$ and $\varepsilon \geqslant 0$. The weak regularity statements of (H1) are not sufficient to prove the decay estimates of (H4) with multiplier arguments as in the introduction. We need to obtain these estimates "by construction" so that we provide here a brief proof of existence of weak solutions satisfying (H1)-(H4). We note that, up to time-dependent scaling terms, (1.15)-(1.16) is similar to the barotropic (quantum) Navier-Stokes system. Construction of weak solution to this system goes back to [28, 15] (Newtonian Navier-Stokes, i.e. non-degenerate viscosity), to [7, 26] (degenerate viscosity) in a framework allowing the presence of vacuum thanks to a suitable setting of the problem, and is by now well-documented, see e.g. [35, 27, 9, 1] and references therein (see also [31]). We emphasize that one of the specificities of [9] is to construct solutions on $\mathbb{R}^d$, instead of $\mathbb{T}^d$ like in most of the references. This approach has been used in [1] in the polytropic case on $\mathbb{R}^d$. Our proof — that weak solutions to (1.15)-(1.16) satisfying (H1)-(H4) do exist — reproduces the strategy of [9] (isothermal case on $\mathbb{R}^d$), which in turn is based on the approach of [35, 27] (polytropic case on the torus). As a consequence, we do not give precise details. We only perform the formal energy estimates justifying our definition of weak solutions, give a scheme of the proof and explain how (H4) is achieved.

4.3.1. *Definition of weak solutions.* The system (1.15)-(1.16) is classically endowed with conservation (1.17) and dissipation estimate (1.20). Such estimates are not sufficient to build up a satisfactory weak solution theory. These pieces of information are complemented with the decay of the by-now called "BD-entropy" (see [6], among others). To construct this new quantity, we differentiate (1.15) with respect to space, and find:

$$(4.1) \qquad \partial_t \left(R \nabla \ln R \right) + \frac{1}{\tau^2} \mathrm{div}(R \nabla \ln(R) \otimes U) + \frac{1}{\tau^2} \mathrm{div}(R \nabla U) = 0.$$

The key-remark from [6] here is that the last term in this equation may combine with the Newtonian tensor in the momentum equation (1.16). So, we multiply (4.1) by ν and combine with (1.16). Denoting $V = U + \nu \nabla \ln(R)$, we obtain:

$$(4.2) \quad \partial_t(RV) + \frac{1}{\tau^2} \mathrm{div}(RV \otimes U) + \frac{\alpha}{2\tau^\alpha} yR + \frac{1}{\tau^{d(\gamma-1)}} \nabla R^\gamma$$
$$= \frac{1}{\tau^2} \mathrm{div} \left(\frac{\varepsilon^2}{2} \mathbb{K}[R] + \nu \sqrt{R} \mathcal{A}[R, U] \right) + \frac{\nu \dot{\tau}}{\tau} \nabla R,$$

where $\mathcal{A} = \sqrt{R}\mathbf{A}U$. We perform then a classical energy estimate on this new equation by multiplying with V/τ^2. The two first terms yield the time-derivative of the kinetic energy associated with V. The other terms are integrating by parts by splitting $V = U + \nu \nabla \ln(R)$ and remarking that, for symmetry reasons, we have:

$$\mathcal{A}[R, U] : \nabla^2 \ln(R) = 0.$$

Eventually, we obtain:

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathcal{E}_{\mathrm{BD}}[R, U] + \mathcal{D}_{\mathrm{BD}}[R, U] = \frac{\alpha \nu d}{2\tau^{2+\alpha}} \int R + \frac{\nu \dot{\tau}}{\tau^3} \int R \, \mathrm{div} \, U,$$

where the BD-entropy is defined by

$$\mathcal{E}_{\mathrm{BD}}[R, U] := \frac{1}{2\tau^2} \int \left(R|U + \nu \nabla \ln R|^2 + \varepsilon^2 |\nabla \sqrt{R}|^2 \right) + \frac{\alpha}{4\tau^\alpha} \int |y|^2 R$$
$$+ \frac{1}{(\gamma - 1)\tau^{d(\gamma-1)}} \int R^\gamma,$$

and the associated nonnegative dissipation is given by

$$\mathcal{D}_{\mathrm{BD}}[R,U] := \frac{\dot{\tau}}{\tau}\left[\frac{1}{\tau^2}\int\left(R|U|^2 + \varepsilon^2|\nabla\sqrt{R}|^2\right) + \frac{\alpha^2}{4\tau^\alpha}\int|y|^2 R + \frac{d}{\tau^{d(\gamma-1)}}\int R^\gamma\right],$$

$$+ \frac{\nu}{\tau^4}\int R|\mathbf{A}U|^2 + \frac{\nu\varepsilon^2}{\tau^4}\int R|\nabla^2\ln R|^2 + \frac{4\nu}{\tau^{d(\gamma-1)+2}}\int|\nabla R^{\gamma/2}|^2.$$

With this further remark we can now set a definition of weak solution on the basis of all the a priori bounded energy/entropy/dissipations:

Definition 4.2. Assume $\nu > 0$, $\gamma > 1$ and $\varepsilon \geqslant 0$. Let $(\sqrt{R_0}, \Lambda_0 = (\sqrt{R}U)_0) \in L^2(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$. We call global weak solution to (1.15)-(1.16), associated to the initial data $(\sqrt{R_0}, \Lambda_0 = (\sqrt{R}U)_0)$, any pair (R,U) such that there exists a collection $(\sqrt{R}, \sqrt{R}U, \mathbb{K}, \mathbb{T})$ satisfying

i) The following regularities:

$$(\langle y\rangle + |U|)\sqrt{R} \in L^\infty_{\mathrm{loc}}\left(0,\infty; L^2(\mathbb{R}^d)\right), \quad \nabla\sqrt{R} \in L^\infty_{\mathrm{loc}}\left(0,\infty; L^2(\mathbb{R}^d)\right),$$

$$\sqrt{R} \in L^\infty_{\mathrm{loc}}(0,\infty; L^{2\gamma}(\mathbb{R}^d)) \quad \nabla R^{\gamma/2} \in L^2_{\mathrm{loc}}(0,\infty; L^2(\mathbb{R}^d)),$$

$$\varepsilon\nabla^2\sqrt{R} \in L^2_{\mathrm{loc}}(0,\infty; L^2(\mathbb{R}^d)), \quad \sqrt{\varepsilon}\nabla R^{1/4} \in L^4_{\mathrm{loc}}(0,\infty; L^4(\mathbb{R}^d)),$$

$$\mathbb{T} \in L^2_{\mathrm{loc}}(0,\infty; L^2(\mathbb{R}^d)),$$

with the compatibility conditions

$$\sqrt{R} \geqslant 0 \text{ a.e. on } (0,\infty)\times\mathbb{R}^d, \quad \sqrt{R}U = 0 \text{ a.e. on } \{\sqrt{R}=0\}.$$

ii) The following equations in $\mathcal{D}'((0,\infty)\times\mathbb{R}^d)$

(4.3)
$$\begin{cases} \partial_t\sqrt{R} + \dfrac{1}{\tau^2}\operatorname{div}(\sqrt{R}U) = \dfrac{1}{2\tau^2}\operatorname{Trace}(\mathbb{T}), \\[2mm] \partial_t(RU) + \dfrac{1}{\tau^2}\operatorname{div}(\sqrt{R}U\otimes\sqrt{R}U) + 2y|\sqrt{R}|^2 + \nabla\left(|\sqrt{R}|^2\right) \\[3mm] \qquad\qquad = \operatorname{div}\left(\dfrac{\varepsilon^2}{2\tau^2}\mathbb{K} + \dfrac{\nu}{\tau^2}\sqrt{R}\mathbb{T}^s\right) + \dfrac{\nu\dot{\tau}}{\tau}\nabla R, \end{cases}$$

with $\mathbb{T}^s$ the symmetric part of $\mathbb{T}$ and the compatibility conditions:

(4.4)
$$\sqrt{R}\mathbb{T} = \nabla(\sqrt{R}\sqrt{R}U) - 2\sqrt{R}U\otimes\nabla\sqrt{R},$$

(4.5)
$$\mathbb{K} = \sqrt{R}\nabla^2\sqrt{R} - \nabla\sqrt{R}\otimes\nabla\sqrt{R}.$$

iii) For any $\psi \in C_0^\infty(\mathbb{R}^d)$,

$$\lim_{t\to 0}\int_{\mathbb{R}^d}\sqrt{R}(t,y)\psi(y)\,\mathrm{d}y = \int_{\mathbb{R}^d}\sqrt{R_0}(y)\psi(y)\,\mathrm{d}y,$$

$$\lim_{t\to 0}\int_{\mathbb{R}^d}\sqrt{R}(t,y)(\sqrt{R}U)(t,y)\psi(y)\,\mathrm{d}y = \int_{\mathbb{R}^d}\sqrt{R_0}(y)\Lambda_0(y)\psi(y)\,\mathrm{d}y.$$

We point out that this definition is readily adapted from [9, Definition 1.1], where the isothermal case $\gamma = 1$ is considered. It is also similar to [1, Definition 2.1]. The third existence statement in **Theorem 1.6** is then a straightforward consequence of:

Proposition 4.3. *Assume* $\nu > 0$, $\gamma > 1$ *and* $\varepsilon \geqslant 0$. *Let* $(\sqrt{R_0}, \Lambda_0 = (\sqrt{R}U)_0) \in L^2(\mathbb{R}^d)\times L^2(\mathbb{R}^d)$ *satisfy the compatibility conditions*

$$\sqrt{R_0} \geqslant 0 \text{ a.e. on } \mathbb{R}^d, \quad (\sqrt{R}U)_0 = 0 \text{ a.e. on } \{\sqrt{R_0}=0\},$$

as well as $\mathcal{E}[R_0,U_0] < \infty$, $\mathcal{E}_{\mathrm{BD}}[R_0,U_0] < \infty$. *There exists at least one global weak solution to* (1.15)-(1.16) *in the sense of Definition 4.2, which satisfies (H1)-(H4) and the conservation of mass.*

We stress that, in this definition, we set:

$$R_0 = \sqrt{R_0}^2, \qquad U_0 = \frac{(\sqrt{R}U)_0}{\sqrt{R_0}} \mathbf{1}_{\sqrt{R_0}>0}.$$

This is the common way to define functions of R and U in such a framework. In particular, the definition of the velocity-field U is satisfactory since we enforce the condition $\sqrt{R}U = 0$ under the condition $\sqrt{R} = 0$ in our construction and assumptions.

4.3.2. *Condition (H4) and roadmap of the proof of Proposition 4.3.* The proof of Proposition 4.3 follows the compactness approach of [9] relying on the key-ingredients introduced in [5], and resumed in, e.g., [22, 35, 27]. We point out that one important novelty of [9] was to treat the isothermal case while the ingredients of [35, 27] handle the precise polytropic case that we consider herein. Consequently, we only point out the roadmap of the proof herein and refer the reader to these previous references for more details on the different ingredients, and how to combine them.

The first step of the proof consists in solving a regularized version of (1.15)-(1.16) on a torus of arbitrary size $\ell > 1$, denoted by $\mathbb{T}_\ell^d$. This regularized version is associated with parameters $r = (r_0, r_1) \in (0, \infty)^2$, $\delta := (\delta_1, \delta_2) \in (0, \infty)^2$, $(\eta_1, \eta_2) \in (0, \infty)^2$, $m > 0$ sufficiently large (see [36]), and involves a "cold-pressure" exponent $k \in (0, \infty)$ that has to be chosen sufficiently large. This regularized system reads:

$$(4.6) \quad \partial_t R + \frac{1}{\tau^2} \operatorname{div}(RU) = \frac{\delta_1}{\tau^2} \Delta R,$$

$$(4.7) \quad \partial_t(RU) + \frac{1}{\tau^2} \operatorname{div}(RU \otimes U) + \frac{\alpha}{2\tau^\alpha} yR + \frac{1}{\tau^{d(\gamma-1)}} \nabla P_c(R)$$

$$+ \frac{r_0}{\tau^2} U + \frac{r_1}{\tau^2} R|U|^2 U + \frac{\delta_1}{\tau^2} (\nabla R \cdot \nabla) U$$

$$= \frac{\varepsilon^2}{2\tau^2} R\nabla\left(\frac{\Delta\sqrt{R}}{\sqrt{R}}\right) + \frac{\nu}{\tau^2} \operatorname{div}(R\mathbf{D}U) + \frac{\nu\dot{\tau}}{\tau} \nabla R + \frac{\delta_2}{\tau^2} \Delta^2 U + \frac{\eta_2}{\tau^2} R\nabla\Delta^{2m+1}R,$$

where:

$$P_c(R) = R^\gamma - \frac{\eta_1}{R^k}.$$

By a suitable truncation/regularization of the initial condition, the regularized system is solved when completed with initial data (R_0, U_0) satisfying:

$$(4.8) \qquad\qquad R_0 \in C^\infty(\mathbb{T}_\ell^d), \quad U_0 \in L^2(\mathbb{T}_\ell^d), \quad \inf_{y \in \mathbb{T}_\ell^d} R_0(y) \geqslant \theta > 0.$$

The remaining steps of the analysis consist in letting successively $|\delta| \to 0$, $|\eta| \to 0$ and then $|r| \to 0$, $\theta \to 0$, $\ell \to \infty$ (and possibly $\varepsilon \to 0$). The following lemma ensures that assumption (H4) is satisfied at the level of the approximation:

Lemma 4.4. *Given initial data* (R_0, U_0) *satisfying* (4.8)*, there exists a global solution* (R, U) *to* (4.6)-(4.7) *associated to* (R_0, U_0) *on the torus* $\mathbb{T}_\ell^d$*, which satisfies moreover the conservation of mass and the decay estimate (H4).*

The property (H4) being stable by weak convergence, the solution we construct inherits this property.

Proof. As in [9, Section 2], existence of solutions to (4.6)-(4.7) (with regularized initial data) is obtained *via* a Faedo-Galerkin approach. Namely, the velocity-field U is first chosen in a finite-dimensional subspace of $L^2(\mathbb{T}_\ell^d)$, Equation (4.7) being projected on this subspace, and (4.6) solved independently via a fixed-point argument. Again, we argue at the level of the finite-dimensional approximation, the same inequalities being satisfied by any limit of these approximations.

Since the continuity equation (4.6) is satisfied pointwise, we have readily:

$$\int_{\mathbb{T}_\ell^d} R(t, \cdot) = \int_{\mathbb{T}_\ell^d} R_0, \quad \forall\, t > 0.$$

Here R_0 should be thought of as the regularized initial data, but the regularization procedure ensures convergence of the mass of the regularized approximation to the mass of R_0. We obtain (1.17).

At the level of the projection, all solutions are smooth in space and C^1 in time. Multiplying (4.7) by U is then fully justified. Similarly to the computation of dissipation estimate for the full system, we obtain (see also [9, Proposition 2.6]) the following decay estimate:

$$(4.9) \qquad \frac{\mathrm{d}}{\mathrm{d}t} \mathcal{E}_{\mathrm{reg}}[R, U] + \mathcal{D}_{\mathrm{reg}}[R, U] = \frac{\alpha d \delta_1}{2\tau^{2+\alpha}} \int R - \frac{\nu \dot{\tau}}{\tau^3} \int R \operatorname{div} U$$

where:

$$\mathcal{E}_{\mathrm{reg}}[R, U] = \frac{1}{2\tau^2} \int_{\mathbb{T}_\ell^d} \left(R|U|^2 + \varepsilon^2 |\nabla \sqrt{R}|^2 + \eta_2 \int_{\mathbb{T}_\ell^d} |\nabla \Delta^m R|^2 \right)$$
$$+ \frac{\alpha}{4\tau^\alpha} \int_{\mathbb{T}_\ell^d} R|y|^2 + \frac{1}{\tau^{d(\gamma-1)}} \int_{\mathbb{T}_\ell^d} \left(\frac{1}{\gamma-1} R^\gamma + \frac{\eta_1}{k+1} \frac{1}{R^k} \right),$$

and

$$\mathcal{D}_{\mathrm{reg}}[R, U] = \frac{\dot{\tau}}{\tau} \left[\frac{1}{\tau^2} \int_{\mathbb{T}_\ell^d} \left(R|U|^2 + \varepsilon^2 |\nabla \sqrt{R}|^2 + \eta_2 |\nabla \Delta^m R|^2 \right) \right.$$
$$+ \frac{\alpha^2}{4\tau^\alpha} \int_{\mathbb{T}_\ell^d} R|y|^2 + \frac{d(\gamma-1)}{\tau^{d(\gamma-1)}} \int_{\mathbb{T}_\ell^d} \left(\frac{1}{\gamma-1} R^\gamma + \frac{\eta_1}{k+1} \frac{1}{R^k} \right) \Bigg]$$
$$+ \frac{\nu}{\tau^4} \int_{\mathbb{T}_\ell^d} R|\mathbf{D}U|^2 + \frac{\delta_2}{\tau^4} \int_{\mathbb{T}_\ell^d} |\Delta U|^2 + \frac{\delta_1 \eta_2}{\tau^4} \int_{\mathbb{T}_\ell^d} |\Delta^{m+1} R|^2$$
$$+ \frac{\delta_1}{\tau^{2+d(\gamma-1)}} \int_{\mathbb{T}_\ell^d} \left(\gamma R^{\gamma-2} + \frac{\eta_1 k}{R^{k+2}} \right) |\nabla R|^2$$
$$+ \frac{r_0}{\tau^4} \int_{\mathbb{T}_\ell^d} |U|^2 + \frac{r_1}{\tau^4} \int_{\mathbb{T}_\ell^d} R|U|^4 + \frac{\delta_1 \varepsilon^2}{2\tau^4} \int_{\mathbb{T}_\ell^d} R|\nabla^2 \ln R|^2.$$

At this point, we adapt the arguments of the introduction. First we remark that the right-hand side RHS of (4.9) satisfies:

$$(4.10) \qquad RHS \leqslant \left(\frac{\alpha d \delta_1}{\tau^{2+\alpha}} + \nu \left(\frac{\dot{\tau}}{\tau} \right)^2 \right) \int_{\mathbb{T}_\ell^d} R_0 + \frac{1}{2} \mathcal{D}_{\mathrm{reg}}[R, U].$$

Again, here R_0 should be the regularized initial data but the approximation procedure ensures convergence of the initial data in a sense that is sufficient to guarantee that all constants involving initial data are bounded by a constant depending on the initial data of the target system (see [9, Section 4.2]). In particular, the mass of the initial data can be assumed to be bounded by a constant C_0 depending only on initial data.

The inequality (4.9) then yields:

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathcal{E}_{\mathrm{reg}}[R, U] + \mathcal{D}_{\mathrm{reg}}[R, U] \leqslant C_0 \left(\frac{\alpha d \delta_1}{\tau^{2+\alpha}} + \nu \left(\frac{\dot{\tau}}{\tau} \right)^2 \right).$$

When $\delta_1 < \nu$, we can bound the right-hand side with a constant C_α depending only on α:

$$\frac{\alpha d \delta_1}{\tau^{2+\alpha}} + \nu \left(\frac{\dot{\tau}}{\tau} \right)^2 \leqslant \frac{C_\alpha \nu}{\tau^2}.$$

Integration in time yields an $L^1(0, \infty)$-bound on $\mathcal{D}_{\mathrm{reg}}[R, U]$. We can then choose $\alpha = \min(2, d(\gamma-1))$ and argue as in the introduction that:

$$\mathcal{D}_{\mathrm{reg}}[R, 0] \geqslant \alpha \frac{\dot{\tau}}{\tau} \mathcal{E}_{\mathrm{reg}}[R, U],$$

to yield that $\mathcal{E}_{\mathrm{reg}}[R, U]$ satisfies (1.25), and reproduce the computations of (1.24). We finally conclude that (H4) is satisfied by remarking that:

$$\mathcal{E} \leqslant \mathcal{E}_{\mathrm{reg}}[R, U], \quad \mathcal{D} \leqslant \mathcal{D}_{\mathrm{reg}}[R, U].$$

$\square$

APPENDIX A. COMPUTATIONS OF FORMAL ENERGY ESTIMATES

In this section, we consider (ρ, u) a solution to (1.1)-(1.2) and justify that at least formally, the decay estimate (1.11) should be satisfied. Define the functional

$$A[\rho, u] := t^2 E[\rho, u] - \int_{\mathbb{R}^d} t\rho u \cdot x + \frac{1}{2} \int_{\mathbb{R}^d} \rho |x|^2$$

$$= \frac{1}{2} \int_{\mathbb{R}^d} \left(\rho |tu - x|^2 + t^2 \varepsilon^2 |\nabla \sqrt{\rho}|^2 \right) + \frac{t^2}{\gamma - 1} \int_{\mathbb{R}^d} \rho^\gamma,$$

where we recall that the energy $E[\rho, u]$ is defined in (1.8). A straightforward computation gives us

$$\frac{\mathrm{d}}{\mathrm{d}t} A[\rho, u] = 2t E[\rho, u] + t^2 \frac{\mathrm{d}}{\mathrm{d}t} E[\rho, u] - t \int_{\mathbb{R}^d} \rho |u|^2 - td \int_{\mathbb{R}^d} \rho^\gamma$$

$$- t\varepsilon^2 \int_{\mathbb{R}^d} |\nabla \sqrt{\rho}|^2 + t\nu \int_{\mathbb{R}^d} \rho \operatorname{div} u.$$

Thanks to (1.7) we then get

$$\frac{\mathrm{d}}{\mathrm{d}t} A[\rho, u] = \frac{t}{\gamma - 1} (2 - d(\gamma - 1)) \int_{\mathbb{R}^d} \rho^\gamma - t^2 D[\rho, u] + t\nu \int_{\mathbb{R}^d} \rho \operatorname{div} u.$$

We now define the functional

$$B[\rho, u] := \frac{1}{t^2} A[\rho, u] = \frac{1}{2} \int_{\mathbb{R}^d} \left(\rho \left| u - \frac{x}{t} \right|^2 + \varepsilon^2 |\nabla \sqrt{\rho}|^2 \right) + \frac{1}{\gamma - 1} \int_{\mathbb{R}^d} \rho^\gamma$$

and we obtain, using previous computation, that

$$\frac{\mathrm{d}}{\mathrm{d}t} B[\rho, u] = \frac{1}{t} \frac{(2 - d(\gamma - 1))}{\gamma - 1} \int_{\mathbb{R}^d} \rho^\gamma - D[\rho, u] + \frac{1}{t} \nu \int_{\mathbb{R}^d} \rho \operatorname{div} u$$

$$- \frac{2}{t^3} \frac{1}{2} \int_{\mathbb{R}^d} \rho |tu - x|^2 - \frac{2}{t^3} \frac{t^2 \varepsilon^2}{2} \int_{\mathbb{R}^d} |\nabla \sqrt{\rho}|^2 - \frac{2}{t^3} \frac{t^2}{\gamma - 1} \int_{\mathbb{R}^d} \rho^\gamma$$

$$= -\frac{d}{t} \int_{\mathbb{R}^d} \rho^\gamma - \frac{1}{t} \int_{\mathbb{R}^d} \rho \left| u - \frac{x}{t} \right|^2 - \frac{\varepsilon^2}{t} \int_{\mathbb{R}^d} |\nabla \sqrt{\rho}|^2 - D[\rho, u] + \frac{1}{t} \nu \int_{\mathbb{R}^d} \rho \operatorname{div} u.$$

We identify in last expression the terms in the definition of $B[\rho, u]$, hence, using that $\frac{1}{t} \nu \int_{\mathbb{R}^d} \rho \operatorname{div} u \leqslant \frac{\nu}{2t^2} \int_{\mathbb{R}^d} \rho + \frac{\nu}{2} \int_{\mathbb{R}^d} \rho |\mathbf{D} u|^2$ and the conservation of mass, we deduce

$$\frac{\mathrm{d}}{\mathrm{d}t} B[\rho, u] \leqslant -\frac{\min(2, d(\gamma - 1))}{t} B[\rho, u] + \frac{C\nu}{t^2} - \frac{1}{2} D[\rho, u]$$

for some constant $C > 0$. Therefore, for $t > 0$, one has

(A.1) $$B[\rho, u](t) \leqslant \frac{C(E_0)}{(1 + t)^{\min(2, d(\gamma - 1))}} + \frac{C\nu}{1 + t}.$$

APPENDIX B. PROOF OF LEMMA 1.1

Proof. The local existence stems directly from the Cauchy-Lipschitz theorem. The only possible obstruction to the global propagation of regularity is the cancellation of τ, which is impossible in view of the relation, obtained after multiplication of (1.13) by $\dot{\tau}$ and integration:

$$\dot{\tau}(t)^2 = 1 - \frac{1}{\tau(t)^\alpha}.$$

This implies $\tau(t) \geqslant 1$. Therefore, $\tau \in C^\infty(\mathbb{R}; \mathbb{R}_+)$. The equation shows that τ is strictly (but not uniformly) convex. If it was bounded, $\tau(t) \leqslant M$, then we would have

$$\ddot{\tau}(t) \geqslant \frac{\alpha}{2M^{1+\alpha}} > 0,$$

hence a contradiction after two integrations. Therefore, $\tau(t_n) \to \infty$ for some sequence $t_n \to \infty$, and since τ is convex, $\tau(t) \to \infty$ as $t \to \infty$. Hence $\dot{\tau}(t)^2 \to 1$, and since τ is

necessarily increasing for $t \geqslant 0$, $\dot{\tau}(t) \to 1$, and the comparison of diverging integrals yields $\tau(t) \underset{t \to \infty}{\sim} t$. $\qquad \square$

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