REGULARIZATION ESTIMATES AND HYDRODYNAMICAL LIMIT FOR THE LANDAU EQUATION

KLEBER CARRAPATOSO, MOHAMAD RACHID, AND ISABELLE TRISTANI

Abstract. In this paper, we study the Landau equation under the Navier-Stokes scaling in the torus for hard and moderately soft potentials. More precisely, we investigate the Cauchy theory in a perturbative framework and establish some new short time regularization estimates for our rescaled nonlinear Landau equation. These estimates are quantified in time and we obtain the instantaneous expected anisotropic gain of regularity (see [54] for the corresponding hypoelliptic estimates on the linearized Landau collision operator). Moreover, the estimates giving the gain of regularity in the velocity variable are uniform in the Knudsen number. Intertwining these new estimates on the Landau equation with estimates on the Navier-Stokes-Fourier system, we are then able to obtain a result of strong convergence towards this fluid system.

Contents

Date: February 11, 2022.
2010 *Mathematics Subject Classification.* 35Q20, 35Q35, 35Q30, 45K05, 76P05, 47H20, 82C40.
Key words and phrases. Landau equation, incompressible Navier-Stokes equation, hydrodynamical limit, hypocoercivity, short time regularization estimates.

1. Introduction

In this paper, we are interested in the link between the Landau and Navier-Stokes equations. It has been a major challenge to establish rigorous links between microscopic and macroscopic equations for many years, this problem goes back to Hilbert [40] and the main goal is to obtain a unified description of gas dynamics. The equations of kinetic theory (including the Boltzmann and the Landau equations) can be seen as an intermediate step between the microscopic and macroscopic scales of description. In order to link the Landau and Navier-Stokes equations, we study a suitable rescaling of the Landau equation, as described in Subsection 1.1.

After giving some preliminary technical results in Section 2, the first part of our paper (Sections 3 and 4) is dedicated to the study of this rescaled Landau equation thanks to hypocoercivity methods, (linear and nonlinear) regularization estimates and sharp nonlinear estimates on the Landau collision operator. We study the Cauchy theory in a close-to-equilibrium framework for this equation and establish new and sharp regularization estimates in short time. The second part of our paper focuses on the aforementioned hydrodynamical limit problem. More precisely, in Section 6, we give a result of strong convergence of the solutions to the Landau equation constructed in the first part of the paper towards strong and global solutions to the incompressible Navier-Stokes-Fourier system. Our approach is reminiscent of the one used in [5, 24] for the hard spheres Boltzmann equation and improves the result obtained in [33, 55] in terms of type of convergence or functional framework in the case of not too soft potentials. Our analysis heavily relies on the estimates on the Landau equation established in the first part of the paper as well as on results of spectral analysis for the linearized Landau equation performed in [60] and presented in Section 5 and some refined estimates on the fluid problem (as in [24]).

1.1. **The kinetic model.**

We start by introducing the Landau equation which models the evolution of charged particles in a plasma through the evolution of the density of particles $f = f(t, x, v)$ which depends on time $t \in \mathbf{R}^+$, position $x \in \mathbf{T}^3$ the 3-dimensional unit periodic box and velocity $v \in \mathbf{R}^3$, when only binary collisions are taken into account. The Landau equation reads:

$$\partial_t f + v \cdot \nabla_x f = \frac{1}{\varepsilon} Q(f, f),$$

where $\varepsilon > 0$ is the Knudsen number which is the inverse of the average number of collisions for each particle per unit of time and Q is the Landau collision operator. It is defined as

$$Q(g, f)(v) = \partial_{v_i} \int_{\mathbf{R}^3} a_{ij}(v - v_*) \left[g(v_*) \partial_{v_j} f(v) - f(v) \partial_{v_j} g(v_*) \right] \mathrm{d}v_*, \tag{1.1}$$

where we use the convention of summation of repeated indices. The matrix a_{ij} is symmetric, semi-positive and is given by

$$a_{ij}(v) = |v|^{\gamma+2} \left(\delta_{ij} - \frac{v_i v_j}{|v|^2} \right), \quad -3 \leqslant \gamma \leqslant 1. \tag{1.2}$$

We have the following classification: We call hard potentials if $\gamma \in (0, 1]$, Maxwellian molecules if $\gamma = 0$, moderately soft potentials if $\gamma \in [-2, 0)$, very soft potentials if $\gamma \in (-3, -2)$ and Coulomb potential if $\gamma = -3$. Hereafter, we shall consider the cases of hard potentials, Maxwellian molecules and moderately soft potentials, i.e.

$$-2 \leqslant \gamma \leqslant 1.$$

The Landau equation preserves mass, momentum and energy. Indeed, at least formally, for any test function φ, we have

$$\begin{aligned} &\int_{\mathbf{R}^3} Q(f, f)(v) \varphi(v) \, \mathrm{d}v \\ &= -\frac{1}{2} \int_{\mathbf{R}^3 \times \mathbf{R}^3} a_{ij}(v - v_*) f(v) f(v_*) \left(\frac{\partial_{v_i} f(v)}{f(v)} - \frac{\partial_{v_i} f(v_*)}{f(v_*)} \right) \left(\partial_{v_j} \varphi(v) - \partial_{v_j} \varphi(v_*) \right) \mathrm{d}v_* \, \mathrm{d}v, \end{aligned} \tag{1.3}$$

from which we deduce that

$$\begin{aligned}(1.4)\quad \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbf{T}^3\times\mathbf{R}^3} f(x,v)\varphi(v)\,\mathrm{d}v\,\mathrm{d}x &= \int_{\mathbf{T}^3\times\mathbf{R}^3}\left[\frac{1}{\varepsilon}Q(f,f)(x,v)-v\cdot\nabla_x f(x,v)\right]\varphi(v)\,\mathrm{d}v\,\mathrm{d}x\\ &= 0 \quad \text{for} \quad \varphi(v)=1,v,|v|^2.\end{aligned}$$

Moreover, the Landau version of the Boltzmann H-theorem asserts that the entropy

$$H(f) := \int_{\mathbf{T}^3\times\mathbf{R}^3} f\,\log f\,\mathrm{d}v\,\mathrm{d}x$$

is non increasing. Indeed, at least formally, since a_{ij} is nonnegative, we have the following inequality for the entropy dissipation $D(f)$:

$$\begin{aligned}D(f) := -\frac{\mathrm{d}}{\mathrm{d}t}H(f) = \frac{1}{2}\int_{\mathbf{T}^3\times\mathbf{R}^3\times\mathbf{R}^3} & a_{ij}(v-v_*)f(v)f(v_*)\\ &\left(\frac{\partial_{v_i}f(v)}{f(v)}-\frac{\partial_{v_i}f(v_*)}{f(v_*)}\right)\left(\frac{\partial_{v_j}f(v)}{f(v)}-\frac{\partial_{v_j}f(v_*)}{f(v_*)}\right)\mathrm{d}v_*\,\mathrm{d}v\,\mathrm{d}x \geqslant 0.\end{aligned}$$

The second part of the H-theorem asserts that local equilibria of the Landau equation are local Maxwellian distributions in velocity. In what follows, we shall consider the following centered normalized Maxwellian independent of time t and space x which is a global equilibrium of our equation defined by

$$M(v) := \frac{1}{(2\pi)^{\frac{3}{2}}}e^{-\frac{|v|^2}{2}}.$$

Taking ε small has the effect of enhancing the role of collisions and thus when ε goes to 0, in view of the above mentioned Landau version of the Boltzmann H-theorem, the solution looks more and more like a local thermodynamical equilibrium. As suggested in previous works (see for example [4]), we consider the following rescaled Landau equation in which an additional dilatation of the macroscopic time scale has been done in order to be able to reach the Navier-Stokes equation in the limit:

$$(1.5)\qquad \partial_t f^\varepsilon + \frac{1}{\varepsilon}v\cdot\nabla_x f^\varepsilon = \frac{1}{\varepsilon^2}Q(f^\varepsilon,f^\varepsilon) \quad \text{in} \quad \mathbf{R}^+\times\mathbf{T}^3\times\mathbf{R}^3\,.$$

To relate the Landau equation to the incompressible Navier-Stokes equation, we look at equation (1.5) under the following linearization of order ε:

$$(1.6)\qquad f^\varepsilon(t,x,v) = M(v) + \varepsilon\sqrt{M}(v)g^\varepsilon(t,x,v).$$

Let us recall that taking ε small in this linearization corresponds to taking a small Mach number, which enables one to get in the limit the incompressible Navier-Stokes equation. If f^ε solves (1.5), then equivalently g^ε solves

$$(1.7)\qquad \partial_t g^\varepsilon + \frac{1}{\varepsilon}v\cdot\nabla_x g^\varepsilon = \frac{1}{\varepsilon^2}Lg^\varepsilon + \frac{1}{\varepsilon}\Gamma(g^\varepsilon,g^\varepsilon) \quad \text{in} \quad \mathbf{R}^+\times\mathbf{T}^3\times\mathbf{R}^3$$

where the nonlinear collision operator Γ is defined by

$$(1.8)\qquad \Gamma(f_1,f_2) := \frac{1}{\sqrt{M}}Q\left(\sqrt{M}f_1,\sqrt{M}f_2\right)$$

and the linearized collision operator L by

$$(1.9)\qquad Lf := \Gamma\left(\sqrt{M},f\right) + \Gamma\left(f,\sqrt{M}\right).$$

Notice that the property (1.3) implies that for any suitable functions f_1 and f_2,

$$(1.10)\qquad \int_{\mathbf{R}^3}\Gamma(f_1,f_2)(v)\,\varphi(v)\,\mathrm{d}v = 0 \quad \text{for} \quad \varphi(v)=\sqrt{M}, v\sqrt{M}, |v|^2\sqrt{M}.$$

We also define the full linearized operator Λ_ε as

$$(1.11)\qquad \Lambda_\varepsilon := \frac{1}{\varepsilon^2}L - \frac{1}{\varepsilon}v\cdot\nabla_x.$$

It is well known (see for example [14]) that the kernel of L is given by
$$\operatorname{Ker} L = \operatorname{Span}\left\{\sqrt{M}, v_1\sqrt{M}, v_2\sqrt{M}, v_3\sqrt{M}, |v|^2\sqrt{M}\right\},$$
and we shall denote by π the orthogonal projector onto $\operatorname{Ker} L$ which is defined by:

$$\begin{aligned}\pi f(v) = \bigg(\int_{\mathbf{R}^3} f(w)\sqrt{M}(w)\,\mathrm{d}w + \int_{\mathbf{R}^3} w f(w)\sqrt{M}(w)\,\mathrm{d}w\cdot v \\ + \int_{\mathbf{R}^3} \frac{|w|^2-3}{3} f(w)\sqrt{M}(w)\,\mathrm{d}w\,\frac{|v|^2-3}{2}\bigg)\sqrt{M}(v).\end{aligned}\tag{1.12}$$

Throughout the paper, we shall also use the following notation: For a given kinetic distribution $f = f(x,v)$, we denote by $f^\perp$ its microscopic part, namely
$$f^\perp := (\operatorname{Id} - \pi) f \tag{1.13}$$
and by (ρ_f, u_f, θ_f) its first macroscopic quantities defined through
$$\rho_f(x) := \int_{\mathbf{R}^3} f(x,v)\sqrt{M}(v)\,\mathrm{d}v, \tag{1.14}$$

$$u_f(x) := \int_{\mathbf{R}^3} v\, f(x,v)\sqrt{M}(v)\,\mathrm{d}v, \tag{1.15}$$
and
$$\theta_f(x) := \frac{1}{3}\int_{\mathbf{R}^3} (|v|^2-3) f(x,v)\sqrt{M}(v)\,\mathrm{d}v, \tag{1.16}$$
so that $\pi f = \left(\rho_f + u_f\cdot v + \theta_f \frac{|v|^2-3}{2}\right)\sqrt{M}$. Using for example Proposition 3.1 from [8], we also have:
$$\operatorname{Ker}\Lambda_\varepsilon = \operatorname{Span}\left\{\sqrt{M}, v_1\sqrt{M}, v_2\sqrt{M}, v_3\sqrt{M}, |v|^2\sqrt{M}\right\}$$
and the projector Π onto $\operatorname{Ker}\Lambda_\varepsilon$ is given by

$$\begin{aligned}\Pi f(v) = \bigg(\int_{\mathbf{T}^3\times\mathbf{R}^3} f(x,w)\sqrt{M}(w)\,\mathrm{d}w\,\mathrm{d}x + \int_{\mathbf{T}^3\times\mathbf{R}^3} f(x,w) w\sqrt{M}(w)\,\mathrm{d}w\,\mathrm{d}x\cdot v \\ + \int_{\mathbf{T}^3\times\mathbf{R}^3} \frac{|w|^2-3}{3} f(x,w)\sqrt{M}(w)\,\mathrm{d}w\,\mathrm{d}x\,\frac{|v|^2-3}{2}\bigg)\sqrt{M}(v).\end{aligned}\tag{1.17}$$

Notice that $\Pi f(v) = \displaystyle\int_{\mathbf{T}^3} \pi f(x,v)\,\mathrm{d}x$.

1.2. **Cauchy theory, decay and regularization for the Landau equation.** We introduce the following H^1-norm in velocity which naturally arises in the study of the Landau equation:
$$\|f\|^2_{H^1_{v,*}} := \|\langle v\rangle^{\frac{\gamma}{2}+1} f\|^2_{L^2_v} + \|\langle v\rangle^{\frac{\gamma}{2}} P_v \nabla_v f\|^2_{L^2_v} + \|\langle v\rangle^{\frac{\gamma}{2}+1}(\operatorname{Id} - P_v)\nabla_v f\|^2_{L^2_v}, \tag{1.18}$$
where P_v stands for the projection on v, namely, $P_v w = \left(w\cdot\frac{v}{|v|}\right)\frac{v}{|v|}$. We define the weighted Sobolev-type spaces $\mathscr{X}$ and $\mathscr{Y}_1$ as the spaces associated to the following norms:
$$\|f\|^2_{\mathscr{X}} := \|\langle v\rangle^{3(\frac{\gamma}{2}+1)} f\|^2_{L^2_{x,v}} + \|\langle v\rangle^{2(\frac{\gamma}{2}+1)}\nabla_x f\|^2_{L^2_{x,v}} + \|\langle v\rangle^{\frac{\gamma}{2}+1}\nabla^2_x f\|^2_{L^2_{x,v}} + \|\nabla^3_x f\|^2_{L^2_{x,v}}, \tag{1.19}$$
and
$$\begin{aligned}\|f\|^2_{\mathscr{Y}_1} := \|\langle v\rangle^{3(\frac{\gamma}{2}+1)} f\|^2_{L^2_x(H^1_{v,*})} + \|\langle v\rangle^{2(\frac{\gamma}{2}+1)}\nabla_x f\|^2_{L^2_x(H^1_{v,*})} \\ + \|\langle v\rangle^{\frac{\gamma}{2}+1}\nabla^2_x f\|^2_{L^2_x(H^1_{v,*})} + \|\nabla^3_x f\|^2_{L^2_x(H^1_{v,*})}.\end{aligned}\tag{1.20}$$
Remark that as in [32, 13], we work with "twisted" Sobolev spaces in which the weights depend on the order of the derivative in x, it allows us to close our nonlinear estimates.

Let us now state our main result on the well-posedness, decay and regularization of the rescaled Landau equation (1.7).

Theorem 1.1. *There is $\eta_0 > 0$ small enough such that for any $\varepsilon \in (0,1)$, if $g^\varepsilon_{\rm in} \in \mathscr{X}$ satisfies*

$$\int_{\mathbf{T}^3\times\mathbf{R}^3} g^\varepsilon_{\rm in}(x,v)\,\varphi(v)\,\mathrm{d}v\,\mathrm{d}x = 0 \quad \textit{for} \quad \varphi(v) = \sqrt{M}, v\sqrt{M}, |v|^2\sqrt{M}, \tag{1.21}$$

and $\|g^\varepsilon_{\rm in}\|_{\mathscr{X}} \leqslant \eta_0$, then the following holds:

(i) There is a unique global solution $g^\varepsilon \in L^\infty(\mathbf{R}_+;\mathscr{X}) \cap L^2(\mathbf{R}_+;\mathscr{Y}_1)$ to (1.7) associated to the initial data $g^\varepsilon_{\rm in}$, which verifies moreover

$$\sup_{t\geqslant 0} e^{2\sigma t}\|g^\varepsilon(t)\|^2_{\mathscr{X}} + \frac{1}{\varepsilon^2}\int_0^\infty e^{2\sigma t}\|(g^\varepsilon(t))^\perp\|^2_{\mathscr{Y}_1}\,\mathrm{d}t + \int_0^\infty e^{2\sigma t}\|g^\varepsilon(t)\|^2_{\mathscr{Y}_1}\,\mathrm{d}t \lesssim \|g^\varepsilon_{\rm in}\|^2_{\mathscr{X}}, \tag{1.22}$$

for any $0 < \sigma < \sigma_0$, where σ_0 is the decay rate of linearized operator Λ_ε given in Proposition 3.2, and where we recall that $(g^\varepsilon)^\perp$ is defined in (1.13).

(ii) In addition, the solution satisfies the following regularization estimates, for all $t > 0$,

$$\|g^\varepsilon(t)\|_{\mathscr{Y}_1} \lesssim \frac{e^{-\sigma t}}{\min(1,\sqrt{t})}\|g^\varepsilon_{\rm in}\|_{\mathscr{X}}, \quad \textit{and} \quad \varepsilon\|\widetilde{\nabla}_x g^\varepsilon(t)\|_{\mathscr{X}} \lesssim \frac{e^{-\sigma t}}{\min(1,t^{3/2})}\|g^\varepsilon_{\rm in}\|_{\mathscr{X}}, \tag{1.23}$$

where $\widetilde{\nabla}_x$ is a weighted anisotropic gradient defined in (2.6).

Remark 1.2. It is worth noticing that the condition (1.21) is equivalent to $g^\varepsilon_{\rm in} \in (\operatorname{Ker}\Lambda_\varepsilon)^\perp$.

Remark 1.3. Let us point out that the results obtained in Theorems 1.1 could be obtained in larger spaces of the type $\mathscr{E} := H^3_x L^2_v(\langle v\rangle^k\sqrt{M})$ for k large enough. More precisely, due to the linearization (1.6), working in spaces like $\mathscr{E}$ means that the original data f^ε lie in polynomially weighted Sobolev spaces, which is more relevant from a physical point of view. We chose to only present the proof in the functional space $\mathscr{X}$ because this functional framework is compatible with the second part of the paper which is about hydrodynamical limit of the Landau equation (1.7). Let us though explain the strategy to perform such an extension of our results from the functional space $\mathscr{X}$ to $\mathscr{E}$. The strategy is the same as the one used in [9] by Briant, Merino-Aceituno and Mouhot where uniform in ε estimates on solutions to the hard-spheres Boltzmann equation have been obtained. The trick is to rewrite the equation (1.7) as an equivalent system of two equations thanks to the splitting of the linearized operator $\Lambda_\varepsilon = \mathcal{A}_\varepsilon + \mathcal{B}_\varepsilon$ introduced in Section 5: We write $g^\varepsilon = g^\varepsilon_1 + g^\varepsilon_2$ with

$$\partial_t g^\varepsilon_1 = \mathcal{B}_\varepsilon g^\varepsilon_1 + \frac{1}{\varepsilon}\Gamma(g^\varepsilon_1,g^\varepsilon_1) + \frac{1}{\varepsilon}\Gamma(g^\varepsilon_1,g^\varepsilon_2) + \frac{1}{\varepsilon}\Gamma(g^\varepsilon_2,g^\varepsilon_1) \quad \text{and} \quad g^\varepsilon_1(t=0) = g^\varepsilon_{\rm in} \in \mathscr{E}$$

and

$$\partial_t g^\varepsilon_2 = \Lambda_\varepsilon g^\varepsilon_2 + \frac{1}{\varepsilon}\Gamma(g^\varepsilon_2,g^\varepsilon_2) + \mathcal{A}_\varepsilon g^\varepsilon_1 \quad \text{and} \quad g^\varepsilon_2(t=0) = 0 \in \mathscr{X}.$$

The first equation can be studied in the large space $\mathscr{E}$ thanks to the nice properties of $\mathcal{B}_\varepsilon$ in all type of spaces and the second one can be studied in the smaller space $\mathscr{X}$ since it starts from 0. Moreover, we have some nice estimates on this equation because the operator $\mathcal{A}_\varepsilon$ enjoys some regularizing properties, it is bounded from $\mathscr{E}$ into $\mathscr{X}$, we can thus use the estimates obtained for the first equation satisfied by g^ε_1. Following those ideas, one can obtain some nice nested a priori estimates on the system, which allow to conclude.

Remark 1.4. Our method should be robust enough to also treat the case of very soft and Coulomb potentials $-3 \leqslant \gamma < -2$ in which the linearized operator does not have a spectral gap (in this case, the inequality (2.10) does not provide coercivity anymore). More precisely, we should be able to obtain a similar result of Theorem 1.1 with the exponential time-decay in (1.22) being replaced by a sub-exponential one, by combining the arguments developed in this paper together with the study made in the case $\varepsilon = 1$ in [12] by Carrapatoso and Mischler. We do not treat this case in the present paper.

The Cauchy theory and the large-time behavior of the Landau equation for $\varepsilon = 1$ have been extensively studied. We here give a small sample of the existing literature: Let us mention [1] for renormalized solutions with defect measure, [19] for the convergence to equilibrium for a priori smooth solutions with general initial data, [32, 50, 13, 12] for strong solutions in a perturbative framework. Concerning the well-posedness of our rescaled equation, it has already been obtained in [33, 54] respectively by Guo and Rachid in Sobolev spaces (respectively in $H^N_{x,v}$ with $N \geqslant 8$ and in $H^3_x L^2_v$) thanks to nonlinear energy methods. In [8], Briant has obtained a similar result in $H^N_{x,v}$ with $N \geqslant 4$ thanks to the so-called H^1-hypocoercivity method at the linear level.

Our global strategy to prove Theorem 1.1-(i) is based on the study of the linearized equation. And then, we go back to the fully nonlinear problem. This is a standard strategy to develop a Cauchy theory in a close-to-equilibrium regime. However, we have to emphasize here that our study is quite involved as explained below.

At the linear level, our strategy is based on a L^2-hypocoercivity method which heavily relies on the micro-macro decomposition and is thus particularly adapted to the study of hydrodynamical problems. Recall that the challenge of hypocoercivity is to understand the interplay between the collision operator that provides dissipativity in the velocity variable and the transport one which is conservative, in order to obtain global dissipativity for the whole linear problem (see [59, 36] for a presentation of this topic). The L^2-hypocoercivity method has been introduced by Hérau [35] (see also [21]) for one dimensional space of collisional invariants and introduced by Guo in [34] for a space of collisional invariants of dimension larger than one (including the Boltzmann and Landau cases). Let us explain into more details the strategy, we first define a norm $|||\cdot|||_{L^2_{x,v}}$ (associated to the scalar product $\langle\!\langle \cdot,\cdot \rangle\!\rangle_{L^2_{x,v}}$) which is equivalent to the usual one $\|\cdot\|_{L^2_{x,v}}$ uniformly in ε and is such that

$$\langle\!\langle \Lambda_\varepsilon f, f \rangle\!\rangle_{L^2_{x,v}} \lesssim -\frac{1}{\varepsilon^2}\|f^\perp\|^2_{L^2_x(H^1_{v,*})} - \|f\|^2_{L^2_{x,v}}.$$

Such a norm is defined in Subsection 3.1 and is inspired by [34] (see also [7]) in which the more complex case of bounded domains with various boundary conditions is treated. Due to the fact that derivatives in x commute with Λ_ε, it is easy to deduce a similar result on the space $H^3_x L^2_v$. However, it is not an easy matter to recover such an energy estimate in larger or smaller spaces than $H^3_x L^2_v$. Actually, the methods presented in [31, 48] by Gualdani, Mischler and Mouhot to develop shrinkage or enlargement arguments at the level of energy estimates is not easily adaptable to rescaled equations if one wants to get uniform estimates in the parameter of rescaling. To develop an enlargement argument, one can use the trick introduced in [31, 48] of splitting the original equation into several ones. This trick was already used in [9, 2] to obtain uniform in ε estimates on the rescaled Boltzmann equation for respectively elastic and inelastic hard spheres in a large class of Sobolev spaces (see also Remark 1.3). However, we do not have such a method of splitting to perform a “shrinkage” argument. In the present paper, we exhibit a norm equivalent to the usual one that provides dissipativity for Λ_ε in a smaller space than $H^3_x L^2_v$ (namely in the space $\mathscr{X}$ defined in (1.19)) and that also preserves the gain of $1/\varepsilon$ on the microscopic part of the solution. This is done in Subsection 3.1. Notice that it is also possible to obtain decay estimates directly on the semigroup associated to Λ_ε thanks to Duhamel formula once one has exhibited a nice splitting of Λ_ε (see Section 5).

We then prove some new and sharp nonlinear estimates on the Landau collision operator (see Subsection 4.1) to be able to develop our Cauchy theory for the whole nonlinear problem in a close-to-equilibrium framework. It is worth mentioning that to prove good a priori estimates on the nonlinear problem, we use the hypocoercive norm defined in Subsection 3.1 and we only perform energy estimates. It is actually important that our analysis does not rely on the use of Duhamel formula because of the rescaling parameter (see the beginning of Section 4 for more details).

The strategy that we use to prove the regularization estimate in Theorem 1.1-(ii) is quite classical for linear hypoelliptic equations and has been introduced by Hérau and Nier [37] for the kinetic Fokker-Planck equation. Such a method has been used for many hypoelliptic equations: In [38] for the fractional kinetic Fokker-Planck equation, in [13, 12] for the linearized Landau equation, in [39] for the linearized Boltzmann equation without cutoff etc... To our knowledge, it is the first time that such a strategy is used for a nonlinear equation (even in the simpler case $\varepsilon = 1$) and for a rescaled equation (with uniform estimates in the rescaling parameter). Roughly speaking, the idea is to introduce a functional with weights in time which is a Lyapunov functional for our equation for small times. From this property, we are then able to recover some regularization estimates quantified in time as stated in Theorem 1.1-(ii). Here, the difficulties are threefold:

- First, we study a nonlinear equation, our computations are thus much more intricate, the idea behind our computations being that we work with small data which allows us to absorbe the nonlinearity. Our proof also requires some new and sharp nonlinear estimates on the collision operator (see Subsection 4.1).
- Then, the functional has to be suitably defined to handle the dependencies in ε. The differences of behaviors between microscopic and macroscopic parts of the solution have to be taken into account and in the spirit of the definition of the H^1-hypocoercive norm of Briant [8], some terms of the functional only involve the microscopic part of the solution (see (4.38) for the definition of the functional).
- Finally, since we want to obtain the optimal gain of regularity (the corresponding hypoelliptic estimates are provided in [54] by Rachid), our functional has to be defined accordingly. For example, in [13] in which the authors were not interested in getting the optimal gain of regularity (and in which only the case $\varepsilon = 1$ was treated), only classical derivation operators were involved in the definition of the functional. Here, the definition of the functional is much more intricate: We have to work with the anisotropic operators $\widetilde{\nabla}_v$ and $\widetilde{\nabla}_x$ defined in (2.6) and our functional also involves additional terms which are necessary to close our estimates.

To end this part, we mention that our proof also provides a regularization estimate in the space variable which is not uniform in ε (see (1.23)). The non-uniformity in ε of such a gain is expected since the transport operator and the linearized collision operator (which gives the gain of regularity in velocity) do not act at the same scale.

1.3. **The fluid model.** In the second part of the paper, we shall prove that the hydrodynamical limit of (1.5) as ε goes to zero is the incompressible Navier-Stokes-Fourier system associated with the Boussinesq equation which writes

$$\begin{cases} \partial_t u + u \cdot \nabla_x u - \nu_1 \Delta_x u = -\nabla_x p \\ \partial_t \theta + u \cdot \nabla_x \theta - \nu_2 \Delta_x \theta = 0 \\ \operatorname{div}_x u = 0 \\ \nabla_x(\rho + \theta) = 0. \end{cases} \tag{1.24}$$

In this system, the temperature θ, the density ρ and the pressure p are scalar unknowns, whereas the velocity u is an unknown vector field. The pressure can actually be eliminated from the equations by applying to the momentum equation the projector $\mathbb{P}$ onto the space of divergence-free vector fields. This projector is bounded over H^ℓ_x for all ℓ, and in L^p_x for all $1 < p < \infty$. To define the viscosity coefficients ν_i (see for example [4]), let us introduce the two unique functions Φ (which is a matrix-valued function) and Ψ (which is a vector-valued function) orthogonal to $\operatorname{Ker} L$ such that

$$\frac{1}{\sqrt{M}} L\left(\sqrt{M}\Phi\right) = \frac{|v|^2}{3}\mathrm{Id} - v \otimes v \quad \text{and} \quad \frac{1}{\sqrt{M}} L\left(\sqrt{M}\Psi\right) = \frac{5-|v|^2}{2}\, v.$$

The viscosity coefficients are then defined by

$$\nu_1 := \frac{1}{10}\int_{\mathbf{R}^3} \Phi : L\left(\sqrt{M}\Phi\right)\sqrt{M}\,\mathrm{d}v \quad \text{and} \quad \nu_2 := \frac{2}{15}\int_{\mathbf{R}^3} \Psi \cdot L\left(\sqrt{M}\Psi\right)\sqrt{M}\,\mathrm{d}v.$$

In what follows, we call *well-prepared data* the class of functions $f \in \operatorname{Ker} L$ that write

$$f(x,v) = \sqrt{M}(v)\left(\rho_f(x) + u_f(x)\cdot v + \frac{|v|^2-3}{2}\theta_f(x)\right) \quad \text{with} \quad \nabla_x \cdot u_f = 0 \quad \text{and} \quad \rho_f + \theta_f = 0 \tag{1.25}$$

where we recall that ρ_f, u_f and θ_f are defined in (1.14), (1.15), (1.16).

It is known that for mean free $(\rho_0, u_0, \theta_0) \in H^3_x$ small enough and satisfying

$$\nabla_x \cdot u_0 = 0 \quad \text{and} \quad \rho_0 + \theta_0 = 0, \tag{1.26}$$

there exists a unique solution $(\rho, u, \theta) \in H^3_x$ to (1.24) defined on $\mathbf{R}^+$ with associated initial data (ρ_0, u_0, θ_0) (see [23, 44, 45, 24]). For such an initial data, we also define g_0 a well-prepared data with (ρ_0, u_0, θ_0) as associated first macroscopic quantities, namely

$$g_0(x,v) := \sqrt{M}(v)\left(\rho_0(x) + u_0(x)\cdot v + \frac{|v|^2-3}{2}\theta_0(x)\right) \quad \text{with} \quad \nabla_x \cdot u_0 = 0 \quad \text{and} \quad \rho_0 + \theta_0 = 0. \tag{1.27}$$

Notice that from the definition of the space $\mathscr{X}$, we in particular have that $g_0 \in \mathscr{X}$ and the mean-free assumption made on (ρ_0, u_0, θ_0) implies that $g_0 \in (\operatorname{Ker}\Lambda_\varepsilon)^\perp$. Notice also that due to the definition of $\mathscr{X}$, the smallness assumption made on (ρ_0, u_0, θ_0) can be translated into a smallness assumption on g_0. Indeed, given the form of g_0, by triangular inequality, it is clear that

$$\|g_0\|_{\mathscr{X}} \lesssim \|(\rho_0, u_0, \theta_0)\|_{H^3_x}.$$

Moreover, since $\left\{\sqrt{M}, \sqrt{M}v_i, \sqrt{M}(|v|^2-3)/2\right\}$ is an orthogonal system in L^2_v, we also have that

$$\|g_0\|_{\mathscr{X}} \gtrsim \|g_0\|_{H^3_x L^2_v} \gtrsim \|(\rho_0, u_0, \theta_0)\|_{H^3_x}.$$

As a consequence, there exists $\eta_1 > 0$ such that if g_0 is of the form (1.27) and satisfies $\|g_0\|_{\mathscr{X}} \leqslant \eta_1$, then there exists $(\rho, u, \theta) \in H^3_x$ defined on $\mathbf{R}^+$ solution to (1.24). We define the kinetic distribution lying in $\operatorname{Ker} L$ with associated macroscopic quantities (ρ, u, θ)

$$g(t,x,v) := \sqrt{M}(v)\left(\rho(t,x) + u(t,x)\cdot v + \frac{|v|^2-3}{2}\theta(t,x)\right). \tag{1.28}$$

We also have the following estimate

$$\|g\|_{L^\infty_t(\mathscr{X})} \lesssim C(\|g_0\|_{\mathscr{X}}) \tag{1.29}$$

where $C(\|g_0\|_{\mathscr{X}})$ is a constant only depending on the $\mathscr{X}$-norm of the data g_0. The aforementioned results on the system (1.24) can be found in [24, Appendix B.3] in which more details and references on the subject are given.

1.4. **Hydrodynamical limit result.** For the statement of the main hydrodynamical result, we first introduce the following notation for functional spaces: If X_1 and X_2 are two function spaces, we say that a function f belongs to $X_1 + X_2$ if there are $f_1 \in X_1$ and $f_2 \in X_2$ such that $f = f_1 + f_2$ and we define

$$\|f\|_{X_1+X_2} := \min_{\substack{f = f_1 + f_2 \\ f_i \in X_i}} \left(\|f_1\|_{X_1} + \|f_2\|_{X_2}\right).$$

Theorem 1.5. *Let $g^\varepsilon_{\rm in} \in \mathscr{X} \cap (\operatorname{Ker}\Lambda_\varepsilon)^\perp$ for $\varepsilon \in (0,1)$ such that $\|g^\varepsilon_{\rm in}\|_{\mathscr{X}} \leqslant \eta_0$ (where η_0 is defined in Theorem 1.1) and $g^\varepsilon \in L^\infty_t(\mathscr{X})$ being the associated solutions of (1.7) with initial data $g^\varepsilon_{\rm in}$ constructed in Theorem 1.1. Consider also $g_0 \in \mathscr{X} \cap (\operatorname{Ker}\Lambda_\varepsilon)^\perp$ such that $\|g_0\|_{\mathscr{X}} \leqslant \eta_1$ and g defined respectively as in (1.27) and (1.28) (where we recall that η_1 has been chosen small enough so that g is defined globally in time).*

There exists $\eta_2 \in (0, \min(\eta_0, \eta_1))$ such that if $\max\left(\|g^\varepsilon_{\rm in}\|_{\mathscr{X}}, \|g_0\|_{\mathscr{X}}\right) \leqslant \eta_2$ and

$$\|g^\varepsilon_{\rm in} - g_0\|_{\mathscr{X}} \xrightarrow[\varepsilon\to 0]{} 0, \tag{1.30}$$

then we have

$$\|g^\varepsilon - g\|_{L^\infty_t(\mathscr{X})} \xrightarrow[\varepsilon\to 0]{} 0. \tag{1.31}$$

If $\max\left(\|g^\varepsilon_{\rm in}\|_{\mathscr{X}}, \|g_0\|_{\mathscr{X}}\right) \leqslant \eta_2$ and

$$\|\pi g^\varepsilon_{\rm in} - g_0\|_{\mathscr{X}} \xrightarrow[\varepsilon\to 0]{} 0, \tag{1.32}$$

then we have

$$\|g^\varepsilon - g\|_{L^1_t(\mathscr{Y}_1)+L^\infty_t(\mathscr{X})} \xrightarrow[\varepsilon\to 0]{} 0. \tag{1.33}$$

Remark 1.6. One can get a rate of convergence in (1.31) and (1.33) if we suppose that g_0 has some additional regularity in x, namely a rate of ε^δ if the regularity is supposed to be $H^{3+\delta}_x$ for $\delta \in (0, 1/2]$. We refer to Theorem 6.1 for a quantitative version of this result.

Remark 1.7. As explained in Remark 1.3, the results of Theorem 1.1 could be obtained in larger spaces $\mathscr{E} = H^3_x L^2_v(\langle v\rangle^k \sqrt{M})$. A similar approach as the one used by Gervais in [26, 25] in which the hard spheres Boltzmann equation is treated in "large" Sobolev spaces, might yield the associated hydrodynamical result.

Remark 1.8. As explained in Remark 1.4, the strategy of the proof of Theorem 1.1 should also work in order to treat the case of very soft and Coulomb potentials. However, in order to obtain the associated hydrodynamical result, our method employs some fine spectral estimates that are known to hold only for the case $-2 \leqslant \gamma \leqslant 1$ by [60]. Therefore, if we are able to extend the results of [60] to the case of very soft and Coulomb potentials, we might then be able to obtain the analogous result of Theorem 1.5.

We first give a short overview of the existing literature on the problem of deriving fluid equations from kinetic ones (we refer to the book by Saint-Raymond [56] for a thorough presentation of the topic). The first justifications of the link between kinetic and fluid equations were formal and based on asymptotic expansions by Hilbert, Chapman, Enskog and Grad (see [40, 16, 29]). The first rigorous convergence proofs based also on asymptotic expansions were given by Caflisch [10] (see also [43] and [17]). In those papers, the limit is justified up to the first singular time for the fluid equation. By using the nonlinear energy method introduced by himself in [32], Guo [33] has justified the limit towards the Navier-Stokes equation and beyond in Hilbert's expansion from Boltzmann and Landau equations (see below for more details on this result).

There has also been some convergence proofs based on spectral analysis in the framework of strong solutions close to equilibrium introduced by Grad [30] and Ukai [57] for the Boltzmann equation. In this respect, we refer to the works by Nishida, Bardos and Ukai, Gallagher and Tristani [53, 5, 24]. These results use the description of the spectrum of the linearized Boltzmann equation in Fourier space in the space variable performed in [52, 15, 22]. Our approach is reminiscent of these ones and relies on the generalization of the paper [22] to several kinetic equations (including the Landau one) made in [60] by Yang and Yu. Notice also that such a spectral result has recently been obtained in [26] by Gervais for the hard-spheres Boltzmann equation in a larger class of Sobolev spaces.

More recently, some uniform in ε estimates on kinetic equations have allowed to prove (at least) weak convergence towards the Navier-Stokes equation. Let us mention [42, 55] in which the cases of Boltzmann equation without cutoff and the Landau equations are treated. In [8, 9], the authors have obtained convergence to equilibrium results for the rescaled Boltzmann equation (and also the Landau equation in [8]) uniformly in the rescaling parameter using respectively hypocoercivity and enlargement methods. In [9], the authors are able to weaken the assumptions on the data down to Sobolev spaces with polynomial weights. We also refer to [2] in which a similar approach combined with perturbative arguments has been used to derive a fluid system from the inelastic Boltzmann equation. Notice that Briant [8] has combined this with Ellis and Pinsky result [22] to recover strong convergence in the case of the Boltzmann equation.

Finally, let us mention that this problem has been extensively studied in the framework of weak solutions, the goal being to obtain solutions for the fluid models from renormalized solutions introduced by DiPerna and Lions in [20] for the Boltzmann equation. We shall not make an extensive presentation of this program as it is out of the realm of this study, but let us mention that it was started by Bardos, Golse and Levermore at the beginning of the nineties in [4, 3] and was continued by those authors, Saint-Raymond, Masmoudi, Lions among others. We mention here a (non exhaustive) list of papers which are part of this program (see [27, 28, 46, 47, 56]).

Let us focus on the Landau equation for which the literature is scarcer. As mentioned above, in [33], Guo justifies the limit the Navier-Stokes limit and beyond in the Hilbert's expansion from (the Boltzmann and) the Landau equations (even for the case of very soft potentials) in the torus by using his nonlinear energy method. Our result on the hydrodynamical limit is reminiscent of the one obtained in [33] for the hard and moderately soft potentials in the sense that we work with strong solutions and we prove a strong convergence result. It is however worth noticing that our functional framework is less restrictive (we only work with 3 derivatives in x and no derivative in v whereas in [33], regularity on 8 derivatives in both variables x and v is required). Moreover, there is an important loss of regularity in the estimates of convergence proven in [33] whereas we only lose δ derivatives in x and there is no loss in v to get a rate of convergence of ε^δ from Landau to Navier-Stokes equation (see Theorem 6.1). In the present paper and in [33], the fluid initial data are supposed to be well-prepared, namely the divergence free condition and the Boussinesq relation (1.26) are supposed to hold. We refer to [41] by Jiang and Xiong for an extension to the case where the fluid part is not supposed to be well-prepared and the creation and propagation of initial layers is studied. In [33, 41], the kinetic initial data is supposed to have a specific form so that there is no creation of kinetic initial layers. Our presentation is slightly different since we do not use Hilbert expansion to study the limit towards the Navier-Stokes equation, our assumption being the following: The projection of the kinetic initial data onto the kernel of the linearized operator L is supposed to converge towards the well-prepared fluid initial data as $\varepsilon \to 0$. Finally, in [55], Rachid obtained a result of weak-$\star$ convergence in $L^\infty_t(H^3_x L^2_v)$ towards the incompressible Navier-Stokes-Fourier system, we have thus strengthen this result for the type of initial data that we consider here. One can notice that the assumptions made on the fluid initial data in [55] are a bit less restrictive since the divergence free condition for u_0 and the Boussinesq relation for ρ_0 and θ_0 are not supposed to hold. The initial layer that such an initial condition creates is absorbed there in the weak convergence. In our framework, we can not hope to absorbe it in a strong convergence framework because the initial layer is propagated over time in the periodic domain (see [41]). Note also that in [24], the authors were able to treat this type of "completely ill-prepared" data only in the case of the whole space since those terms have some dispersive properties in the whole space.

Let us describe into more details our strategy to obtain strong convergence. It is inspired by the ones used in [5, 8, 24]. Indeed, as in [24], using the spectral analysis performed in [60] by Yang and Yu, in order to prove our main convergence result, we reformulate the fluid equation in a kinetic fashion and we then study the equation satisfied by the difference satisfied between the kinetic and the fluid solutions. However, let us point out that we are not able to perform a fixed point argument as in the aforementioned paper. This is due to the fact that the structure of the Landau bilinear operator is more complicated than the hard-spheres Boltzmann one. Indeed, there is an anisotropic loss of derivatives and weights in the nonlinear estimates which prevents us from closing a fixed point estimate. To circumvent this difficulty, we use some new a priori estimates on the solution of the linearized rescaled Landau equation and on the nonlinear rescaled Landau equation (1.7) that are uniform in the Knudsen number and that have been presented in Theorem 1.1. By intertwining these refined and sharp kinetic estimates and fluid mechanics ones, we are able to prove a result of strong convergence from the solutions of the Landau equation to the Navier-Stokes one as stated in Theorem 1.5.

1.5. **Outline of the paper.** In Section 2, we give some technical results on the Landau collision operator that will be used all along the paper. In Section 3, we develop hypocoercivity and regularization estimates for the linearized problem. In Section 4, we develop our perturbative Cauchy theory for the whole nonlinear problem as well as some regularization estimates on it. In Section 5, we develop some new estimates on the linearized problem that are useful to prove our hydrodynamical result, which is proven in Section 6.

Acknowledgements. The authors thank Frédéric Hérau for his valuable advice and fruitful discussions. This work has been partially supported by the Projects EFI: ANR-17-CE40-0030 (K.C. and I.T.) and SALVE: ANR-19-CE40-0004 (I.T.) of the French National Research Agency (ANR).

2. Preliminary results

In this section, we present some technical results and tools that will be useful throughout the paper.

2.1. **Collision operator.** Recalling the definition of the matrix $a = (a_{ij})_{ij}$ in (1.2) and that we use the convention of summation of repeated indices through the paper, we define the following quantities

$$
\begin{aligned}
b_i(v) &= \partial_j a_{ij}(v) = -2\,|v|^\gamma\, v_i,\\
c(v) &= \partial_{ij} a_{ij}(v) = -2(\gamma+3)\,|v|^\gamma,
\end{aligned}
$$

in such a way that one can rewrite the Landau operator (1.1) as

$$
\begin{aligned}
Q(g,f) &= (a_{ij} * g)\partial_{v_i,v_j} f - (c*g)f\\
&= \partial_{v_i}\left\{(a_{ij}*g)\partial_{v_j} f - (b_i * g) f\right\},
\end{aligned}
\tag{2.1}
$$

where $*$ denotes the convolution in the velocity variable v.

We now state a technical lemma in which we provide a rewriting of the nonlinear operator Γ defined in (1.8) and the linearized collision operator L defined in (1.9).

Lemma 2.1. *There holds*

$$
\begin{aligned}
\Gamma(f_1,f_2) = \partial_{v_i}&\left\{\left(a_{ij} * [\sqrt{M} f_1]\right)\partial_{v_j} f_2\right\} - \partial_{v_i}\left\{\left(b_i * [\sqrt{M} f_1]\right) f_2\right\}\\
&- \left(a_{ij} * [\sqrt{M} f_1]\right) v_i \partial_{v_j} f_2 + \frac{1}{4}\left(a_{ij} * [\sqrt{M} f_1]\right) v_i v_j f_2\\
&- \frac{1}{2}\left(a_{ii} * [\sqrt{M} f_1]\right) f_2,
\end{aligned}
\tag{2.2}
$$

and

$$
\begin{aligned}
Lf = \partial_{v_i}&\left\{(a_{ij}*M)\,\partial_{v_j} f\right\} + \left\{-\frac{1}{4}(a_{ij}*M)\,v_i v_j + \frac{1}{2}\partial_{v_i}[(a_{ij}*M)\,v_j]\right\} f\\
&+ \left(a_{ij} * [\sqrt{M} f]\right) v_i v_j \sqrt{M} - \left(a_{ii} * [\sqrt{M} f]\right)\sqrt{M} - \left(c * [\sqrt{M} f]\right)\sqrt{M}.
\end{aligned}
\tag{2.3}
$$

Proof. From the definition of Γ in (1.8) and using the formulation (2.1) for Q, we first obtain

$$
\Gamma(g_1,g_2) = \frac{1}{\sqrt{M}}\partial_{v_i}\left\{\left(a_{ij} * [\sqrt{M} g_1]\right)\partial_{v_j}(\sqrt{M} g_2) - \left(b_i * [\sqrt{M} g_1]\right)\sqrt{M} g_2\right\}.
$$

By writing $\partial_{v_j}(\sqrt{M} g_2) = \sqrt{M}\partial_{v_j} g_2 - \frac{1}{2} v_j \sqrt{M} g_2$ we thus obtain

$$
\begin{aligned}
\Gamma(g_1,g_2) = \frac{1}{\sqrt{M}}\partial_{v_i}&\left\{\left(a_{ij} * [\sqrt{M} g_1]\right)\sqrt{M}\partial_{v_j} g_2\right\} - \frac{1}{2}\frac{1}{\sqrt{M}}\partial_{v_i}\left\{\left(a_{ij} * [\sqrt{M} g_1]\right) v_j \sqrt{M} g_2\right\}\\
&- \frac{1}{\sqrt{M}}\partial_{v_i}\left\{\left(b_i * [\sqrt{M} g_1]\right)\sqrt{M} g_2\right\}.
\end{aligned}
$$

Applying the derivative ∂_{v_i} inside the brackets to the term $\sqrt{M}$ we then get

$$
\begin{aligned}
\Gamma(g_1,g_2) = \partial_{v_i}&\left\{\left(a_{ij} * [\sqrt{M} g_1]\right)\partial_{v_j} g_2\right\} - \frac{1}{2}\left(a_{ij} * [\sqrt{M} g_1]\right) v_i \partial_{v_j} g_2\\
&- \frac{1}{2}\partial_{v_i}\left\{\left(a_{ij} * [\sqrt{M} g_1]\right) v_j g_2\right\} + \frac{1}{4}\left(a_{ij} * [\sqrt{M} g_1]\right) v_i v_j g_2\\
&- \partial_{v_i}\left\{\left(b_i * [\sqrt{M} g_1]\right) g_2\right\} + \frac{1}{2}\left(b_i * [\sqrt{M} g_1]\right) v_i g_2.
\end{aligned}
$$

Finally, we apply the derivative to the third term in the right-hand side of the above equation and using that $a_{ij}=a_{ji}$ for any i, j, we get

$$\begin{aligned}\Gamma(g_1,g_2) &= \partial_{v_i}\left\{\left(a_{ij}*[\sqrt{M}g_1]\right)\partial_{v_j}g_2\right\} - \frac12\left(a_{ij}*[\sqrt{M}g_1]\right)v_i\partial_{v_j}g_2\\ &\quad -\frac12\left(b_j*[\sqrt{M}g_1]\right)v_jg_2 - \frac12\left(a_{ii}*[\sqrt{M}g_1]\right)g_2 - \frac12\left(a_{ij}*[\sqrt{M}g_1]\right)v_j\partial_{v_i}g_2\\ &\quad +\frac14\left(a_{ij}*[\sqrt{M}g_1]\right)v_iv_jg_2 - \partial_{v_i}\left\{\left(b_i*[\sqrt{M}g_1]\right)g_2\right\} + \frac12\left(b_i*[\sqrt{M}g_1]\right)v_ig_2\\ &= \partial_{v_i}\left\{\left(a_{ij}*[\sqrt{M}g_1]\right)\partial_{v_j}g_2\right\} - \left(a_{ij}*[\sqrt{M}g_1]\right)v_i\partial_{v_j}g_2\\ &\quad -\frac12\left(a_{ii}*[\sqrt{M}g_1]\right)g_2 + \frac14\left(a_{ij}*[\sqrt{M}g_1]\right)v_iv_jg_2 - \partial_{v_i}\left\{\left(b_i*[\sqrt{M}g_1]\right)g_2\right\}\end{aligned}$$

which completes the proof of (2.2).

We now prove (2.3). From (2.2) we get

$$\begin{aligned}\Gamma(\sqrt{M},f) &= \partial_{v_i}\left\{(a_{ij}*M)\,\partial_{v_j}f\right\} - \partial_{v_i}\left\{(b_i*M)\,f\right\}\\ &\quad -(a_{ij}*M)\,v_i\partial_{v_j}f + \frac14(a_{ij}*M)\,v_iv_jf - \frac12(a_{ii}*M)\,f.\end{aligned}$$

For the second term in the right-hand side of above equation, we observe that

$$-\partial_{v_i}\left\{(b_i*M)\,f\right\} = -(c*M)\,f - (b_i*M)\,\partial_{v_i}f,$$

as well as

$$-(b_i*M)\,\partial_{v_i}f = -\left(\partial_{v_j}a_{ij}*M\right)\partial_{v_i}f = (a_{ij}*v_jM)\,\partial_{v_i}f = (a_{ij}*M)\,v_j\partial_{v_i}f$$

by using that $a_{ij}(v-w)(v_i-w_i)=0$. We also remark, using that $a_{ij}(v-w)(v_i-w_i)(v_j-w_j)=0$,

$$\begin{aligned}-(c*M) &= -(a_{ij}*\partial_{v_i}\partial_{v_j}M) = (a_{ij}*\delta_{ij}M) - (a_{ij}*v_iv_jM)\\ &= (a_{ii}*M) - (a_{ij}*M)v_iv_j,\end{aligned}$$

as well as

$$\partial_{v_i}[(a_{ij}*M)v_j] = -(a_{ij}*M)v_iv_j + (a_{ii}*M).$$

Putting together previous equalities, we finally obtain

$$\Gamma(\sqrt{M},f) = \partial_{v_i}\left\{(a_{ij}*M)\,\partial_{v_j}f\right\} + \left\{-\frac14(a_{ij}*M)\,v_iv_j + \frac12\partial_{v_i}[(a_{ij}*M)v_j]\right\}f.$$

Moreover, from (2.2) we get

$$\begin{aligned}\Gamma(f,\sqrt{M}) &= \partial_{v_i}\left\{\left(a_{ij}*[\sqrt{M}f]\right)\partial_{v_j}\sqrt{M}\right\} - \partial_{v_i}\left\{\left(b_i*[\sqrt{M}f]\right)\sqrt{M}\right\}\\ &\quad -\left(a_{ij}*[\sqrt{M}f_1]\right)v_i\partial_{v_j}\sqrt{M} + \frac14\left(a_{ij}*[\sqrt{M}f]\right)v_iv_j\sqrt{M}\\ &\quad -\frac12\left(a_{ii}*[\sqrt{M}f]\right)\sqrt{M}\end{aligned}$$

so that, developing the derivatives ∂_{v_i}, we get

$$\begin{aligned}\Gamma(f,\sqrt{M}) &= \left(b_j*[\sqrt{M}f]\right)\partial_{v_j}\sqrt{M} + \left(a_{ij}*[\sqrt{M}f]\right)\partial_{v_i}\partial_{v_j}\sqrt{M}\\ &\quad -\left(c*[\sqrt{M}f]\right)\sqrt{M} - \left(b_i*[\sqrt{M}f]\right)\partial_{v_i}\sqrt{M}\\ &\quad -\left(a_{ij}*[\sqrt{M}f]\right)v_i\partial_{v_j}\sqrt{M} + \frac14\left(a_{ij}*[\sqrt{M}f]\right)v_iv_j\sqrt{M}\\ &\quad -\frac12\left(a_{ii}*[\sqrt{M}f]\right)\sqrt{M}.\end{aligned}$$

Observing that

$$\partial_{v_j}\sqrt{M} = -\frac12 v_j\sqrt{M} \quad\text{and}\quad \partial_{v_i}\partial_{v_j}\sqrt{M} = -\frac12\delta_{ij}\sqrt{M} + \frac14 v_iv_j\sqrt{M},$$

we then get

$$\Gamma(f,\sqrt{M}) = \Big(a_{ij} * [\sqrt{M} f]\Big)\, v_i v_j \sqrt{M} - \Big(a_{ii} * [\sqrt{M} f]\Big) \sqrt{M} - \Big(c * [\sqrt{M} f]\Big) \sqrt{M},$$

which concludes the proof. $\square$

For $v \in \mathbf{R}^3$, we define the symmetric matrix $\mathbf{A}(v) = (A_{ij}(v))_{1\leqslant i,j\leqslant 3}$ whose coefficients are given by

$$A_{ij}(v) = (a_{ij} *_v M)\,(v).$$

We can decompose $\mathbf{A}(v)$ as $\mathbf{A}(v) = \mathbf{B}^\top(v)\mathbf{B}(v)$, where $\mathbf{B}(v) = (B_{ij}(v))_{1\leqslant i,j\leqslant 3}$ is a matrix with real-valued smooth entries (see [54]). Recall also that from [18], for $v \in \mathbf{R}^3 \setminus \{0\}$, the matrix $\mathbf{A}(v)$ has a simple eigenvalue $\ell_1(v) > 0$ associated with the eigenvector v and a double eigenvalue $\ell_2(v) > 0$ associated with the eigenspace $v^\perp$. Moreover, when $|v| \to +\infty$, we have

$$\ell_1(v) \sim 2\langle v\rangle^{\gamma} \quad \text{and} \quad \ell_2(v) \sim \langle v\rangle^{\gamma+2}. \tag{2.4}$$

As such, one can write that for any $v \in \mathbf{R}^3 \setminus \{0\}$,

$$\mathbf{A}(v) = \ell_1(v)\frac{v}{|v|}\otimes\frac{v}{|v|} + \ell_2(v)\left(\mathrm{Id} - \frac{v}{|v|}\otimes\frac{v}{|v|}\right),$$

where Id denotes the identity matrix and

$$\mathbf{B}(v) = \sqrt{\ell_1}(v)\frac{v}{|v|}\otimes\frac{v}{|v|} + \sqrt{\ell_2}(v)\left(\mathrm{Id} - \frac{v}{|v|}\otimes\frac{v}{|v|}\right). \tag{2.5}$$

In what follows, we will use the following differential operators

$$\widetilde{\nabla}_v := \mathbf{B}(v)\nabla_v \quad \text{and} \quad \widetilde{\nabla}_x := \mathbf{B}(v)\nabla_x \tag{2.6}$$

as well as their adjoint operators in L^2_v given by, for $F : \mathbf{R}^d \to \mathbf{R}^d$,

$$(\widetilde{\nabla}_v)^* F = -\nabla_v \cdot (\mathbf{B}(v)F) \quad \text{and} \quad (\widetilde{\nabla}_x)^* F = -\widetilde{\nabla}_x \cdot F. \tag{2.7}$$

Using the formulation (2.3) of the linearized collision operator L, we can therefore rewrite it as

$$\begin{aligned} Lf = &-\widetilde{\nabla}_v^*\widetilde{\nabla}_v f - \left\{\frac{1}{4}|\mathbf{B}(v)v|^2 - \frac{1}{2}\nabla_v\cdot\Big[\mathbf{B}^\top(v)\mathbf{B}(v)v\Big]\right\} f \\ &- \Big\{-\Big(a_{ij} * \sqrt{M} f\Big)\, v_i v_j + \Big(a_{ii} * \sqrt{M} f\Big) + \Big(c * \sqrt{M} f\Big)\Big\}\sqrt{M}. \end{aligned} \tag{2.8}$$

The functions A_{ij}, B_{ij} verify the following properties (see for example [33, 54]): For any multi-index $\alpha \in \mathbf{N}^3$ we have, for all $v \in \mathbf{R}^3$,

$$|\partial_v^\alpha A_{ij}(v)| \lesssim \langle v\rangle^{\gamma+2-|\alpha|}, \quad |\partial_v^\alpha B_{ij}(v)| \lesssim \langle v\rangle^{\frac{\gamma}{2}+1-|\alpha|}. \tag{2.9}$$

From [18, 32, 49, 51], we also know that L has a spectral gap, more precisely, there is $\sigma_L > 0$ such that

$$\langle Lf, f\rangle_{L^2_v} \leqslant -\sigma_L \|f - \pi f\|^2_{H^1_{v,*}} \tag{2.10}$$

for any $f \in \mathrm{Dom}(L)$, where we recall that π is the projector onto the kernel of L defined in (1.12) and the $H^1_{v,*}$-norm is defined in (1.18). Notice that in the case $-2 \leqslant \gamma \leqslant 1$ which we study in the present paper, the previous inequality is indeed a coercivity estimate because $\|\cdot\|_{H^1_{v,*}} \geqslant \|\cdot\|_{L^2_v}$.

2.2. **Functional spaces.** We first notice that there exist some universal positive constants C_1 and C_2 such that the $H^1_{v,*}$-norm in velocity defined in (1.18) satisfies

$$C_1\|f\|^2_{H^1_{v,*}} \leqslant \|\langle v\rangle^{\frac{\gamma}{2}+1} f\|^2_{L^2_v} + \|\widetilde\nabla_v f\|^2_{L^2_v} \leqslant C_2\|f\|^2_{H^1_{v,*}}$$

where $\widetilde\nabla_v$ is defined in (2.6). This comes from [18, 32], more precisely, we use the smoothness of the eigenvalues $\ell_1(v)$ and $\ell_2(v)$ in $v \in \mathbf{R}^3 \setminus \{0\}$, their asymptotic behavior at infinity (2.4) and the fact that if $|v|<1$, they are bounded from below by a positive constant independent of v. In order to lighten the notations, in what follows, we will still denote

$$\|f\|^2_{H^1_{v,*}} = \|\langle v\rangle^{\frac{\gamma}{2}+1} f\|^2_{L^2_v} + \|\widetilde\nabla_v f\|^2_{L^2_v}.$$

We also introduce some H^2 norm in velocity defined through the following norm:

$$\|f\|^2_{H^2_{v,*}} := \|\langle v\rangle^{\gamma+2} f\|^2_{L^2_v} + \|\widetilde\nabla_v(\langle v\rangle^{\frac{\gamma}{2}+1} f)\|^2_{L^2_v} + \|\langle v\rangle^{\frac{\gamma}{2}+1}\widetilde\nabla_v f\|^2_{L^2_v} + \|\widetilde\nabla_v\widetilde\nabla_v f\|^2_{L^2_v}.$$

Similarly to the definitions of $\mathscr{X}$ and $\mathscr{Y}_1$, we define the weighted Sobolev-type space $\mathscr{Y}_2$ as the space associated to the norm

$$\begin{aligned}\|f\|^2_{\mathscr{Y}_2} := &\|\langle v\rangle^{3(\frac{\gamma}{2}+1)} f\|^2_{L^2_x(H^2_{v,*})} + \|\langle v\rangle^{2(\frac{\gamma}{2}+1)}\nabla_x f\|^2_{L^2_x(H^2_{v,*})}\\ &+ \|\langle v\rangle^{\frac{\gamma}{2}+1}\nabla^2_x f\|^2_{L^2_x(H^2_{v,*})} + \|\nabla^3_x f\|^2_{L^2_x(H^2_{v,*})}.\end{aligned} \tag{2.11}$$

We then introduce the spaces $\mathscr{Z}^\varepsilon_i$ for $i=1,2$ that involve derivatives in x: $\mathscr{Z}^\varepsilon_1$ is associated with the norm

$$\|f\|^2_{\mathscr{Z}^\varepsilon_1} := \|f\|^2_{\mathscr{X}} + \|f\|^2_{\mathscr{Y}_1} + \varepsilon^2\|\widetilde\nabla_x f\|^2_{\mathscr{X}}, \tag{2.12}$$

and $\mathscr{Z}^\varepsilon_2$ is associated with

$$\|f\|^2_{\mathscr{Z}^\varepsilon_2} := \|f\|^2_{\mathscr{Z}^\varepsilon_1} + \|f\|^2_{\mathscr{Y}_2} + \varepsilon^4\|\widetilde\nabla^2_x f\|^2_{\mathscr{X}}. \tag{2.13}$$

For the sequel, it is worth noticing that if $f \in \mathscr{Z}^\varepsilon_2$, then $\varepsilon\widetilde\nabla_v\widetilde\nabla_x f \in \mathscr{X}$. Indeed, a simple computation based on integrations by parts shows that

$$\varepsilon\|\widetilde\nabla_v\widetilde\nabla_x f\|_{\mathscr{X}} \lesssim \varepsilon^2\|\widetilde\nabla^2_x f\|_{\mathscr{X}} + \|f\|_{\mathscr{Y}_2}.$$

For $i=1,2$, we also define the associated dual spaces $(\mathscr{Y}_i)'$ and $(\mathscr{Z}^\varepsilon_i)'$ with $\mathscr{X}$ as a pivot space, more precisely, they are associated with the following norms:

$$\|f\|_{\mathscr{Y}_i'} := \sup_{\|\varphi\|_{\mathscr{Y}_i}\leqslant 1} \langle f,\varphi\rangle_{\mathscr{X}} \tag{2.14}$$

and

$$\|f\|_{(\mathscr{Z}^\varepsilon_i)'} := \sup_{\|\varphi\|_{\mathscr{Z}^\varepsilon_i}\leqslant 1} \langle f,\varphi\rangle_{\mathscr{X}} \tag{2.15}$$

where $\langle\cdot,\cdot\rangle_{\mathscr{X}}$ is the scalar product associated to $\|\cdot\|_{\mathscr{X}}$ defined in (1.19). Notice that we have the following interpolation result:

$$[\mathscr{X}, (\mathscr{Z}^\varepsilon_2)']_{1/2,2} = (\mathscr{Z}^\varepsilon_1)'. \tag{2.16}$$

The notation used above is the classical one of real interpolation (see [6]). For sake of completeness, we briefly recall the meaning of this notation. For C and D two Banach spaces which are both embedded in the same Hausdorff topological vector space, for any $z \in C+D$, we define the K-function by

$$K(t,z) := \inf_{z=c+d}\left(\|c\|_C + t\|d\|_D\right), \quad \forall\, t>0.$$

The space $[C,D]_{\theta,p}$ for $\theta\in(0,1)$ and $p\in[1,+\infty]$ is then defined by:

$$[C,D]_{\theta,p} := \left\{z \in C+D,\ t\mapsto K(t,z)/t^\theta \in L^p\,(\mathrm{d}t/t)\right\}.$$

2.3. **Basic estimates.** We gather in this subsection some basic estimates concerning the collision operator L that will be useful in the sequel.

In order to simplify, we recall the formulation of the operator L in (2.8) and we introduce the function

$$\psi(v) := \frac{1}{4}|\mathbf{B}(v)v|^2 - \frac{1}{2}\nabla_v \cdot \Big[\mathbf{B}^\top(v)\mathbf{B}(v)v\Big] \tag{2.17}$$

as well as the operators

$$L_1 f := -\widetilde{\nabla}_v^* \widetilde{\nabla}_v f - \psi f \tag{2.18}$$

and

$$L_2 f := -\Big\{-\Big(a_{ij} * \sqrt{M} f\Big)\, v_i v_j + \Big(a_{ii} * \sqrt{M} f\Big) + \Big(c * \sqrt{M} f\Big)\Big\}\sqrt{M} \tag{2.19}$$

so that we have $L = L_1 + L_2$.

We start with some basic commutator estimates.

Lemma 2.2. *For any suitable function $f = f(x,v)$ and any $(x,v) \in \mathbf{T}^3 \times \mathbf{R}^3$, there holds:*

(i) $[\widetilde{\nabla}_v, v\cdot\nabla_x] f(x,v) = \widetilde{\nabla}_x f(x,v)$.

(ii) For any $\alpha \in \mathbf{R}$ and $1 \leqslant j \leqslant 3$, one has

$$|[\langle v\rangle^\alpha, \widetilde{\nabla}_{v_j}]f|(x,v) = |[\langle v\rangle^\alpha, \widetilde{\nabla}^*_{v_j}]f|(x,v) \lesssim \langle v\rangle^{\frac{\gamma}{2}+\alpha-1}|f|(x,v).$$

(iii) For any $1 \leqslant i,j \leqslant 3$, one has

$$|[\widetilde{\nabla}_{v_i}, \widetilde{\nabla}_{v_j}]f|(x,v) \lesssim \langle v\rangle^{\gamma+1}|\nabla_v f|(x,v).$$

(iv) For any $1 \leqslant i,j \leqslant 3$, one has

$$|[\widetilde{\nabla}_{v_i}, \widetilde{\nabla}^*_{v_j}]f|(x,v) \lesssim \langle v\rangle^{\gamma+1}|\nabla_v f|(x,v) + \langle v\rangle^{\gamma}|f|(x,v).$$

(v) For any $1 \leqslant i,j \leqslant 3$, one has

$$|[\widetilde{\nabla}_{v_i}, \widetilde{\nabla}_{x_j}]f|(x,v) = |[\widetilde{\nabla}^*_{v_i}, \widetilde{\nabla}_{x_j}]f|(x,v) \lesssim \langle v\rangle^{\gamma+1}|\nabla_x f|(x,v).$$

(vi) For any $\alpha \in \mathbf{R}$, one has

$$\Big|\Big[[\langle v\rangle^\alpha, \widetilde{\nabla}^*_{v_j}], \widetilde{\nabla}_{v_j}\Big] f\Big|(x,v) \lesssim \langle v\rangle^{\gamma+\alpha-1}|f|(x,v).$$

(vii) For any $1 \leqslant i,j,k \leqslant 3$, one has

$$\Big|\Big[[\widetilde{\nabla}_{v_i}, \widetilde{\nabla}_{v_j}], \widetilde{\nabla}_{v_k}\Big] f\Big|(x,v) \lesssim \langle v\rangle^{\frac{3\gamma}{2}+1}|\nabla_v f|(x,v).$$

and

$$\Big|\Big[[\widetilde{\nabla}_{v_i}, \widetilde{\nabla}^*_{v_j}], \widetilde{\nabla}_{v_k}\Big] f\Big|(x,v) \lesssim \langle v\rangle^{\frac{3\gamma}{2}+1}|\nabla_v f|(x,v) + \langle v\rangle^{\frac{3\gamma}{2}}|f|(x,v).$$

(viii) For any $1 \leqslant i,j,k \leqslant 3$, one has

$$\Big|\Big[[\widetilde{\nabla}_{x_i}, \widetilde{\nabla}_{v_j}], \widetilde{\nabla}_{v_k}\Big] f\Big|(x,v) = \Big|\Big[[\widetilde{\nabla}_{x_i}, \widetilde{\nabla}^*_{v_j}], \widetilde{\nabla}_{v_k}\Big] f\Big|(x,v) \lesssim \langle v\rangle^{\frac{3\gamma}{2}+1}|\nabla_x f|(x,v).$$

Proof. Recall that we denote $\mathbf{B}(v) = (B_{ij})_{1\leqslant i,j\leqslant 3}$ and that we have

$$\widetilde{\nabla}_{v_i} f = B_{im}\partial_{v_m} f, \quad \widetilde{\nabla}_{x_i} f = B_{im}\partial_{x_m} f, \quad \widetilde{\nabla}^*_{v_i} f = -\partial_{v_m}(B_{im} f), \quad \widetilde{\nabla}^*_{x_i} f = -B_{im}\partial_{x_m} f = -\widetilde{\nabla}_{x_i} f.$$

(i) We have

$$\begin{aligned}
[\widetilde{\nabla}_{v_i}, v_\ell \partial_{x_\ell}] f &= B_{im}\partial_{v_m}(v_\ell\partial_{x_\ell} f) - v_\ell\partial_{x_\ell}(B_{im}\partial_{v_m} f)\\
&= B_{i\ell}\partial_{x_\ell} f + B_{im} v_\ell \partial_{v_m}\partial_{x_\ell} f - v_\ell B_{im}\partial_{x_\ell}\partial_{v_m} f\\
&= \widetilde{\nabla}_{x_i} f.
\end{aligned}$$

(ii) We easily compute

$$[\langle v\rangle^\alpha, \widetilde{\nabla}_{v_j}] f = -(\widetilde{\nabla}_{v_j}\langle v\rangle^\alpha) f$$

as well as

$$[\langle v\rangle^\alpha, \widetilde{\nabla}^*_{v_j}]f = (\widetilde{\nabla}_{v_j}\langle v\rangle^\alpha)f.$$

We conclude the proof by remarking that $\widetilde{\nabla}_{v_j}\langle v\rangle^\alpha = \alpha(\mathbf{B}(v)v)_j\langle v\rangle^{\alpha-2}$ and using that $|\mathbf{B}(v)v| \lesssim \langle v\rangle^{\frac{\gamma}{2}+1}$ thanks to the definition (2.5) of $\mathbf{B}(v)$.

(iii) We easily compute

$$\begin{aligned}
[\widetilde{\nabla}_{v_i}, \widetilde{\nabla}_{v_j}]f &= B_{im}\partial_{v_m}(B_{j\ell}\partial_{v_\ell}f) - B_{j\ell}\partial_{v_\ell}(B_{im}\partial_{v_m}f)\\
&= B_{im}B_{j\ell}\partial_{v_m}\partial_{v_\ell}f + B_{im}(\partial_{v_m}B_{j\ell})\partial_{v_\ell}f - B_{j\ell}B_{im}\partial_{v_\ell}\partial_{v_m}f - B_{j\ell}(\partial_{v_\ell}B_{im})\partial_{v_m}f\\
&= B_{im}(\partial_{v_m}B_{j\ell})\partial_{v_\ell}f - B_{j\ell}(\partial_{v_\ell}B_{im})\partial_{v_m}f,
\end{aligned}$$

and we conclude the proof using (2.9).

(iv) We have

$$\begin{aligned}
[\widetilde{\nabla}_{v_i}, \widetilde{\nabla}^*_{v_j}]f &= -B_{im}\partial_{v_m}\partial_{v_\ell}(B_{j\ell}f) + \partial_{v_\ell}(B_{j\ell}B_{im}\partial_{v_m}f)\\
&= -B_{im}B_{j\ell}\partial_{v_m}\partial_{v_\ell}f - B_{im}(\partial_{v_m}B_{j\ell})\partial_{v_\ell}f\\
&\quad - B_{im}(\partial_{v_m}\partial_{v_\ell}B_{j\ell})f - B_{im}(\partial_{v_\ell}B_{j\ell})\partial_{v_m}f\\
&\quad + B_{im}(\partial_{v_\ell}B_{j\ell})\partial_{v_m}f + B_{j\ell}(\partial_{v_\ell}B_{im})\partial_{v_m}f + B_{j\ell}B_{im}\partial_{v_\ell}\partial_{v_m}f\\
&= -B_{im}(\partial_{v_m}B_{j\ell})\partial_{v_\ell}f + B_{j\ell}(\partial_{v_\ell}B_{im})\partial_{v_m}f - B_{im}(\partial_{v_m}\partial_{v_\ell}B_{j\ell})f.
\end{aligned}$$

We can simplify last expression by relabelling the indices m and ℓ of the second term, which gives

$$[\widetilde{\nabla}_{v_i}, \widetilde{\nabla}^*_{v_j}]f = [B_{jm}(\partial_{v_m}B_{i\ell}) - B_{im}(\partial_{v_m}B_{j\ell})]\,\partial_{v_\ell}f - B_{im}(\partial_{v_m}\partial_{v_\ell}B_{j\ell})f.$$

We then conclude the proof by using using (2.9).

(v) We have

$$\begin{aligned}
[\widetilde{\nabla}_{v_i}, \widetilde{\nabla}_{x_j}]f &= B_{im}\partial_{v_m}(B_{j\ell}\partial_{x_\ell}f) - B_{j\ell}\partial_{x_\ell}(B_{im}\partial_{v_m}f)\\
&= B_{im}B_{j\ell}\partial_{v_m}\partial_{x_\ell}f + B_{im}(\partial_{v_m}B_{j\ell})\partial_{x_\ell}f - B_{j\ell}B_{im}\partial_{x_\ell}\partial_{v_m}f\\
&= B_{im}(\partial_{v_m}B_{j\ell})\partial_{x_\ell}f
\end{aligned}$$

as well as

$$\begin{aligned}
-[\widetilde{\nabla}^*_{v_i}, \widetilde{\nabla}_{x_j}]f &= \partial_{v_m}(B_{im}B_{j\ell}\partial_{x_\ell}f) - B_{j\ell}\partial_{x_\ell}\partial_{v_m}(B_{im}f)\\
&= B_{j\ell}\partial_{x_\ell}\partial_{v_k}(B_{im}f) + B_{im}(\partial_{v_m}B_{j\ell})\partial_{x_\ell}f - B_{j\ell}\partial_{x_\ell}\partial_{v_m}(B_{im}f)\\
&= B_{im}(\partial_{v_m}B_{j\ell})\partial_{x_\ell}f.
\end{aligned}$$

We then obtain the estimate by using (2.9).

(vi) Thanks to the proof of item (ii), we write

$$\begin{aligned}
[\langle v\rangle^\alpha, \widetilde{\nabla}^*_{v_j}]\widetilde{\nabla}_{v_j}f &= (\widetilde{\nabla}_{v_j}\langle v\rangle^\alpha)\widetilde{\nabla}_{v_j}f\\
&= \widetilde{\nabla}_{v_j}((\widetilde{\nabla}_{v_j}\langle v\rangle^\alpha)f) - (\widetilde{\nabla}_{v_j}(\widetilde{\nabla}_{v_j}\langle v\rangle^\alpha))f\\
&= \widetilde{\nabla}_{v_j}[\langle v\rangle^\alpha, \widetilde{\nabla}^*_{v_j}]f - (\widetilde{\nabla}_{v_j}(\widetilde{\nabla}_{v_j}\langle v\rangle^\alpha))f.
\end{aligned}$$

We conclude the proof by using $\widetilde{\nabla}_{v_j}\langle v\rangle^\alpha = \alpha(\mathbf{B}(v)v)_j\,\langle v\rangle^{\alpha-2}$ and the upper bound (2.9).

(vii) Thanks to the proof of item (iii), we first write

$$\begin{aligned}
[\widetilde{\nabla}_{v_i}, \widetilde{\nabla}_{v_j}]\widetilde{\nabla}_{v_k}f &= B_{im}(\partial_{v_m}B_{j\ell})\partial_{v_\ell}(B_{kp}\partial_{v_p}f) - B_{j\ell}(\partial_{v_\ell}B_{im})\partial_{v_m}(B_{kp}\partial_{v_p}f)\\
&= B_{im}(\partial_{v_m}B_{j\ell})\widetilde{\nabla}_{v_k}\partial_{v_\ell}f + B_{im}(\partial_{v_m}B_{j\ell})(\partial_{v_\ell}B_{kp})\partial_{v_p}f\\
&\quad - B_{j\ell}(\partial_{v_\ell}B_{im})\widetilde{\nabla}_{v_k}\partial_{v_m}f - B_{j\ell}(\partial_{v_\ell}B_{im})(\partial_{v_m}B_{kp})\partial_{v_p}f.
\end{aligned}$$

We then obtain

$$
\begin{aligned}
[\widetilde\nabla_{v_i}, \widetilde\nabla_{v_j}]\widetilde\nabla_{v_k} f &= \widetilde\nabla_{v_k}[B_{im}(\partial_{v_m} B_{j\ell})\partial_{v_\ell} f] - (\widetilde\nabla_{v_k}[B_{im}(\partial_{v_m} B_{j\ell})])\partial_{v_\ell} f \\
&\quad - \widetilde\nabla_{v_k}[B_{j\ell}(\partial_{v_\ell} B_{im})\partial_{v_m} f] + (\widetilde\nabla_{v_k}[B_{j\ell}(\partial_{v_\ell} B_{im})])\partial_{v_m} f \\
&\quad + [B_{im}(\partial_{v_m} B_{j\ell})(\partial_{v_\ell} B_{kp}) - B_{j\ell}(\partial_{v_\ell} B_{im})(\partial_{v_m} B_{kp})]\partial_{v_p} f \\
&= \widetilde\nabla_{v_k}[\widetilde\nabla_{v_i}, \widetilde\nabla_{v_j}] f - (\widetilde\nabla_{v_k}[B_{im}(\partial_{v_m} B_{j\ell})])\partial_{v_\ell} f + (\widetilde\nabla_{v_k}[B_{j\ell}(\partial_{v_\ell} B_{im})])\partial_{v_m} f \\
&\quad + [B_{im}(\partial_{v_m} B_{j\ell})(\partial_{v_\ell} B_{kp}) - B_{j\ell}(\partial_{v_\ell} B_{im})(\partial_{v_m} B_{kp})]\partial_{v_p} f
\end{aligned}
$$

and we conclude the proof of the first estimate using (2.9). The second estimate is obtained in a similar way by using the computation of item (iv), thus we omit it.

(viii) Thanks to the proof of item (v), we write

$$
\begin{aligned}
[\widetilde\nabla_{x_i}, \widetilde\nabla_{v_j}]\widetilde\nabla_{v_k} f &= -B_{jm}(\partial_{v_m} B_{i\ell})\partial_{x_\ell}\widetilde\nabla_{v_k} f \\
&= -\widetilde\nabla_{v_k}\left(B_{jm}(\partial_{v_m} B_{i\ell})\partial_{x_\ell} f\right) - \left(\widetilde\nabla_{v_k}[B_{jm}(\partial_{v_m} B_{i\ell})]\right)\partial_{x_\ell} f \\
&= \widetilde\nabla_{v_k}[\widetilde\nabla_{x_i}, \widetilde\nabla_{v_j}^*] f - \left(\widetilde\nabla_{v_k}[B_{jm}(\partial_{v_m} B_{i\ell})]\right)\partial_{x_\ell} f.
\end{aligned}
$$

The estimate then follows from (2.9).

□

Using the above result, we shall now compute some commutators related to the L_1 term defined in (2.18) of the collision operator L.

Lemma 2.3. *For any suitable function $f = f(x,v)$ and any $(x,v) \in \mathbf{T}^3 \times \mathbf{R}^3$, there holds*

(i) For any $\alpha \in \mathbf{R}$, one has

$$
[\langle v\rangle^\alpha, L_1] f = -\widetilde\nabla_{v_\ell}^*[\langle v\rangle^\alpha, \widetilde\nabla_{v_\ell}] f - \widetilde\nabla_{v_\ell}[\langle v\rangle^\alpha, \widetilde\nabla_{v_\ell}^*] f - \Big[[\langle v\rangle^\alpha, \widetilde\nabla_{v_\ell}^*], \widetilde\nabla_{v_\ell}\Big] f.
$$

(ii) For any $1 \leqslant k \leqslant 3$, one has

$$
[\widetilde\nabla_{v_k}, L_1] f = -\widetilde\nabla_{v_\ell}^*[\widetilde\nabla_{v_k}, \widetilde\nabla_{v_\ell}] f - \widetilde\nabla_{v_\ell}[\widetilde\nabla_{v_k}, \widetilde\nabla_{v_\ell}^*] f - \Big[[\widetilde\nabla_{v_k}, \widetilde\nabla_{v_\ell}^*], \widetilde\nabla_{v_\ell}\Big] f - (\widetilde\nabla_{v_k}\psi) f.
$$

(iii) For any $1 \leqslant k \leqslant 3$, one has

$$
[\widetilde\nabla_{x_k}, L_1] f = -\widetilde\nabla_{v_\ell}^*[\widetilde\nabla_{x_k}, \widetilde\nabla_{v_\ell}] f - \widetilde\nabla_{v_\ell}[\widetilde\nabla_{x_k}, \widetilde\nabla_{v_\ell}^*] f - \Big[[\widetilde\nabla_{x_k}, \widetilde\nabla_{v_\ell}^*], \widetilde\nabla_{v_\ell}\Big] f
$$

and

$$
[\langle v\rangle^{\frac{\gamma}{2}}\partial_{x_k}, L_1] f = -\widetilde\nabla_{v_\ell}^*[\langle v\rangle^{\frac{\gamma}{2}}\partial_{x_k}, \widetilde\nabla_{v_\ell}] f - \widetilde\nabla_{v_\ell}[\langle v\rangle^{\frac{\gamma}{2}}\partial_{x_k}, \widetilde\nabla_{v_\ell}^*] f - \Big[[\langle v\rangle^{\frac{\gamma}{2}}\partial_{x_k}, \widetilde\nabla_{v_\ell}^*], \widetilde\nabla_{v_\ell}\Big] f.
$$

Proof. (i) We first write

$$
\langle v\rangle^\alpha (L_1 f) = -\langle v\rangle^\alpha \widetilde\nabla_{v_\ell}^*\widetilde\nabla_{v_\ell} f - \psi\langle v\rangle^\alpha f
$$

and

$$
L_1(\langle v\rangle^\alpha f) = -\widetilde\nabla_{v_\ell}^*\widetilde\nabla_{v_\ell}(\langle v\rangle^\alpha f) - \psi\langle v\rangle^\alpha f.
$$

We now observe that

$$
\begin{aligned}
\langle v\rangle^\alpha \widetilde\nabla_{v_\ell}^*\widetilde\nabla_{v_\ell} f &= \widetilde\nabla_{v_\ell}^*\langle v\rangle^\alpha\widetilde\nabla_{v_\ell} f + [\langle v\rangle^\alpha, \widetilde\nabla_{v_\ell}^*]\widetilde\nabla_{v_\ell} f \\
&= \widetilde\nabla_{v_\ell}^*\widetilde\nabla_{v_\ell}(\langle v\rangle^\alpha f) + \widetilde\nabla_{v_\ell}^*[\langle v\rangle^\alpha, \widetilde\nabla_{v_\ell}] f + \widetilde\nabla_{v_\ell}[\langle v\rangle^\alpha, \widetilde\nabla_{v_\ell}^*] f + \Big[[\langle v\rangle^\alpha, \widetilde\nabla_{v_\ell}^*], \widetilde\nabla_{v_\ell}\Big] f,
\end{aligned}
$$

which completes the proof.

(ii) We first compute

$$
\widetilde\nabla_{v_k}(L_1 f) = -\widetilde\nabla_{v_k}\widetilde\nabla_{v_\ell}^*\widetilde\nabla_{v_\ell} f - \widetilde\nabla_{v_k}(\psi f)
$$

and

$$
L_1(\widetilde\nabla_{v_k} f) = -\widetilde\nabla_{v_\ell}^*\widetilde\nabla_{v_\ell}(\widetilde\nabla_{v_k} f) - \psi\widetilde\nabla_{v_k} f.
$$

We conclude the proof by observing that

$$\begin{aligned}\widetilde\nabla_{v_k}\widetilde\nabla^*_{v_\ell}\widetilde\nabla_{v_\ell} f &= \widetilde\nabla^*_{v_\ell}\widetilde\nabla_{v_k}\widetilde\nabla_{v_\ell} f + [\widetilde\nabla_{v_k},\widetilde\nabla^*_{v_\ell}]\widetilde\nabla_{v_\ell} f\\ &= \widetilde\nabla^*_{v_\ell}\widetilde\nabla_{v_\ell}\widetilde\nabla_{v_k} f + \widetilde\nabla^*_{v_\ell}[\widetilde\nabla_{v_k},\widetilde\nabla_{v_\ell}] f + \widetilde\nabla_{v_\ell}[\widetilde\nabla_{v_k},\widetilde\nabla^*_{v_\ell}] f + \Big[[\widetilde\nabla_{v_k},\widetilde\nabla^*_{v_\ell}],\widetilde\nabla_{v_\ell}\Big] f\end{aligned}$$

and writing $\widetilde\nabla_{v_k}(\psi f) = \psi\widetilde\nabla_{v_k} f + (\widetilde\nabla_{v_k}\psi) f$.

(iii) We compute

$$\widetilde\nabla_{x_k}(L_1 f) = -\widetilde\nabla_{x_k}\widetilde\nabla^*_{v_\ell}\widetilde\nabla_{v_\ell} f - \psi\widetilde\nabla_{x_k} f$$

and

$$L_1(\widetilde\nabla_{x_k} f) = -\widetilde\nabla^*_{v_\ell}\widetilde\nabla_{v_\ell}\widetilde\nabla_{x_k} f - \psi\widetilde\nabla_{x_k} f.$$

We now observe that

$$\begin{aligned}\widetilde\nabla_{x_k}\widetilde\nabla^*_{v_\ell}\widetilde\nabla_{v_\ell} f &= \widetilde\nabla^*_{v_\ell}\widetilde\nabla_{x_k}\widetilde\nabla_{v_\ell} f + [\widetilde\nabla_{x_k},\widetilde\nabla^*_{v_\ell}]\widetilde\nabla_{v_\ell} f\\ &= \widetilde\nabla^*_{v_\ell}\widetilde\nabla_{v_\ell}\widetilde\nabla_{x_k} f + \widetilde\nabla^*_{v_\ell}[\widetilde\nabla_{x_k},\widetilde\nabla_{v_\ell}] f + \widetilde\nabla_{v_\ell}[\widetilde\nabla_{x_k},\widetilde\nabla^*_{v_\ell}] f + \Big[[\widetilde\nabla_{x_k},\widetilde\nabla^*_{v_\ell}],\widetilde\nabla_{v_\ell}\Big] f\end{aligned}$$

which gives the first estimate.

The computation for $[\langle v\rangle^{\frac{\gamma}{2}}\partial_{x_k}, L_1]$ can be obtained in a similar fashion, thus we omit it. $\square$

Lemma 2.4. *For any suitable function $f = f(v)$ and any $v\in\mathbf{R}^3$ there holds:*

(i) For any $i,j\in\{1,2,3\}$ one has

$$|(a_{ij}*f)(v)| + |(a_{ij}*f)(v)v_i| + |(a_{ij}*f)(v)v_iv_j| \lesssim \langle v\rangle^{\gamma+2}\|\langle v\rangle^7 f\|_{L^2_v}$$

(ii) For any $i,j,\ell\in\{1,2,3\}$ one has

$$|(\partial_{v_\ell}a_{ij}*f)(v)| + |(b_i*f)(v)| \lesssim \langle v\rangle^{\gamma+1}\|\langle v\rangle^4 f\|_{L^2_v}$$

and

$$|\left(\partial_{v_\ell}a_{ij}*f\right)(v)v_i| + |\left(\partial_{v_\ell}a_{ij}*f\right)(v)v_iv_j| \lesssim \langle v\rangle^{\gamma+2}\|\langle v\rangle^8 f\|_{L^2_v}$$

(iii) If $\gamma\geqslant 0$, for any $\ell\in\{1,2,3\}$ one has

$$|\left(c*f\right)(v)| + |\left(\partial_{v_\ell}b_i*f\right)(v)| \lesssim \langle v\rangle^{\gamma}\|\langle v\rangle^3 f\|_{L^2_v}.$$

(iv) If $\gamma\in[-2,0)$, for any $\ell\in\{1,2,3\}$ one has

$$|\left(c*f\right)(v)| + |\left(\partial_{v_\ell}b_i*f\right)(v)| \lesssim \langle v\rangle^{\gamma}\|\langle v\rangle^3 f\|_{L^4_v}.$$

Proof. All estimates in point (i) come from [13, Lemma 3.4]. The estimates on the term $|(b_i*f)(v)|$ in point (ii) and on the term $|(c*f)(v)|$ in (iii)–(iv) also come from [13, Lemma 3.4], and the proof of the estimates on $|(\partial_{v_\ell}a_{ij}*f)(v)|$ in (ii) and on $|\left(\partial_{v_\ell}b_i*f\right)(v)|$ in (iii)–(iv) follow the same lines since $|\partial_{v_\ell}a_{ij}|\lesssim|v|^{\gamma+1}$ and $|\partial_{v_\ell}b_i|\lesssim|v|^{\gamma}$.

We thus only prove the second estimate in (ii). Using that $a_{ij}(v-v_*)v_iv_j = a_{ij}(v-v_*)(v_*)_i(v_*)_j$, we observe that

$$\begin{aligned}\left(\partial_{v_\ell}a_{ij}*f\right)(v)v_iv_j &= (a_{ij}*\partial_{v_\ell}f)(v)v_iv_j\\ &= (a_{ij}*v_iv_j\partial_{v_\ell}f)(v)\\ &= (\partial_{v_\ell}a_{ij}*[v_iv_jf])(v) - (a_{ij}*[\partial_{v_\ell}(v_iv_j)f])(v).\end{aligned}$$

Using the the estimate in (i) and the first estimate in (ii), we thus deduce

$$|(\partial_{v_\ell}a_{ij}*f)(v)v_iv_j| \lesssim \langle v\rangle^{\gamma+1}\|\langle v\rangle^6 f\|_{L^2_v} + \langle v\rangle^{\gamma+2}\|\langle v\rangle^8 f\|_{L^2_v}.$$

$\square$

Lemma 2.5. *For any function $f=f(v)$ smooth enough and $\alpha\in\mathbf{R}$, there holds:*

$$\|(c*(\sqrt{M}f))\,\langle v\rangle^{\alpha}\sqrt{M}\|_{L^2_v} \lesssim \|M^{\frac14}f\|_{L^2_v}.$$

Proof. See [13, Proof of Lemma 2.12]. $\square$

We are now able to obtain some upper bounds on the term L_2 defined in (2.19) of the collision operator L as follows.

Lemma 2.6. *For any function $f = f(x,v)$ smooth enough, there holds:*

(i) For any $\alpha \in \mathbf{R}$, one has

$$\|\langle v\rangle^\alpha L_2 f\|_{L^2_{x,v}} \lesssim \|M^{\frac14} f\|_{L^2_{x,v}}.$$

(ii) For any $1 \leqslant k \leqslant 3$, one has

$$\|\widetilde\nabla_{v_k}(L_2 f)\|_{L^2_{x,v}} \lesssim \|M^{\frac14} f\|_{L^2_{x,v}} + \|M^{\frac14}\widetilde\nabla_v f\|_{L^2_{x,v}}.$$

(iii) For any $1 \leqslant k \leqslant 3$, one has

$$\|\widetilde\nabla_{x_k}(L_2 f)\|_{L^2_{x,v}} \lesssim \|M^{\frac14}\widetilde\nabla_x f\|_{L^2_{x,v}}.$$

Proof. (i) We write

$$\langle v\rangle^\alpha L_2 f = -\left\{-\left(a_{ij} *_v \sqrt{M} f\right) v_i v_j + \left(a_{ii} *_v \sqrt{M} f\right) + \left(c *_v \sqrt{M} f\right)\right\} \langle v\rangle^\alpha \sqrt{M}$$

from which, thanks to Lemma 2.4, we deduce

$$|\langle v\rangle^\alpha L_2 f| \lesssim \|M^{\frac14} f\|_{L^2_v} \langle v\rangle^{\gamma+\alpha+2}\sqrt{M} + |(c *_v \sqrt{M} f)| \langle v\rangle^\alpha \sqrt{M}.$$

We conclude the proof by taking the L^2 norm of the last estimate and using Lemma 2.5.

(ii) Writing $\widetilde\nabla_{v_k} = B_{k\ell}\partial_{v_\ell}$ we compute

$$\begin{aligned}\widetilde\nabla_{v_k}(L_2 f) = -B_{k\ell}\Big\{&-\left(a_{ij} *_v \partial_{v_\ell}(\sqrt{M} f)\right) v_i v_j - \left(a_{ij} *_v \sqrt{M} f\right)\partial_{v_\ell}(v_i v_j)\\ &+ \left(a_{ii} *_v \partial_{v_\ell}(\sqrt{M} f)\right) + \left(c *_v \partial_{v_\ell}(\sqrt{M} f)\right)\Big\}\sqrt{M}\\ &- \left\{-\left(a_{ij} *_v \sqrt{M} f\right) v_i v_j + \left(a_{ii} *_v \sqrt{M} f\right) + \left(c *_v \sqrt{M} f\right)\right\}\widetilde\nabla_{v_k}\sqrt{M}.\end{aligned}$$

Thanks to Lemmas 2.4 and 2.5, we obtain that

$$\|\widetilde\nabla_{v_k}(L_2 f)(x,\cdot)\|_{L^2_v} \lesssim \|M^{\frac14} f(x,\cdot)\|_{L^2_v} + \|M^{\frac14}\widetilde\nabla_v f(x,\cdot)\|_{L^2_v},$$

and we conclude by integrating in x this last estimate.

(iii) Writing $\widetilde\nabla_{x_k} = B_{k\ell}\partial_{x_\ell}$ we compute

$$\widetilde\nabla_{x_k}(L_2 f) = -B_{k\ell}\Big\{-\left(a_{ij} *_v \sqrt{M}\partial_{x_\ell} f\right) v_i v_j + \left(a_{ii} *_v \sqrt{M}\partial_{x_\ell} f\right) + \left(c *_v \sqrt{M}\partial_{x_\ell} f\right)\Big\}\sqrt{M},$$

and we obtain the wanted result thanks to Lemmas 2.4 and 2.5. ☐

3. Estimates on the linearized problem

For the rest of the paper, the parameter ε will belong to $(0,1)$. We recall that the functional spaces $\mathscr{X}$, $\mathscr{Y}_1$ and $\mathscr{Z}_1^\varepsilon$ are respectively defined in (1.19), (1.20) and (2.12), that the operator Λ_ε is given in (1.11) and that Π is the projector onto the kernel of Λ_ε (see (1.17)). We consider $U^\varepsilon(t)$ the semigroup associated with Λ_ε and study its decay and regularization properties.

Theorem 3.1. *For any $\sigma \in (0,\sigma_0)$, we have:*

$$\|U^\varepsilon(t)(\mathrm{Id} - \Pi)\|_{\mathscr{X}\to\mathscr{X}} \lesssim e^{-\sigma t},$$

$$\|U^\varepsilon(t)(\mathrm{Id} - \Pi)\|_{\mathscr{X}\to\mathscr{Y}_1} \lesssim \frac{e^{-\sigma t}}{\min(1,\sqrt{t})} \quad \textit{and} \quad \|U^\varepsilon(t)(\mathrm{Id} - \Pi)\|_{\mathscr{X}\to\mathscr{Z}_1^\varepsilon} \lesssim \frac{e^{-\sigma t}}{\min(1,t^{3/2})},$$

where σ_0 is defined in Proposition 3.2.

Our proof is based on hypocoercivity tricks and thus on direct energy estimates on the whole problem $\partial_t f = \Lambda_\varepsilon f$. Our method to prove this theorem is of particular interest when one wants to extend the analysis to the whole nonlinear problem (see the paragraph at the beginning of Section 4). Indeed, it is based on a micro-macro decomposition of the solution and thus allows to identify the different behaviors of the microscopic and macroscopic parts of the solution. We will use this approach in Section 4 in which we develop a Cauchy theory for the nonlinear Landau equation as well as some regularization estimates on it.

Let us finally notice that we are actually able to prove Theorem 3.1 by using a splitting of Λ_ε as presented in Section 5. By establishing nice estimates on each part of the splitting, we are then able to recover the wanted estimates on the whole semigroup $U^\varepsilon(t)$ thanks to Duhamel formula. Such an analysis does not allow us to develop our Cauchy theory and regularization estimates for the nonlinear problem but will be useful to study our problem of hydrodynamical limit, we thus postpone it to Section 5.

3.1. **Hypocoercivity estimates.** In this part, we state some hypocoercivity results for our linearized operator Λ_ε defined in (1.11). The first one provides a result of hypocoercivity in $L^2_{x,v}$ and the proof is a mere adaptation of the one provided in [7, Theorem 5.1] in the more complicated case of bounded domains with various boundary conditions. For sake of completeness, we give the proof of Proposition 3.2 in Appendix A.

Proposition 3.2. *There exists a norm $|||\cdot|||_{L^2_{x,v}}$ on $L^2_{x,v}$ (with associated scalar product $\langle\langle\cdot,\cdot\rangle\rangle_{L^2_{x,v}}$) equivalent to the standard norm $\|\cdot\|_{L^2_{x,v}}$ which satisfies the following property: For any $f \in \mathrm{Dom}\,\Lambda_\varepsilon \cap (\mathrm{Ker}\,\Lambda_\varepsilon)^\perp$,*

$$\langle\langle \Lambda_\varepsilon f, f\rangle\rangle_{L^2_{x,v}} \leqslant -\sigma_0 |||f|||^2_{L^2_{x,v}} - \kappa_0 \|f\|^2_{L^2_x(H^1_{v,*})} - \frac{\kappa_0}{\varepsilon^2}\|f^\perp\|^2_{L^2_x(H^1_{v,*})},$$

for some constructive constants $0<\sigma_0<\sigma_L$ (where σ_L is defined in (2.10)), $\kappa_0>0$ and where $f^\perp$ is defined in (1.13).

Roughly speaking, the norm $|||\cdot|||_{L^2_{x,v}}$ is of the following form

$$|||f|||^2_{L^2_{x,v}} = \|f\|^2_{L^2_{x,v}} + \varepsilon \sum_{i=1}^3 \eta_i \left\langle \partial_{x_i}\Delta_x^{-1}\pi f, \widetilde{\pi}_i f\right\rangle_{L^2_x}$$

where $\widetilde{\pi}_i : L^2_{x,v} \to L^2_x$ is some suitable moment operator, the inverse laplacian Δ_x^{-1} is suitably defined and the constants η_i are chosen to be small enough (see (A.19) in the proof). The norm $|||\cdot|||_{L^2_{x,v}}$ thus depends on ε but is equivalent to the usual norm $\|\cdot\|_{L^2_{x,v}}$ uniformly in ε, this explains the fact that we do not mention the dependency in ε in our notation.

In the following proposition, we provide a result of hypocoercivity in $\mathscr{X}$. Notice that obtaining hypocoercivity in $H^3_x L^2_v$ is a straightforward consequence of the previous proposition since derivatives in x commute with Λ_ε. Due to the presence of additional weights in the definition of the space $\mathscr{X}$, we have to exhibit a new norm equivalent to the usual one for which we can recover a suitable energy estimate.

Proposition 3.3. *There exists a norm $|||\cdot|||_{\mathscr{X}}$ on $\mathscr{X}$ (with associated scalar product denoted by $\langle\langle\cdot,\cdot\rangle\rangle_{\mathscr{X}}$) equivalent to the standard norm $\|\cdot\|_{\mathscr{X}}$ which satisfies the following property: For any $f \in \mathrm{Dom}\,\Lambda_\varepsilon \cap (\mathrm{Ker}\,\Lambda_\varepsilon)^\perp$,*

$$\langle\langle \Lambda_\varepsilon f, f\rangle\rangle_{\mathscr{X}} \leqslant -\sigma |||f|||^2_{\mathscr{X}} - \kappa\|f\|^2_{\mathscr{Y}_1} - \frac{\kappa}{\varepsilon^2}\|f^\perp\|^2_{\mathscr{Y}_1}, \tag{3.1}$$

for any $0<\sigma<\sigma_0$ (where σ_0 is defined in Proposition 3.2) and for some $\kappa>0$. As a consequence there holds, for all $t\geqslant 0$,

$$\|U^\varepsilon(t)(\mathrm{Id}-\Pi)\|_{\mathscr{X}\to\mathscr{X}} \leqslant Ce^{-\sigma t}. \tag{3.2}$$

Proof. We define the inner product $\langle\!\langle \cdot,\cdot \rangle\!\rangle_{\mathscr{X}}$ on $\mathscr{X}$ by

(3.3)
$$\langle\!\langle f,g \rangle\!\rangle_{\mathscr{X}} := \sum_{i=0}^{2} \left(\delta \left\langle \langle v\rangle^{(3-i)(\frac{\gamma}{2}+1)} \nabla_x^i f^\perp, \langle v\rangle^{(3-i)(\frac{\gamma}{2}+1)} \nabla_x^i g^\perp \right\rangle_{L^2_{x,v}} + \left\langle\!\left\langle \nabla_x^i f, \nabla_x^i g \right\rangle\!\right\rangle_{L^2_{x,v}} \right) + \left\langle\!\left\langle \nabla_x^3 f, \nabla_x^3 g \right\rangle\!\right\rangle_{L^2_{x,v}}$$

so that its associated norm is given by

$$|\!|\!| f |\!|\!|_{\mathscr{X}}^2 := \sum_{i=0}^{2} \left(\delta \| \langle v\rangle^{(3-i)(\frac{\gamma}{2}+1)} \nabla_x^i f^\perp \|_{L^2_{x,v}}^2 + |\!|\!| \nabla_x^i f |\!|\!|_{L^2_{x,v}}^2 \right) + |\!|\!| \nabla_x^3 f |\!|\!|_{L^2_{x,v}}^2 \tag{3.4}$$

for some constant $\delta \in (0,1)$ to be chosen small enough, and where $\langle\!\langle \cdot,\cdot \rangle\!\rangle_{L^2_{x,v}}$ and $|\!|\!| \cdot |\!|\!|_{L^2_{x,v}}$ are defined in Proposition 3.2 (see (A.19)). We first observe that this norm is equivalent to the norm $\|\cdot\|_{\mathscr{X}}$.

Let $\sigma' \in (\sigma, \sigma_0)$ be fixed and $f \in \mathscr{X} \cap (\operatorname{Ker} \Lambda_\varepsilon)^\perp$. We shall prove

$$\langle\!\langle \Lambda_\varepsilon f, f \rangle\!\rangle_{\mathscr{X}} \leqslant -\sigma' |\!|\!| f |\!|\!|_{\mathscr{X}}^2 - \frac{\kappa'}{\varepsilon^2} \| f^\perp \|_{\mathscr{Y}_1}^2 \tag{3.5}$$

for some constant $\kappa' > 0$. Since there exists a constant $c_0 > 0$ such that

$$\|\pi f\|_{\mathscr{Y}_1}^2 \leqslant c_0 \|\pi f\|_{H^3_x L^2_v}^2 \leqslant c_0 \|f\|_{H^3_x L^2_v}^2 \leqslant c_0 \|f\|_{\mathscr{X}}^2,$$

the previous inequality readily implies (3.1) with some constant $\kappa = \min\left(\frac{\sigma'-\sigma}{c_0}, \frac{\kappa'}{2}\right)$ by decomposing the first term with $\sigma' = \sigma + (\sigma' - \sigma)$. Estimate (3.2) is then a direct consequence of (3.1).

Step 1. We first compute

$$\langle\!\langle \Lambda_\varepsilon f, f \rangle\!\rangle_{\mathscr{X}} = \sum_{i=0}^{2} \delta \left\langle \langle v\rangle^{(3-i)(\frac{\gamma}{2}+1)} \nabla_x^i (\Lambda_\varepsilon f)^\perp, \langle v\rangle^{(3-i)(\frac{\gamma}{2}+1)} \nabla_x^i f^\perp \right\rangle_{L^2_{x,v}} + \sum_{i=0}^{3} \left\langle\!\left\langle \nabla_x^i \Lambda_\varepsilon f, \nabla_x^i f \right\rangle\!\right\rangle_{L^2_{x,v}},$$

and we observe that, thanks to Proposition 3.2 and the fact that ∇_x commutes with Λ_ε, we already have

$$\sum_{i=0}^{3} \left\langle\!\left\langle \nabla_x^i \Lambda_\varepsilon f, \nabla_x^i f \right\rangle\!\right\rangle_{L^2_{x,v}} \leqslant -\sum_{i=0}^{3} \left\{ \sigma_0 |\!|\!| \nabla_x^i f |\!|\!|_{L^2_{x,v}}^2 + \kappa_0 \|\nabla_x^i f\|_{L^2_x(H^1_{v,*})}^2 + \frac{\kappa_0}{\varepsilon^2} \|\nabla_x^i f^\perp\|_{L^2_x(H^1_{v,*})}^2 \right\}.$$

Step 2. We first observe that

$$(\Lambda_\varepsilon f)^\perp = (\mathrm{Id} - \pi)\Lambda_\varepsilon f = \frac{1}{\varepsilon^2}(\mathrm{Id} - \pi) L f - \frac{1}{\varepsilon}(\mathrm{Id} - \pi)(v \cdot \nabla_x f),$$

from which we obtain, since $\pi L = 0$ and $Lf = Lf^\perp$, that

$$(\Lambda_\varepsilon f)^\perp = \frac{1}{\varepsilon^2} L f^\perp - \frac{1}{\varepsilon} \left\{ v \cdot \nabla_x f^\perp + v \cdot \nabla_x (\pi f) - \pi (v \cdot \nabla_x f) \right\}. \tag{3.6}$$

We therefore get, for any $i \in \{0,1,2\}$ and using that the transport operator $v \cdot \nabla_x$ is skew-adjoint,

$$\left\langle \langle v\rangle^{(3-i)(\frac{\gamma}{2}+1)} \nabla_x^i (\Lambda_\varepsilon f)^\perp, \langle v\rangle^{(3-i)(\frac{\gamma}{2}+1)} \nabla_x^i f^\perp \right\rangle_{L^2_{x,v}} =: \frac{1}{\varepsilon^2} R_1^i - \frac{1}{\varepsilon} R_2^i + \frac{1}{\varepsilon} R_3^i$$

with

$$R_1^i := \left\langle \langle v\rangle^{(3-i)(\frac{\gamma}{2}+1)}\, \nabla_x^i L f^\perp, \langle v\rangle^{(3-i)(\frac{\gamma}{2}+1)}\, \nabla_x^i f^\perp \right\rangle_{L^2_{x,v}},$$
$$R_2^i := \left\langle \langle v\rangle^{(3-i)(\frac{\gamma}{2}+1)}\, \nabla_x^i (v\cdot\nabla_x(\pi f)), \langle v\rangle^{(3-i)(\frac{\gamma}{2}+1)}\, \nabla_x^i f^\perp \right\rangle_{L^2_{x,v}},$$
$$R_3^i := \left\langle \langle v\rangle^{(3-i)(\frac{\gamma}{2}+1)}\, \nabla_x^i (\pi(v\cdot\nabla_x f)), \langle v\rangle^{(3-i)(\frac{\gamma}{2}+1)}\, \nabla_x^i f^\perp \right\rangle_{L^2_{x,v}},$$

and we treat each term separately. For simplicity we denote $\omega_i = \langle v\rangle^{(3-i)(\frac{\gamma}{2}+1)}$ in the sequel.

Step 3. We deal with the term R_1^i. Since ∇_x commutes with L we have

$$\begin{aligned} R_1^i &= \left\langle L(\omega_i\nabla_x^i f^\perp), \omega_i\nabla_x^i f^\perp\right\rangle_{L^2_{x,v}} + \left\langle [\omega_i, L_1]\nabla_x^i f^\perp, \omega_i\nabla_x^i f^\perp\right\rangle_{L^2_{x,v}} \\ &\quad + \left\langle [\omega_i, L_2]\nabla_x^i f^\perp, \omega_i\nabla_x^i f^\perp\right\rangle_{L^2_{x,v}} \\ &=: R_{11}^i + R_{12}^i + R_{13}^i, \end{aligned}$$

where we recall that L_1 and L_2 are defined in (2.18) and (2.19), respectively. Thanks to the spectral gap estimate (2.10), we have

$$\begin{aligned} R_{11}^i &\leqslant -\sigma_L \|\omega_i\nabla_x^i f^\perp - \pi(\omega_i\nabla_x^i f^\perp)\|^2_{L^2_x(H^1_{v,*})} \\ &\leqslant -\sigma_L \|\omega_i\nabla_x^i f^\perp\|^2_{L^2_x(H^1_{v,*})} + C\|\nabla_x^i f^\perp\|^2_{L^2_{x,v}}, \end{aligned}$$

for some constant $C>0$. From Lemma 2.3 we get

$$\begin{aligned} R_{12}^i = &- \left\langle [\omega_i, \widetilde\nabla_{v_\ell}]\nabla_x^i f^\perp, \widetilde\nabla_{v_\ell}(\omega_i\nabla_x^i f^\perp)\right\rangle_{L^2_{x,v}} - \left\langle [\omega_i, \widetilde\nabla^*_{v_\ell}]\nabla_x^i f^\perp, \widetilde\nabla^*_{v_\ell}(\omega_i\nabla_x^i f^\perp)\right\rangle_{L^2_{x,v}} \\ &- \left\langle \left[[\omega_i, \widetilde\nabla^*_{v_\ell}], \widetilde\nabla_{v_\ell}\right] \nabla_x^i f^\perp, \omega_i\nabla_x^i f^\perp\right\rangle_{L^2_{x,v}}. \end{aligned}$$

Using Lemma 2.2 and Cauchy-Schwarz inequality, we first get

$$\begin{aligned} |R_{12}^i| &\leqslant C\|\widetilde\nabla_v(\omega_i\nabla_x^i f^\perp)\|_{L^2_{x,v}} \|\langle v\rangle^{\frac{\gamma}{2}-1}\omega_i\nabla_x^i f^\perp\|_{L^2_{x,v}} + C\|\widetilde\nabla_v^*(\omega_i\nabla_x^i f^\perp)\|_{L^2_{x,v}} \|\langle v\rangle^{\frac{\gamma}{2}-1}\omega_i\nabla_x^i f^\perp\|_{L^2_{x,v}} \\ &\quad + C\|\langle v\rangle^{\frac{\gamma}{2}-\frac12}\omega_i\nabla_x^i f^\perp\|^2_{L^2_{x,v}}. \end{aligned}$$

Using Young's inequality to write $\langle v\rangle^{\frac{\gamma}{2}-\frac12}\omega_i \leqslant \eta\langle v\rangle^{\frac{\gamma}{2}+1}\omega_i + C_\eta$ for any $\eta>0$ (and similarly for $\langle v\rangle^{\frac{\gamma}{2}-1}\omega_i$), and observing that

$$\|\langle v\rangle^{\frac{\gamma}{2}+1} g\|_{L^2_{x,v}} + \|\widetilde\nabla_v g\|_{L^2_{x,v}} + \|\widetilde\nabla_v^* g\|_{L^2_{x,v}} \lesssim \|g\|_{L^2_x(H^1_{v,*})},$$

we deduce from the previous estimate that

$$|R_{12}^i| \leqslant (\sigma_L - \sigma'')\|\omega_i\nabla_x^i f^\perp\|^2_{L^2_x(H^1_{v,*})} + C\|\nabla_x^i f^\perp\|^2_{L^2_{x,v}}$$

for any $\sigma'' \in (\sigma', \sigma_0)$ by using Young's inequality again and taking $\eta>0$ small enough.

For the term R_{13}^i, we use Lemma 2.6 to obtain

$$|R_{13}^i| \leqslant C\|\nabla_x^i f^\perp\|^2_{L^2_{x,v}}.$$

Gathering previous estimates we finally get

$$R_1^i \leqslant -\sigma''\|\omega_i\nabla_x^i f^\perp\|^2_{L^2_x(H^1_{v,*})} + C\|\nabla_x^i f^\perp\|^2_{L^2_{x,v}}.$$

Step 4. We deal with the terms R_2^i and R_3^i. Observing that

$$\|\omega_i\nabla_x^i(v\cdot\nabla_x(\pi f))\|_{L^2_{x,v}} + \|\omega_i\nabla_x^i(\pi(v\cdot\nabla_x f))\|_{L^2_{x,v}} \lesssim \|\nabla_x^{i+1} f\|_{L^2_{x,v}}$$

we obtain

$$\begin{aligned} |R_2^i| + |R_3^i| &\leqslant C\|\nabla_x^{i+1} f\|_{L^2_{x,v}} \|\omega_i\nabla_x^i f^\perp\|_{L^2_{x,v}} \\ &\leqslant \frac{(\sigma''-\sigma''')}{\varepsilon}\|\omega_i\nabla_x^i f^\perp\|^2_{L^2_x(H^1_{v,*})} + C\varepsilon\|\nabla_x^{i+1} f\|^2_{L^2_{x,v}}, \end{aligned}$$

for any $\sigma''' \in (\sigma', \sigma'')$, where we have used Young's inequality and that $\|\cdot\|_{L^2_{x,v}}$ and $|||\cdot|||_{L^2_{x,v}}$ are equivalent in last line.

Step 5. Gathering previous estimates we then obtain

$$\begin{aligned}\langle\langle \Lambda_\varepsilon f, f\rangle\rangle_{\mathscr{X}} \leqslant &-\sum_{i=0}^{3}\left\{\sigma_0 |||\nabla_x^i f|||^2_{L^2_{x,v}} + \kappa_0 \|\nabla_x^i f\|^2_{L^2_x(H^1_{v,*})} + \frac{\kappa_0}{\varepsilon^2}\|\nabla_x^i f^\perp\|^2_{L^2_x(H^1_{v,*})}\right\}\\ &+\sum_{i=0}^{2}\left\{-\frac{\sigma'''\delta}{\varepsilon^2}\|\omega_i\nabla_x^i f^\perp\|^2_{L^2_x(H^1_{v,*})} + \frac{C\delta}{\varepsilon^2}\|\nabla_x^i f^\perp\|^2_{L^2_{x,v}} + C\delta |||\nabla_x^{i+1} f|||^2_{L^2_{x,v}}\right\}.\end{aligned}$$

Recalling that $\|\cdot\|_{L^2_{x,v}} \leqslant \|\cdot\|_{L^2_x(H^1_{v,*})}$, it follows

$$\begin{aligned}\langle\langle \Lambda_\varepsilon f, f\rangle\rangle_{\mathscr{X}} \leqslant &-\sigma_0 |||f|||^2_{L^2_{x,v}} - (\sigma_0 - C\delta)\sum_{i=1}^{3} |||\nabla_x^i f|||^2_{L^2_{x,v}} - \frac{\sigma'}{\varepsilon^2}\sum_{i=0}^{2}\delta\|\omega_i\nabla_x^i f^\perp\|^2_{L^2_{x,v}}\\ &-\frac{(\kappa_0 - C\delta)}{\varepsilon^2}\sum_{i=0}^{3}\|\nabla_x^i f^\perp\|^2_{L^2_x(H^1_{v,*})} - \frac{(\sigma''' - \sigma')}{\varepsilon^2}\sum_{i=0}^{2}\delta\|\omega_i\nabla_x^i f^\perp\|^2_{L^2_x(H^1_{v,*})}.\end{aligned}$$

We then choose $\delta \in (0,1)$ small enough such that $\sigma_0 - C\delta \geqslant \sigma'$ and $\kappa_0 - C\delta > 0$, therefore we obtain (3.5) with $\kappa' = \min((\sigma''' - \sigma')\delta, \kappa_0 - C\delta) > 0$, which completes the proof. ☐

3.2. **Regularization estimates.**

Proposition 3.4. *The solution $f(t) = U^\varepsilon(t) f_{\rm in}$ to the equation*

$$\begin{cases}\partial_t f = \Lambda_\varepsilon f\\ f(0) = f_{\rm in} \in \mathscr{X} \cap (\operatorname{Ker}\Lambda_\varepsilon)^\perp\end{cases}$$

satisfies, for all $t > 0$,

$$\|f(t)\|_{\mathscr{Y}_1} \leqslant C\frac{e^{-\sigma t}}{\min(1,\sqrt{t})}\|f_{\rm in}\|_{\mathscr{X}} \tag{3.7}$$

and

$$\|f(t)\|_{\mathscr{X}_1^\varepsilon} \leqslant C\frac{e^{-\sigma t}}{\min(1,t^{3/2})}\|f_{\rm in}\|_{\mathscr{X}}, \tag{3.8}$$

for any $0 < \sigma < \sigma_0$ (where σ_0 is defined in Proposition 3.2).

Remark 3.5. Notice that thanks to the second inequality, one can in particular recover a gain of one derivative in the spatial variable (with the associated anisotropic gain of weight in velocity), at the price of loosing a $1/\varepsilon$. As already mentioned, this is explained by the fact that the gain comes from the transport operator which does not act as the same scale as the collision operator in velocity. Notice also that in [9], the authors were facing a similar singularity in ε when wanting to obtain a gain of regularity in the spatial variable for the hard-spheres Boltzmann equation. The latter equation is not hypoelliptic but thanks to a suitable use of averaging lemmas, the authors were also able to obtain regularization properties in the spatial variable with the same singularity in ε.

Proof of Proposition 3.4. We shall prove that for any $t \in (0,1]$ there holds

$$\|f(t)\|_{\mathscr{Y}_1} \leqslant \frac{C}{\sqrt{t}}\|f_{\rm in}\|_{\mathscr{X}} \tag{3.9}$$

and

$$\|f(t)\|_{\mathscr{X}_1^\varepsilon} \leqslant \frac{C}{t^{3/2}}\|f_{\rm in}\|_{\mathscr{X}}, \tag{3.10}$$

which readily imply (3.7) and (3.8) thanks to the exponential decay of U^ε in $\mathscr{X}$ from Proposition 3.3.

Step 1. Define the functional

$$\mathscr{U}_\varepsilon(t,f) = |||f|||^2_{\mathscr{X}} + \alpha_1 t\left(\|\widetilde{\nabla}_v f^\perp\|^2_{\mathscr{X}} + K\|\langle v\rangle^{\frac{\gamma}{2}+1} f^\perp\|^2_{\mathscr{X}}\right)$$

$$+\,\varepsilon\alpha_2 t^2 \left\langle \widetilde\nabla_v f, \widetilde\nabla_x f\right\rangle_{\mathscr X} + \varepsilon^2\alpha_3 t^3 \left(\|\widetilde\nabla_x f\|_{\mathscr X}^2 + K\|\langle v\rangle^{\frac{\gamma}{2}}\nabla_x f\|_{\mathscr X}^2\right),$$

with constants $0<\alpha_3\ll\alpha_2\ll\alpha_1\ll 1$ so that $\alpha_2\leqslant\sqrt{\alpha_1\alpha_3}$ and $K>0$. The constants α_i will be chosen small enough and K large enough during the proof.

We easily observe that

$$|\!|\!| f|\!|\!|_{\mathscr X}^2 + t\left(\|\widetilde\nabla_v f^\perp\|_{\mathscr X}^2 + \|\langle v\rangle^{\frac{\gamma}{2}+1} f^\perp\|_{\mathscr X}^2\right) + \varepsilon^2 t^3 \|\widetilde\nabla_x f\|_{\mathscr X}^2 \lesssim \mathscr U_\varepsilon(t,f).$$

Remarking that $\|f\|_{\mathscr Y_1}\lesssim \|f^\perp\|_{\mathscr Y_1}+\|\pi f\|_{\mathscr X}$, we thus obtain

$$\|f\|_{\mathscr Y_1} \lesssim \|\widetilde\nabla_v f^\perp\|_{\mathscr X}^2 + \|\langle v\rangle^{\frac{\gamma}{2}+1} f^\perp\|_{\mathscr X} + \|\pi f\|_{\mathscr X},$$

from which we deduce the following lower bounds: For all $t\in[0,1]$ there holds

$$t\|f\|_{\mathscr Y_1}^2 \lesssim \mathscr U_\varepsilon(t,f) \quad\text{and}\quad t^3\|f\|_{\mathscr X_1^\varepsilon}^2 \lesssim \mathscr U_\varepsilon(t,f). \tag{3.11}$$

Therefore, in order to prove (3.9) and (3.10), it is sufficient to show that

$$\frac{\mathrm d}{\mathrm dt}\mathscr U_\varepsilon(t,f)\leqslant 0,\quad \forall\, t\in[0,1],$$

which we shall do next. We thus compute

$$\begin{aligned}\frac{\mathrm d}{\mathrm dt}\mathscr U_\varepsilon(t,f) &= \frac{\mathrm d}{\mathrm dt}|\!|\!| f|\!|\!|_{\mathscr X}^2 + \alpha_1\Big(K\|\langle v\rangle^{\frac{\gamma}{2}+1}f^\perp\|_{\mathscr X}^2 + \|\widetilde\nabla_v f^\perp\|_{\mathscr X}^2\Big)\\ &\quad + \alpha_1 t\frac{\mathrm d}{\mathrm dt}\Big(K\|\langle v\rangle^{\frac{\gamma}{2}+1}f^\perp\|_{\mathscr X}^2 + \|\widetilde\nabla_v f^\perp\|_{\mathscr X}^2\Big)\\ &\quad + 2\alpha_2\varepsilon t\left\langle \widetilde\nabla_v f,\widetilde\nabla_x f\right\rangle_{\mathscr X} + \alpha_2\varepsilon t^2\frac{\mathrm d}{\mathrm dt}\left\langle \widetilde\nabla_v f,\widetilde\nabla_x f\right\rangle_{\mathscr X}\\ &\quad + 3\alpha_3\varepsilon^2 t^2\Big(\|\widetilde\nabla_x f\|_{\mathscr X}^2 + K\|\langle v\rangle^{\frac{\gamma}{2}}\nabla_x f\|_{\mathscr X}^2\Big)\\ &\quad + \alpha_3\varepsilon^2 t^3\frac{\mathrm d}{\mathrm dt}\Big(\|\widetilde\nabla_x f\|_{\mathscr X}^2 + K\|\langle v\rangle^{\frac{\gamma}{2}}\nabla_x f\|_{\mathscr X}^2\Big)\end{aligned} \tag{3.12}$$

and we estimate each term separately in the sequel. In order to simplify, we introduce the notations $g_i=\nabla_x^i f$, $g_i^\perp=(\nabla_x^i f)^\perp=\nabla_x^i f^\perp$ and $\omega_i=\langle v\rangle^{(3-i)(\frac{\gamma}{2}+1)}$ so that

$$\|f\|_{\mathscr X}^2=\sum_{i=0}^3\|\omega_i g_i\|_{L^2_{x,v}}^2,\qquad \|f\|_{\mathscr Y_1}^2=\sum_{i=0}^3\|\omega_i g_i\|_{L^2_x(H^1_{v,*})}^2,\qquad \|f\|_{\mathscr Y_2}^2=\sum_{i=0}^3\|\omega_i g_i\|_{L^2_x(H^2_{v,*})}^2.$$

Step 2. From Proposition 3.3, we already have

$$\frac{\mathrm d}{\mathrm dt}|\!|\!| f|\!|\!|_{\mathscr X}^2 \leqslant -\kappa_0\|f\|_{\mathscr Y_1}^2 - \frac{\kappa_0}{\varepsilon^2}\|f^\perp\|_{\mathscr Y_1}^2, \tag{3.13}$$

for some constant $\kappa_0>0$.

Step 3. We deal in this step with the time-derivative of the term $(K\|\langle v\rangle^{\frac{\gamma}{2}+1}f^\perp\|_{\mathscr X}^2 + \|\widetilde\nabla_v f^\perp\|_{\mathscr X}^2)$. We split the computations into two parts.

Step 3.1. We first compute

$$\frac12\frac{\mathrm d}{\mathrm dt}\|\langle v\rangle^{\frac{\gamma}{2}+1}f^\perp\|_{\mathscr X}^2 = \frac12\sum_{i=0}^3\frac{\mathrm d}{\mathrm dt}\|\omega_i\langle v\rangle^{\frac{\gamma}{2}+1}g_i^\perp\|_{L^2_{x,v}}^2.$$

Observing that $f^\perp$ satisfies the equation $\partial_t f^\perp=(\Lambda_\varepsilon f)^\perp$, with $(\Lambda_\varepsilon f)^\perp$ given by (3.6), using that the transport operator is skew-adjoint and that derivatives in x commute with π and Λ_ε, for any $i\in\{0,1,2,3\}$ we obtain

$$\frac12\frac{\mathrm d}{\mathrm dt}\|\omega_i\langle v\rangle^{\frac{\gamma}{2}+1}g_i^\perp\|_{L^2_{x,v}}^2 = \left\langle \omega_i\langle v\rangle^{\frac{\gamma}{2}+1}\partial_t g_i^\perp, \omega_i\langle v\rangle^{\frac{\gamma}{2}+1}g_i^\perp\right\rangle_{L^2_{x,v}} =: \frac{1}{\varepsilon^2}J_1^i - \frac1\varepsilon J_2^i + \frac1\varepsilon J_3^i$$

with

$$
\begin{aligned}
J_1^i &:= \left\langle \omega_i \langle v\rangle^{\frac{\gamma}{2}+1} L g_i^\perp, \omega_i \langle v\rangle^{\frac{\gamma}{2}+1} g_i^\perp \right\rangle_{L^2_{x,v}} \\
J_2^i &:= \left\langle \omega_i \langle v\rangle^{\frac{\gamma}{2}+1} (v\cdot\nabla_x(\pi g_i)), \omega_i \langle v\rangle^{\frac{\gamma}{2}+1} g_i^\perp \right\rangle_{L^2_{x,v}} \\
J_3^i &:= \left\langle \omega_i \langle v\rangle^{\frac{\gamma}{2}+1} (\pi(v\cdot\nabla_x g_i)), \omega_i \langle v\rangle^{\frac{\gamma}{2}+1} g_i^\perp \right\rangle_{L^2_{x,v}}.
\end{aligned}
$$

For the first term, we write

$$
\begin{aligned}
J_1^i &= \left\langle L(\omega_i \langle v\rangle^{\frac{\gamma}{2}+1} g_i^\perp) + [\omega_i \langle v\rangle^{\frac{\gamma}{2}+1}, L_1] g_i^\perp + [\omega_i \langle v\rangle^{\frac{\gamma}{2}+1}, L_2] g_i^\perp, \omega_i \langle v\rangle^{\frac{\gamma}{2}+1} g_i^\perp \right\rangle_{L^2_{x,v}} \\
&=: J_{11}^i + J_{12}^i + J_{13}^i,
\end{aligned}
$$

where we recall that L_1 and L_2 are defined in (2.18) and (2.19), respectively. Thanks to the spectral gap estimate (2.10), one has

$$
\begin{aligned}
J_{11}^i &\leqslant -2\kappa \|\omega_i \langle v\rangle^{\frac{\gamma}{2}+1} g_i^\perp - \pi(\omega_i \langle v\rangle^{\frac{\gamma}{2}+1} g_i^\perp)\|^2_{L^2_x(H^1_{v,*})} \\
&\leqslant -2\kappa \|\omega_i \langle v\rangle^{\frac{\gamma}{2}+1} g_i^\perp\|^2_{L^2_x(H^1_{v,*})} + C\|g_i^\perp\|^2_{L^2_{x,v}},
\end{aligned}
$$

for some constants $\kappa, C > 0$. From Lemma 2.3, we get

$$
\begin{aligned}
J_{12}^i = &- \left\langle [\omega_i \langle v\rangle^{\frac{\gamma}{2}+1}, \widetilde\nabla_{v_\ell}] g_i^\perp, \widetilde\nabla_{v_\ell}(\omega_i \langle v\rangle^{\frac{\gamma}{2}+1} g_i^\perp) \right\rangle_{L^2_{x,v}} \\
&- \left\langle [\omega_i \langle v\rangle^{\frac{\gamma}{2}+1}, \widetilde\nabla^*_{v_\ell}] g_i^\perp, \widetilde\nabla^*_{v_\ell}(\omega_i \langle v\rangle^{\frac{\gamma}{2}+1} g_i^\perp) \right\rangle_{L^2_{x,v}} \\
&- \left\langle \left[[\omega_i \langle v\rangle^{\frac{\gamma}{2}+1}, \widetilde\nabla^*_{v_\ell}], \widetilde\nabla_{v_\ell} \right] g_i^\perp, \omega_i \langle v\rangle^{\frac{\gamma}{2}+1} g_i^\perp \right\rangle_{L^2_{x,v}}.
\end{aligned}
$$

Using Lemma 2.2 and observing that $\|\widetilde\nabla_v h\|_{L^2_{x,v}} + \|\widetilde\nabla_v^* h\|_{L^2_{x,v}} \lesssim \|h\|_{L^2_x(H^1_{v,*})}$, we then obtain

$$
\begin{aligned}
|J_{12}^i| &\leqslant C\|\omega_i \langle v\rangle^{\frac{\gamma}{2}+1} g_i^\perp\|_{L^2_x(H^1_{v,*})} \|\langle v\rangle^{\frac{\gamma}{2}-1}\omega_i \langle v\rangle^{\frac{\gamma}{2}+1} g_i^\perp\|_{L^2_{x,v}} + C\|\langle v\rangle^{\frac{\gamma}{2}-\frac12}\omega_i \langle v\rangle^{\frac{\gamma}{2}+1} g_i^\perp\|^2_{L^2_{x,v}} \\
&\leqslant \kappa \|\omega_i \langle v\rangle^{\frac{\gamma}{2}+1} g_i^\perp\|^2_{L^2_x(H^1_{v,*})} + C\|\omega_i \langle v\rangle^{\frac{\gamma}{2}+1} g_i^\perp\|^2_{L^2_{x,v}}
\end{aligned}
$$

where we have used that $\gamma \leqslant 1$ and Young's inequality in last line. For the term J_{13}^i, we use Lemma 2.6 to obtain

$$
|J_{13}^i| \leqslant C\|\omega_i g_i^\perp\|^2_{L^2_{x,v}}.
$$

We deal with the terms J_2^i and J_3^i. Observing that

$$
\|\omega_i \langle v\rangle^{\frac{\gamma}{2}+1}(v\cdot\nabla_x(\pi g_i))\|_{L^2_{x,v}} + \|\omega_i \langle v\rangle^{\frac{\gamma}{2}+1}(\pi(v\cdot\nabla_x g_i))\|_{L^2_{x,v}} \lesssim \|\omega_i \widetilde\nabla_x g_i\|_{L^2_{x,v}},
$$

we obtain

$$
|J_2^i| + |J_3^i| \leqslant C\|\omega_i \widetilde\nabla_x g_i\|_{L^2_{x,v}} \|\omega_i \langle v\rangle^{\frac{\gamma}{2}+1} g_i^\perp\|_{L^2_{x,v}}.
$$

Gathering previous estimates and using that $\| \langle v\rangle^{\gamma+\frac12} f^\perp\|_{\mathscr X} \lesssim \|f^\perp\|_{\mathscr Y_1}$, we obtain

$$
\frac12 \frac{\mathrm d}{\mathrm dt}\|\langle v\rangle^{\frac{\gamma}{2}+1} f^\perp\|^2_{\mathscr X} \leqslant -\frac{\kappa}{\varepsilon^2}\|\langle v\rangle^{\frac{\gamma}{2}+1} f^\perp\|^2_{\mathscr Y_1} + \frac{C}{\varepsilon^2}\|f^\perp\|^2_{\mathscr Y_1} + \frac{C}{\varepsilon}\|f^\perp\|_{\mathscr Y_1}\|\widetilde\nabla_x f\|_{\mathscr X} \tag{3.14}
$$

for some constants $\kappa, C > 0$.

Step 3.2. We now compute

$$
\begin{aligned}
\frac12 \frac{\mathrm d}{\mathrm dt}\|\widetilde\nabla_v f^\perp\|^2_{\mathscr X} &= \frac12 \sum_{i=0}^3 \frac{\mathrm d}{\mathrm dt}\|\omega_i \widetilde\nabla_v g_i^\perp\|^2_{L^2_{x,v}} \\
&=: \sum_{i=0}^3 \left(\frac{1}{\varepsilon^2} I_1^i - \frac{1}{\varepsilon} I_2^i + \frac{1}{\varepsilon} I_3^i - \frac{1}{\varepsilon} I_4^i \right)
\end{aligned}
$$

with

$$
\begin{aligned}
I_1^i &:= \left\langle \omega_i \widetilde\nabla_{v_k} L g_i^\perp, \omega_i \widetilde\nabla_{v_k} g_i^\perp \right\rangle_{L^2_{x,v}} \\
I_2^i &:= \left\langle \omega_i \widetilde\nabla_{v_k} (v\cdot\nabla_x(\pi g_i)), \omega_i \widetilde\nabla_{v_k} g_i^\perp \right\rangle_{L^2_{x,v}} \\
I_3^i &:= \left\langle \omega_i \widetilde\nabla_{v_k} (\pi(v\cdot\nabla_x g_i)), \omega_i \widetilde\nabla_{v_k} g_i^\perp \right\rangle_{L^2_{x,v}} \\
I_4^i &:= \left\langle \omega_i \widetilde\nabla_{v_k} (v\cdot\nabla_x g_i^\perp), \omega_i \widetilde\nabla_{v_k} g_i^\perp \right\rangle_{L^2_{x,v}} .
\end{aligned}
$$

For the first term, we write

$$
\begin{aligned}
I_1^i = &\left\langle L(\omega_i \widetilde\nabla_{v_k} g_i^\perp), \omega_i \widetilde\nabla_{v_k} g_i^\perp \right\rangle_{L^2_{x,v}} + \left\langle [\omega_i, L_1]\widetilde\nabla_{v_k} g_i^\perp, \omega_i \widetilde\nabla_{v_k} g_i^\perp \right\rangle_{L^2_{x,v}} \\
&+ \left\langle \omega_i[\widetilde\nabla_{v_k}, L_1] g_i^\perp, \omega_i \widetilde\nabla_{v_k} g_i^\perp \right\rangle_{L^2_{x,v}} + \left\langle [\omega_i, L_2]\widetilde\nabla_{v_k} g_i^\perp, \omega_i \widetilde\nabla_{v_k} g_i^\perp \right\rangle_{L^2_{x,v}} \\
&+ \left\langle \omega_i[\widetilde\nabla_{v_k}, L_2] g_i^\perp, \omega_i \widetilde\nabla_{v_k} g_i^\perp \right\rangle_{L^2_{x,v}} \\
=:&\; I_{11}^i + I_{12}^i + I_{13}^i + I_{14}^i + I_{15}^i.
\end{aligned}
$$

Thanks to the spectral gap estimate (2.10), one has

$$
\begin{aligned}
I_{11}^i &\leqslant -2\kappa \|\omega_i \widetilde\nabla_{v_k} g_i^\perp - \pi(\omega_i \widetilde\nabla_{v_k} g_i^\perp)\|^2_{L^2_x(H^1_{v,*})} \\
&\leqslant -2\kappa \|\omega_i \widetilde\nabla_{v_k} g_i^\perp\|^2_{L^2_x(H^1_{v,*})} + C\|g_i^\perp\|^2_{L^2_{x,v}},
\end{aligned}
$$

for some constants $\kappa, C > 0$. From Lemma 2.3, we get

$$
\begin{aligned}
I_{12}^i = &-\left\langle [\omega_i, \widetilde\nabla_{v_\ell}]\widetilde\nabla_{v_k} g_i^\perp, \widetilde\nabla_{v_\ell}(\omega_i \widetilde\nabla_{v_k} g_i^\perp)\right\rangle_{L^2_{x,v}} - \left\langle [\omega_i, \widetilde\nabla^*_{v_\ell}]\widetilde\nabla_{v_k} g_i^\perp, \widetilde\nabla^*_{v_\ell}(\omega_i \widetilde\nabla_{v_k} g_i^\perp)\right\rangle_{L^2_{x,v}} \\
&- \left\langle \left[[\omega_i, \widetilde\nabla^*_{v_\ell}], \widetilde\nabla_{v_\ell}\right] \widetilde\nabla_{v_k} g_i^\perp, \omega_i \widetilde\nabla_{v_k} g_i^\perp \right\rangle_{L^2_{x,v}},
\end{aligned}
$$

and

$$
\begin{aligned}
I_{13}^i = &-\left\langle [\widetilde\nabla_{v_k}, \widetilde\nabla_{v_\ell}] g_i^\perp, \widetilde\nabla_{v_\ell}(\omega_i^2 \widetilde\nabla_{v_k} g_i^\perp)\right\rangle_{L^2_{x,v}} - \left\langle [\widetilde\nabla_{v_k}, \widetilde\nabla^*_{v_\ell}] g_i^\perp, \widetilde\nabla^*_{v_\ell}(\omega_i^2 \widetilde\nabla_{v_k} g_i^\perp)\right\rangle_{L^2_{x,v}} \\
&- \left\langle \omega_i \left[[\widetilde\nabla_{v_k}, \widetilde\nabla^*_{v_\ell}], \widetilde\nabla_{v_\ell}\right] g_i^\perp, \omega_i \widetilde\nabla_{v_k} g_i^\perp \right\rangle_{L^2_{x,v}} - \left\langle \omega_i (\widetilde\nabla_{v_k}\psi) g_i^\perp, \omega_i \widetilde\nabla_{v_k} g_i^\perp \right\rangle_{L^2_{x,v}}.
\end{aligned}
$$

For I_{12}^i, using Lemma 2.2 and observing that $\|\widetilde\nabla_v h\|_{L^2_{x,v}} + \|\widetilde\nabla^*_v h\|_{L^2_{x,v}} \lesssim \|h\|_{L^2_x(H^1_{v,*})}$, we obtain

$$
|I_{12}^i| \leqslant C\|\omega_i \widetilde\nabla_{v_k} g_i^\perp\|_{L^2_x(H^1_{v,*})} \|\langle v\rangle^{\frac{\gamma}{2}-1}\omega_i \widetilde\nabla_{v_k} g_i^\perp\|_{L^2_{x,v}} + C\|\langle v\rangle^{\frac{\gamma}{2}-\frac12}\omega_i \widetilde\nabla_{v_k} g_i^\perp\|^2_{L^2_{x,v}}.
$$

For I_{13}^i, we first observe that writing

$$
\begin{aligned}
\widetilde\nabla_{v_\ell}(\omega_i^2 h) &= \omega_i \widetilde\nabla_{v_\ell}(\omega_i h) + (\widetilde\nabla_{v_\ell}\omega_i)\omega_i h \\
\widetilde\nabla^*_{v_\ell}(\omega_i^2 h) &= -\omega_i \widetilde\nabla_{v_\ell}(\omega_i h) - \left[(\partial_{v_p} B_{\ell p})\omega_i + (\widetilde\nabla_{v_\ell}\omega_i)\right]\omega_i h
\end{aligned}
$$

and using the bounds (2.9), we have

$$
\|\omega_i^{-1}\widetilde\nabla_{v_\ell}(\omega_i^2 h)\|_{L^2_{x,v}} + \|\omega_i^{-1}\widetilde\nabla^*_{v_\ell}(\omega_i^2 h)\|_{L^2_{x,v}} \lesssim \|\omega_i h\|_{L^2_x(H^1_{v,*})}. \tag{3.15}
$$

Therefore, using Lemma 2.2 and noticing that

$$
|\widetilde\nabla_v \psi| \lesssim \langle v\rangle^{\frac{\gamma}{2}+1} |\nabla_v\psi| \lesssim \langle v\rangle^{\frac{\gamma}{2}+1}\langle v\rangle^{\gamma+1},
$$

we obtain

$$
\begin{aligned}
|I_{13}^i| \leqslant\; &C\|\omega_i \widetilde\nabla_{v_k} g_i^\perp\|_{L^2_x(H^1_{v,*})} \|\omega_i \langle v\rangle^{\gamma+1}\nabla_v g_i^\perp\|_{L^2_{x,v}} + C\|\omega_i \widetilde\nabla_{v_k} g_i^\perp\|_{L^2_x(H^1_{v,*})} \|\omega_i\langle v\rangle^{\gamma} g_i^\perp\|_{L^2_{x,v}} \\
&+ C\|\omega_i\langle v\rangle^\gamma \nabla_v g_i^\perp\|_{L^2_{x,v}} \|\langle v\rangle^{\frac{\gamma}{2}+1}\omega_i \widetilde\nabla_{v_k} g_i^\perp\|_{L^2_{x,v}} \\
&+ C\|\omega_i\langle v\rangle^{\gamma-1} g_i^\perp\|_{L^2_{x,v}} \|\langle v\rangle^{\frac{\gamma}{2}+1}\omega_i \widetilde\nabla_{v_k} g_i^\perp\|_{L^2_{x,v}} \\
&+ C\|\omega_i\langle v\rangle^{\gamma+1} g_i^\perp\|_{L^2_{x,v}} \|\langle v\rangle^{\frac{\gamma}{2}+1}\omega_i \widetilde\nabla_{v_k} g_i^\perp\|_{L^2_{x,v}}.
\end{aligned}
$$

Putting together the estimates for I^i_{12} and I^i_{13} and using Young's inequality, it follows

$$|I^i_{12}| + |I^i_{13}| \leqslant \kappa \|\omega_i \widetilde{\nabla}_{v_k} g_i^\perp\|^2_{L^2_x(H^1_{v,*})} + C\|\omega_i \langle v\rangle^{\gamma+1} g_i^\perp\|^2_{L^2_{x,v}} + C\|\omega_i\langle v\rangle^{\gamma+1}\nabla_v g_i^\perp\|^2_{L^2_{x,v}}.$$

For the terms I^i_{14} and I^i_{15}, Lemma 2.6 yields

$$|I^i_{14}| + |I^i_{15}| \leqslant C\|\omega_i g_i^\perp\|^2_{L^2_{x,v}} + C\|\omega_i \widetilde{\nabla}_v g_i^\perp\|^2_{L^2_{x,v}}.$$

The terms I^i_2 and I^i_3 can be dealt as before in Step 3.1, and we obtain

$$|I^i_2| + |I^i_3| \leqslant C\|\omega_i \widetilde{\nabla}_x g_i\|_{L^2_{x,v}} \|\omega_i \widetilde{\nabla}_{v_k} g_i^\perp\|_{L^2_{x,v}}.$$

For the remainder term I^i_4, we use Lemma 2.2 and the fact that the transport operator is skew-symmetric to obtain

$$I^i_4 = \left\langle \omega_i \widetilde{\nabla}_{x_k} g_i^\perp, \omega_i \widetilde{\nabla}_{v_k} g_i^\perp\right\rangle_{L^2_{x,v}} \leqslant \|\omega_i \widetilde{\nabla}_{x_k} g_i^\perp\|_{L^2_{x,v}} \|\omega_i \widetilde{\nabla}_{v_k} g_i^\perp\|_{L^2_{x,v}}.$$

Gathering previous estimates, we obtain

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\widetilde{\nabla}_v f^\perp\|^2_{\mathscr{X}} \leqslant -\frac{\kappa}{\varepsilon^2}\|\widetilde{\nabla}_v f^\perp\|^2_{\mathscr{Y}_1} + \frac{C}{\varepsilon^2}\|\langle v\rangle^{\gamma+1} f^\perp\|^2_{\mathscr{X}} + \frac{C}{\varepsilon^2}\|\langle v\rangle^{\gamma+1}\nabla_v f^\perp\|^2_{\mathscr{X}} \\ + \frac{C}{\varepsilon}\|\widetilde{\nabla}_v f^\perp\|_{\mathscr{X}} \|\widetilde{\nabla}_x f\|_{\mathscr{X}}.\end{aligned} \tag{3.16}$$

Finally, we observe that $\|\langle v\rangle^{\gamma+1} f^\perp\|^2_{\mathscr{X}} + \|\langle v\rangle^{\gamma+1}\nabla_v f^\perp\|^2_{\mathscr{X}} \lesssim \|\langle v\rangle^{\frac{\gamma}{2}+1} f^\perp\|_{\mathscr{Y}_1}$ and thus, gathering (3.14) and (3.16) and taking $K > 0$ large enough, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(K\|\langle v\rangle^{\frac{\gamma}{2}+1} f^\perp\|^2_{\mathscr{X}} + \|\widetilde{\nabla}_v f^\perp\|^2_{\mathscr{X}}\right) \\ &\qquad \leqslant -\frac{\kappa_1}{\varepsilon^2}\|f^\perp\|^2_{\mathscr{Y}_2} + \frac{C}{\varepsilon^2}\|f^\perp\|^2_{\mathscr{Y}_1} + \frac{C}{\varepsilon}\|f^\perp\|_{\mathscr{Y}_1}\|\widetilde{\nabla}_x f\|_{\mathscr{X}}\end{aligned} \tag{3.17}$$

for some constants $\kappa_1, C > 0$.

Step 4. We deal in this step with the time-derivative of the term $\langle \widetilde{\nabla}_v f, \widetilde{\nabla}_x f\rangle_{\mathscr{X}}$. We compute

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left\langle \widetilde{\nabla}_v f, \widetilde{\nabla}_x f\right\rangle_{\mathscr{X}} &= \sum_{i=0}^3 \frac{\mathrm{d}}{\mathrm{d}t}\left\langle \omega_i \widetilde{\nabla}_v g_i, \omega_i \widetilde{\nabla}_x g_i\right\rangle_{L^2_{x,v}} \\ &=: \sum_{i=0}^3 \left(\frac{1}{\varepsilon^2}R^i_1 - \frac{1}{\varepsilon}R^i_2 + \frac{1}{\varepsilon^2}S^i_1 - \frac{1}{\varepsilon}S^i_2\right)\end{aligned}$$

with

$$\begin{aligned}R^i_1 &:= \left\langle \omega_i \widetilde{\nabla}_{v_k}(Lg_i^\perp), \omega_i \widetilde{\nabla}_{x_k} g_i\right\rangle_{L^2_{x,v}} \\ S^i_1 &:= \left\langle \omega_i \widetilde{\nabla}_{v_k} g_i, \omega_i \widetilde{\nabla}_{x_k}(Lg_i^\perp)\right\rangle_{L^2_{x,v}} \\ R^i_2 &:= \left\langle \omega_i \widetilde{\nabla}_{v_k}(v\cdot\nabla_x g_i), \omega_i \widetilde{\nabla}_{x_k} g_i\right\rangle_{L^2_{x,v}} \\ S^i_2 &:= \left\langle \omega_i \widetilde{\nabla}_{v_k} g_i, \omega_i \widetilde{\nabla}_{x_k}(v\cdot\nabla_x g_i)\right\rangle_{L^2_{x,v}}.\end{aligned}$$

Recalling that $\widetilde{\nabla}^*_{x_k} = -\widetilde{\nabla}_{x_k}$, we then write

$$\begin{aligned}R^i_1 + S^i_1 &= \left\langle Lg_i^\perp, \widetilde{\nabla}^*_{v_k}(\omega_i^2 \widetilde{\nabla}_{x_k} g_i) - \widetilde{\nabla}_{x_k}(\omega_i^2 \widetilde{\nabla}_{v_k} g_i)\right\rangle_{L^2_{x,v}} \\ &\leqslant \|\omega_i L g_i^\perp\|_{L^2_{x,v}}\left(\|\omega_i^{-1}\widetilde{\nabla}^*_{v_k}(\omega_i^2\widetilde{\nabla}_{x_k} g_i)\|_{L^2_{x,v}} + \|\omega_i^{-1}\widetilde{\nabla}_{x_k}(\omega_i^2\widetilde{\nabla}_{v_k} g_i)\|_{L^2_{x,v}}\right).\end{aligned}$$

Observe that

$$\omega_i^{-1}\widetilde{\nabla}_{x_k}(\omega_i^2 \widetilde{\nabla}_{v_k} g_i) = \widetilde{\nabla}_{v_k}(\omega_i \widetilde{\nabla}_{x_k} g_i) + [\omega_i, \widetilde{\nabla}_{v_k}]\widetilde{\nabla}_{x_k} g_i + \omega_i[\widetilde{\nabla}_{x_k}, \widetilde{\nabla}_{v_k}] g_i$$

so that using Lemma 2.2, we get

$$\|\omega_i^{-1}\widetilde{\nabla}_{x_k}(\omega_i^2\widetilde{\nabla}_{v_k} g_i)\|_{L^2_{x,v}} \lesssim \|\omega_i \widetilde{\nabla}_x g_i\|_{L^2_x(H^1_{v,*})}. \tag{3.18}$$

Writing $\omega_i L g_i^\perp = \omega_i L_1 g_i^\perp + \omega_i L_2 g_i^\perp$ and using the explicit formula (2.18) of L_1 together with the bound of Lemma 2.6 for L_2, we obtain

$$\|\omega_i L g_i^\perp\|_{L^2_{x,v}} \lesssim \|\omega_i g_i^\perp\|_{L^2_x(H^2_{v,*})}. \tag{3.19}$$

Together with (3.15), we hence get

$$|R_1^i + S_1^i| \leqslant C\|\omega_i g_i^\perp\|_{L^2_x(H^2_{v,*})}\|\omega_i \widetilde\nabla_x g_i\|_{L^2_x(H^1_{v,*})}.$$

For the remainder terms, we observe that using Lemma 2.2 and the fact that the transport operator is skew-adjoint,

$$\begin{aligned} R_2^i + S_2^i &= \|\omega_i \widetilde\nabla_{x_k} g_i\|^2_{L^2_{x,v}} + \left\langle \omega_i v\cdot\nabla_x(\widetilde\nabla_{v_k} g_i), \omega_i \widetilde\nabla_{x_k} g_i\right\rangle_{L^2_{x,v}} \\ &\quad + \left\langle \omega_i \widetilde\nabla_{v_k} g_i, \omega_i v\cdot\nabla_x(\widetilde\nabla_{x_k} g_i)\right\rangle_{L^2_{x,v}} = \|\omega_i \widetilde\nabla_{x_k} g_i\|^2_{L^2_{x,v}}. \end{aligned}$$

Gathering previous estimates we obtain

$$\frac{\mathrm d}{\mathrm dt}\left\langle \widetilde\nabla_v f, \widetilde\nabla_x f\right\rangle_{\mathscr X} \leqslant \frac{C}{\varepsilon^2}\|f^\perp\|_{\mathscr Y_2}\|\widetilde\nabla_x f\|_{\mathscr Y_1} - \frac{1}{\varepsilon}\|\widetilde\nabla_x f\|^2_{\mathscr X} \tag{3.20}$$

for some constant $C>0$.

Step 5. We deal in this step with the derivative in time of the term $(K\|\langle v\rangle^{\frac{\gamma}{2}}\nabla_x f\|^2_{\mathscr X} + \|\widetilde\nabla_x f\|^2_{\mathscr X})$. We first compute, using that the transport operator is skew-adjoint,

$$\begin{aligned} \frac12\frac{\mathrm d}{\mathrm dt}\|\langle v\rangle^{\frac{\gamma}{2}}\nabla_x f\|^2_{\mathscr X} &= \frac12\sum_{i=0}^3 \frac{\mathrm d}{\mathrm dt}\|\omega_i\langle v\rangle^{\frac{\gamma}{2}}\nabla_x g_i\|^2_{L^2_{x,v}} \\ &=: \sum_{i=0}^3 \frac{1}{\varepsilon^2}\left\langle \omega_i\langle v\rangle^{\frac{\gamma}{2}}\partial_{x_k}(Lg_i), \omega_i\langle v\rangle^{\frac{\gamma}{2}}\partial_{x_k} g_i\right\rangle_{L^2_{x,v}}. \end{aligned}$$

Since ∂_{x_k} commutes with L, we can argue as for the term J_1^i in Step 3.1 above to obtain

$$\left\langle \omega_i\langle v\rangle^{\frac{\gamma}{2}}\partial_{x_k}(Lg_i), \omega_i\langle v\rangle^{\frac{\gamma}{2}}\partial_{x_k} g_i\right\rangle_{L^2_{x,v}} \leqslant -\kappa\|\omega_i\langle v\rangle^{\frac{\gamma}{2}}\nabla_x g_i\|^2_{L^2_x(H^1_{v,*})} + C\|\omega_i\widetilde\nabla_x g_i\|^2_{L^2_{x,v}},$$

for some constants $\kappa, C>0$, therefore

$$\frac12\frac{\mathrm d}{\mathrm dt}\|\langle v\rangle^{\frac{\gamma}{2}}\nabla_x f\|^2_{\mathscr X} \leqslant -\frac{\kappa}{\varepsilon^2}\|\langle v\rangle^{\frac{\gamma}{2}}\nabla_x f\|^2_{\mathscr Y_1} + \frac{C}{\varepsilon^2}\|\widetilde\nabla_x f\|^2_{\mathscr X}. \tag{3.21}$$

Using that the transport operator is skew-adjoint and commutes with $\widetilde\nabla_{x_k}$, we now compute

$$\begin{aligned} \frac12\frac{\mathrm d}{\mathrm dt}\|\widetilde\nabla_x f\|^2_{\mathscr X} &= \frac12\sum_{i=0}^3\frac{\mathrm d}{\mathrm dt}\|\omega_i\widetilde\nabla_x g_i\|^2_{L^2_{x,v}} \\ &= \sum_{i=0}^3\frac{1}{\varepsilon^2}\left\langle \omega_i\widetilde\nabla_{x_k}(Lg_i), \omega_i\widetilde\nabla_{x_k} g_i\right\rangle_{L^2_{x,v}} =: \sum_{i=0}^3\frac{1}{\varepsilon^2}N^i. \end{aligned}$$

We then write

$$\begin{aligned} N^i &= \left\langle L(\omega_i\widetilde\nabla_{x_k} g_i), \omega_i\widetilde\nabla_{x_k} g_i\right\rangle_{L^2_{x,v}} + \left\langle [\omega_i, L_1]\widetilde\nabla_{x_k} g_i, \omega_i\widetilde\nabla_{v_k} g_i\right\rangle_{L^2_{x,v}} \\ &\quad + \left\langle \omega_i[\widetilde\nabla_{x_k}, L_1]g_i, \omega_i\widetilde\nabla_{x_k} g_i\right\rangle_{L^2_{x,v}} + \left\langle [\omega_i, L_2]\widetilde\nabla_{x_k} g_i, \omega_i\widetilde\nabla_{x_k} g_i\right\rangle_{L^2_{x,v}} \\ &\quad + \left\langle \omega_i[\widetilde\nabla_{x_k}, L_2]g_i, \omega_i\widetilde\nabla_{x_k} g_i\right\rangle_{L^2_{x,v}} \\ &=: N_1^i + N_2^i + N_3^i + N_4^i + N_5^i. \end{aligned}$$

Thanks to the spectral gap estimate (2.10), one has

$$\begin{aligned} N_1^i &\leqslant -2\kappa\|\omega_i\widetilde\nabla_{x_k} g_i - \pi(\omega_i\widetilde\nabla_{x_k} g_i)\|^2_{L^2_x(H^1_{v,*})} \\ &\leqslant -2\kappa\|\omega_i\widetilde\nabla_{x_k} g_i^\perp\|^2_{L^2_x(H^1_{v,*})} + C\|\omega_i\widetilde\nabla_{x_k} g_i\|^2_{L^2_{x,v}}, \end{aligned}$$

for some constants $\kappa, C > 0$. From Lemma 2.3, we get

$$N_2^i = -\left\langle [\omega_i, \widetilde{\nabla}_{v_\ell}]\widetilde{\nabla}_{x_k} g_i, \widetilde{\nabla}_{v_\ell}(\omega_i \widetilde{\nabla}_{x_k} g_i)\right\rangle_{L^2_{x,v}} - \left\langle [\omega_i, \widetilde{\nabla}^*_{v_\ell}]\widetilde{\nabla}_{x_k} g_i, \widetilde{\nabla}^*_{v_\ell}(\omega_i \widetilde{\nabla}_{x_k} g_i)\right\rangle_{L^2_{x,v}} - \left\langle \left[[\omega_i, \widetilde{\nabla}^*_{v_\ell}], \widetilde{\nabla}_{v_\ell}\right] \widetilde{\nabla}_{x_k} g_i, \omega_i \widetilde{\nabla}_{x_k} g_i\right\rangle_{L^2_{x,v}},$$

and

$$N_3^i = -\left\langle [\widetilde{\nabla}_{x_k}, \widetilde{\nabla}_{v_\ell}] g_i, \widetilde{\nabla}_{v_\ell}(\omega_i^2 \widetilde{\nabla}_{x_k} g_i)\right\rangle_{L^2_{x,v}} - \left\langle [\widetilde{\nabla}_{x_k}, \widetilde{\nabla}^*_{v_\ell}] g_i, \widetilde{\nabla}^*_{v_\ell}(\omega_i^2 \widetilde{\nabla}_{x_k} g_i)\right\rangle_{L^2_{x,v}} - \left\langle \omega_i \left[[\widetilde{\nabla}_{x_k}, \widetilde{\nabla}^*_{v_\ell}], \widetilde{\nabla}_{v_\ell}\right] g_i, \omega_i \widetilde{\nabla}_{x_k} g_i\right\rangle_{L^2_{x,v}}.$$

Arguing in a similar way as in Step 3.2 above (for the term I_{12}^i and I_{13}^i), it follows

$$|N_2^i| + |N_3^i| \leqslant \kappa \|\omega_i \widetilde{\nabla}_{x_k} g_i\|^2_{L^2_x(H^1_{v,*})} + C\|\omega_i \langle v\rangle^{\gamma+1}\nabla_x g_i\|^2_{L^2_{x,v}}.$$

For the terms N_4^i and N_5^i, Lemma 2.6 yields

$$|N_4^i| + |N_5^i| \leqslant C\|\omega_i \widetilde{\nabla}_x g_i\|^2_{L^2_{x,v}}.$$

Gathering previous estimates, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\widetilde{\nabla}_x f\|^2_{\mathscr{X}} \leqslant -\frac{\kappa}{\varepsilon^2}\|\widetilde{\nabla}_x f\|^2_{\mathscr{Y}_1} + \frac{C}{\varepsilon^2}\|\langle v\rangle^{\gamma+1}\nabla_x f\|^2_{\mathscr{X}}. \tag{3.22}$$

Finally, we observe that $\|\langle v\rangle^{\gamma+1}\nabla_x f\|^2_{\mathscr{X}} \lesssim \|\langle v\rangle^{\frac{\gamma}{2}}\nabla_x f\|_{\mathscr{Y}_1}$ and thus, gathering (3.21) and (3.22) and taking $K > 0$ large enough, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(K\|\langle v\rangle^{\frac{\gamma}{2}}\nabla_x f\|^2_{\mathscr{X}} + \|\widetilde{\nabla}_x f^\perp\|^2_{\mathscr{X}}\right) \leqslant -\frac{\kappa_2}{\varepsilon^2}\|\widetilde{\nabla}_x f\|^2_{\mathscr{Y}_1} + \frac{C}{\varepsilon^2}\|\widetilde{\nabla}_x f\|^2_{\mathscr{X}} \tag{3.23}$$

for some constants $\kappa_2, C > 0$.

Step 6. Conclusion. Gathering (3.13)–(3.17)–(3.20)–(3.23), we thus obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{U}_\varepsilon(t,f) &\leqslant -\kappa_0\|f\|^2_{\mathscr{Y}_1} - \frac{\kappa_0}{\varepsilon^2}\|f^\perp\|^2_{\mathscr{Y}_1} + \alpha_1\left(K\|\langle v\rangle^{\frac{\gamma}{2}+1}f^\perp\|^2_{\mathscr{X}} + \|\widetilde{\nabla}_v f^\perp\|^2_{\mathscr{X}}\right)\\
&\quad + \alpha_1 t\left(-\frac{\kappa_1}{\varepsilon^2}\|f^\perp\|^2_{\mathscr{Y}_2} + \frac{C}{\varepsilon^2}\|f^\perp\|^2_{\mathscr{Y}_1} + \frac{C}{\varepsilon}\|f^\perp\|_{\mathscr{Y}_1}\|\widetilde{\nabla}_x f\|_{\mathscr{X}}\right)\\
&\quad + 2\alpha_2\varepsilon t\left\langle \widetilde{\nabla}_v f, \widetilde{\nabla}_x f\right\rangle_{\mathscr{X}} + \alpha_2\varepsilon t^2\left(\frac{C}{\varepsilon^2}\|f^\perp\|_{\mathscr{Y}_2}\|\widetilde{\nabla}_x f\|_{\mathscr{Y}_1} - \frac{1}{\varepsilon}\|\widetilde{\nabla}_x f\|^2_{\mathscr{X}}\right)\\
&\quad + 3\alpha_3\varepsilon^2 t^2\left(\|\widetilde{\nabla}_x f\|^2_{\mathscr{X}} + K\|\langle v\rangle^{\frac{\gamma}{2}}\nabla_x f\|^2_{\mathscr{X}}\right)\\
&\quad + \alpha_3\varepsilon^2 t^3\left(-\frac{\kappa_2}{\varepsilon^2}\|\widetilde{\nabla}_x f\|^2_{\mathscr{Y}_1} + \frac{C}{\varepsilon^2}\|\widetilde{\nabla}_x f\|^2_{\mathscr{X}}\right).\end{aligned}$$

Observe that

$$K\|\langle v\rangle^{\frac{\gamma}{2}+1}f^\perp\|^2_{\mathscr{X}} + \|\widetilde{\nabla}_v f^\perp\|^2_{\mathscr{X}} \lesssim \|f^\perp\|^2_{\mathscr{Y}_1}$$

and

$$\|\widetilde{\nabla}_x f\|^2_{\mathscr{X}} + K\|\langle v\rangle^{\frac{\gamma}{2}}\nabla_x f\|^2_{\mathscr{X}} \lesssim \|\widetilde{\nabla}_x f\|^2_{\mathscr{X}}$$

and also that, thanks to Young's inequality, there holds

$$\begin{aligned}\frac{\alpha_1 t C}{\varepsilon}\|f^\perp\|_{\mathscr{Y}_1}\|\widetilde{\nabla}_x f\|_{\mathscr{X}} &\leqslant \frac{\alpha_2}{4}t^2\|\widetilde{\nabla}_x f\|^2_{\mathscr{X}} + C\frac{\alpha_1^2}{\alpha_2}\frac{1}{\varepsilon^2}\|f^\perp\|^2_{\mathscr{Y}_1}\\
\frac{\alpha_2 t^2 C}{\varepsilon}\|f^\perp\|_{\mathscr{Y}_2}\|\widetilde{\nabla}_x f\|_{\mathscr{Y}_1} &\leqslant \frac{\alpha_3\kappa_2}{2}t^3\|\widetilde{\nabla}_x f\|^2_{\mathscr{Y}_1} + C\frac{\alpha_2^2}{\alpha_3}\frac{t}{\varepsilon^2}\|f^\perp\|^2_{\mathscr{Y}_2}\\
2\alpha_2\varepsilon t\left\langle \widetilde{\nabla}_x f, \widetilde{\nabla}_v f\right\rangle_{\mathscr{X}} &\leqslant \frac{\alpha_2}{4}t^2\|\widetilde{\nabla}_x f\|^2_{\mathscr{X}} + C\alpha_2\varepsilon^2\|f\|^2_{\mathscr{Y}_1}.\end{aligned}$$

We therefore deduce that, for any $t \in [0,1]$, there holds

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{U}_\varepsilon(t,f) \leqslant & -(\kappa_0 - C\alpha_2)\,\|f\|^2_{\mathscr{Y}_1} - \frac{1}{\varepsilon^2}\left(\kappa_0 - C\alpha_1 - C\frac{\alpha_1^2}{\alpha_2}\right)\|f^\perp\|^2_{\mathscr{Y}_1} \\ & - \frac{t}{\varepsilon^2}\left(\alpha_1\kappa_1 - C\frac{\alpha_2^2}{\alpha_3}\right)\|f^\perp\|^2_{\mathscr{Y}_2} - t^2\left(\frac{\alpha_2}{2} - C\alpha_3\right)\|\widetilde{\nabla}_x f\|^2_{\mathscr{X}} \\ & - \frac{\alpha_3\kappa_2 t^3}{2}\|\widetilde{\nabla}_x f\|^2_{\mathscr{Y}_1}.\end{aligned} \tag{3.24}$$

We now choose $\alpha_1 = \eta$, $\alpha_2 = \eta^{\frac{3}{2}}$, and $\alpha_3 = \eta^{\frac{5}{3}}$, with $\eta \in (0,1)$ small enough such that each quantity appearing inside the parentheses in above inequality is positive. We hence obtain that $\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{U}_\varepsilon(t,f) \leqslant 0$ for any $t \in [0,1]$, which concludes the proof as explained in Step 1. □

4. Cauchy theory and regularization estimates for the nonlinear problem

In this section, we provide a Cauchy theory for (1.7) for small initial data as well as some new regularization estimates for this equation. Notice that our proofs are based on the results developed in Subsection 3.1. It is actually crucial to be able to avoid the use of Duhamel formula to obtain nice estimates on the nonlinear problem because of the singularity in ε that is in front of the nonlinear term in (1.7). Our strategy is to perform direct energy estimates with the norm $|||\cdot|||_{\mathscr{X}}$ introduced in Subsection 3.1 (see (A.19) and (3.4) for the precise definition) and exploit the facts that $\Gamma(f,g) = (\Gamma(f,g))^\perp$ and $\langle \Gamma(f,g), h\rangle_{L^2_{x,v}} = \langle \Gamma(f,g), h^\perp\rangle_{L^2_{x,v}}$ so that

$$\langle\!\langle \Gamma(f,g), h\rangle\!\rangle_{L^2_{x,v}} = \left\langle \Gamma(f,g), h^\perp\right\rangle_{L^2_{x,v}}$$

and thus

$$\begin{aligned}\langle\!\langle \Gamma(f,g), h\rangle\!\rangle_{\mathscr{X}} = & \sum_{i=0}^{2} \delta\left\langle \langle v\rangle^{(3-i)(\frac{\gamma}{2}+1)}\,\nabla_x^i\Gamma(f,g), \langle v\rangle^{(3-i)(\frac{\gamma}{2}+1)}\,\nabla_x^i h^\perp\right\rangle_{L^2_{x,v}} \\ & + \sum_{i=0}^{3}\left\langle \nabla_x^i\Gamma(f,g), \nabla_x^i h^\perp\right\rangle_{L^2_{x,v}},\end{aligned}$$

where we recall that $\delta \in (0,1)$ is a small enough constant chosen in Proposition 3.3. Notice also that we used the particular form of $|||\cdot|||_{L^2_{x,v}}$ defined in (A.19) and the fact that $\pi\Gamma(f,g) = 0$. Rearranging terms, we then deduce

$$\begin{aligned}\langle\!\langle \Gamma(f,g), h\rangle\!\rangle_{\mathscr{X}} = & \,\delta\left\langle \Gamma(f,g), h^\perp\right\rangle_{\mathscr{X}} + \sum_{i=0}^{2}\left\langle \nabla_x^i\Gamma(f,g), \nabla_x^i h^\perp\right\rangle_{L^2_{x,v}} \\ & + (1-\delta)\left\langle \nabla_x^3\Gamma(f,g), \nabla_x^3 h^\perp\right\rangle_{L^2_{x,v}}.\end{aligned} \tag{4.1}$$

4.1. Bilinear estimates for the Landau operator.

In this part, we start by establishing some new and sharp nonlinear estimates on the Landau collision operator, we recall that the matrix $\mathbf{B}(v)$ is defined in (2.5) and that the spaces $\mathscr{X}$, $\mathscr{Y}_1$, $\mathscr{Y}_2$ and $\mathscr{Y}_1'$ are respectively defined in (1.19), (1.20), (2.11) and (2.14).

We start by establishing some convolution estimates for the coefficients a_{ij} and b_i:

Lemma 4.1. *For any suitable function $f = f(v)$, vector fields $G = G(v), H = H(v)$ and $\ell \in \{1,2,3\}$ there holds, for any $v \in \mathbf{R}^3$:*

$$|(a_{ij} * f)G_iH_j|(v) \lesssim \|\langle v\rangle^7 f\|_{L^2_v}|\mathbf{B}(v)G(v)||\mathbf{B}(v)H(v)| \tag{4.2}$$

$$|(\partial_{v_\ell}a_{ij} * f)G_iH_j|(v) \lesssim \|\langle v\rangle^8 f\|_{L^2_v}|\mathbf{B}(v)G(v)||\mathbf{B}(v)H(v)| \tag{4.3}$$

$$|(\partial_{v_\ell}b_i * f)G_i|(v) \lesssim \|\langle v\rangle^3 f\|_{L^2_v}\langle v\rangle^{\frac{\gamma}{2}}|\mathbf{B}(v)G(v)| \quad \textit{if} \quad 0 \leqslant \gamma \leqslant 1 \tag{4.4}$$

$$|(\partial_{v_\ell}b_i * f)G_i|(v) \lesssim \left(\|\langle v\rangle^4 f\|_{H^1_v}\langle v\rangle^{-1} + \|\langle v\rangle^4 f\|_{L^2_v}\right)\langle v\rangle^{\frac{\gamma}{2}}|\mathbf{B}(v)G(v)| \quad \textit{if} \quad -2 \leqslant \gamma < 0. \tag{4.5}$$

Proof. We split the proof into three steps.

Step 1. Proof of (4.2). We only prove the estimate for $|v| \geqslant 1$, the case $|v| < 1$ being trivial. Recalling that P_v denotes the projection onto v, we decompose

$$G_i = (P_v)_i(G) + (\mathrm{Id} - P_v)_i(G) = v_i \left(G \cdot \frac{v}{|v|^2} \right) + (\mathrm{Id} - P_v)_i(G)$$

and

$$H_j = (P_v)_j(H) + (\mathrm{Id} - P_v)_j(H) = v_j \left(H \cdot \frac{v}{|v|^2} \right) + (\mathrm{Id} - P_v)_j(H).$$

We thus obtain

$$\begin{aligned}(a_{ij} * f)(v) G_i(v) H_j(v) &= (a_{ij} * f)(v) v_i v_j \left(G(v) \cdot \frac{v}{|v|^2} \right) \left(H(v) \cdot \frac{v}{|v|^2} \right) \\ &\quad + (a_{ij} * f)(v) v_i \left(G(v) \cdot \frac{v}{|v|^2} \right) (\mathrm{Id} - P_v)_j(H(v)) \\ &\quad + (a_{ij} * f)(v) v_j (\mathrm{Id} - P_v)_i(G(v)) \left(H(v) \cdot \frac{v}{|v|^2} \right) \\ &\quad + (a_{ij} * f)(v) (\mathrm{Id} - P_v)_i(G(v)) (\mathrm{Id} - P_v)_j(H(v)).\end{aligned}$$

Using Lemma 2.4, we estimate each term of the previous splitting. First,

$$\begin{aligned}&\left| (a_{ij} * f)(v) v_i v_j \left(G(v) \cdot \frac{v}{|v|^2} \right) \left(H(v) \cdot \frac{v}{|v|^2} \right) \right| \\ &\qquad \lesssim \|\langle v \rangle^7 f\|_{L^2_v} \langle v \rangle^{\frac{\gamma}{2}} |G(v)| \langle v \rangle^{\frac{\gamma}{2}} |H(v)|.\end{aligned}$$

Then,

$$\begin{aligned}&\left| (a_{ij} * f)(v) v_i \left(G(v) \cdot \frac{v}{|v|^2} \right) (\mathrm{Id} - P_v)_j(H(v)) \right| \\ &\qquad \lesssim \|\langle v \rangle^7 f\|_{L^2_v} \langle v \rangle^{\frac{\gamma}{2}} |G(v)| \langle v \rangle^{\frac{\gamma}{2}+1} |(\mathrm{Id} - P_v) H(v)|\end{aligned}$$

and

$$\begin{aligned}&\left| (a_{ij} * f)(v) v_j (\mathrm{Id} - P_v)_i(G(v)) \left(H(v) \cdot \frac{v}{|v|^2} \right) \right| \\ &\qquad \lesssim \|\langle v \rangle^7 f\|_{L^2_v} \langle v \rangle^{\frac{\gamma}{2}+1} |(\mathrm{Id} - P_v) G(v)| \langle v \rangle^{\frac{\gamma}{2}} |H(v)|.\end{aligned}$$

Finally,

$$\begin{aligned}&|(a_{ij} * f)(v) (\mathrm{Id} - P_v)_i(G(v)) (\mathrm{Id} - P_v)_j(H(v))| \\ &\qquad \lesssim \|\langle v \rangle^7 f\|_{L^2_v} \langle v \rangle^{\frac{\gamma}{2}+1} |(\mathrm{Id} - P_v) G(v)| \langle v \rangle^{\frac{\gamma}{2}+1} |(\mathrm{Id} - P_v) H(v)|.\end{aligned}$$

We conclude the proof of (4.2) by gathering previous estimates and recalling that

$$|\mathbf{B}(v) G(v)| \lesssim \langle v \rangle^{\frac{\gamma}{2}} |P_v G(v)| + \langle v \rangle^{\frac{\gamma}{2}+1} |(\mathrm{Id} - P_v) G(v)| \lesssim |\mathbf{B}(v) G(v)|.$$

Step 2. Proof of (4.3). The proof of (4.3) is similar to the one of (4.2) by using the bounds on $(\partial_{v_\ell} a_{ij} * f)$, $(\partial_{v_\ell} a_{ij} * f)\, v_i$, $(\partial_{v_\ell} a_{ij} * f)\, v_j$ and $(\partial_{v_\ell} a_{ij} * f)\, v_i v_j$ given by Lemma 2.4. We thus skip it.

Step 3. Proof of (4.4) *and* (4.5). Again, we only prove the estimate for $|v| \geqslant 1$. Recall that $|\partial_{v_\ell} b_i| \lesssim |v|^\gamma$. If $0 \leqslant \gamma \leqslant 1$, then using Lemma 2.4, we have

$$|(\partial_{v_\ell} b_i * f)(v) G_i(v)| \lesssim \|\langle v \rangle^3 f\|_{L^2_v} \langle v \rangle^\gamma |G(v)|.$$

If $-2 \leqslant \gamma < 0$, we use the above decomposition of G to write

$$(\partial_{v_\ell} b_i * f)(v) G_i(v) = (\partial_{v_\ell} b_i * f)(v) v_i \left(G(v) \cdot \frac{v}{|v|^2} \right) + (\partial_{v_\ell} b_i * f)(v) (\mathrm{Id} - P_v)_i(G(v)).$$

Remarking that $(\partial_{v_\ell} b_i * f) = (b_i * \partial_{v_\ell} f)$, we also observe that

$$\begin{aligned}(b_i * \partial_{v_\ell} f)(v) v_i &= (\partial_{v_j} a_{ij} * \partial_{v_\ell} f)(v) v_i \\ &= (a_{ij} * v_i \partial_{v_j} \partial_{v_\ell} f)(v) \\ &= (b_i * (v_i \partial_{v_\ell} f))(v) - (a_{ii} * \partial_{v_\ell} f)(v),\end{aligned}$$

from which we obtain

$$\begin{aligned}(b_i * \partial_{v_\ell} f)(v) v_i &= (b_i * \partial_{v_\ell}[v_i f])(v) - (b_i * (\partial_{v_\ell} v_i) f))(v) - (a_{ii} * \partial_{v_\ell} f)(v) \\ &= (\partial_{v_\ell} b_i * [v_i f])(v) - (b_\ell * f)(v) - (\partial_{v_\ell} a_{ii} * f)(v).\end{aligned}$$

From Lemma 2.4 and using classical Sobolev embeddings, we have

$$|(\partial_{v_\ell} b_i * f)(v)| \lesssim \langle v\rangle^\gamma \|\langle v\rangle^3 f\|_{L^4_v} \lesssim \langle v\rangle^\gamma \|\langle v\rangle^3 f\|_{H^1_v}.$$

Therefore, using once more Lemma 2.4, we obtain

$$|(\partial_{v_\ell} b_i * f)(v) v_i| \lesssim \langle v\rangle^\gamma \|\langle v\rangle^4 f\|_{H^1_v} + \langle v\rangle^{\gamma+1} \|\langle v\rangle^4 f\|_{L^2_v}.$$

Hence,

$$\begin{aligned}&|(\partial_{v_\ell} b_i * f)(v) G_i(v)| \\ &\quad \lesssim \|\langle v\rangle^4 f\|_{H^1_v} \Big(\langle v\rangle^\gamma |(\mathrm{Id} - P_v) G(v)| + \langle v\rangle^{\gamma-1} |G(v)| \Big) + \|\langle v\rangle^4 f\|_{L^2_v} \langle v\rangle^\gamma |G(v)|,\end{aligned}$$

which concludes the proof. □

We shall now establish bilinear estimates for the nonlinear operator Γ in Propositions 4.2, 4.3, 4.4 and 4.5 below. Recall from (2.2) that

$$\Gamma(g_1, g_2) = \Gamma_1(g_1, g_2) + \Gamma_2(g_1, g_2) + \Gamma_3(g_1, g_2) + \Gamma_4(g_1, g_2) + \Gamma_5(g_1, g_2)$$

with

$$\Gamma_1(g_1, g_2) = \partial_{v_i} \left\{ \left(a_{ij} * [\sqrt{M} g_1] \right) \partial_{v_j} g_2 \right\}, \tag{4.6}$$

$$\Gamma_2(g_1, g_2) = -\partial_{v_i} \left\{ \left(b_i * [\sqrt{M} g_1] \right) g_2 \right\}, \tag{4.7}$$

$$\Gamma_3(g_1, g_2) = -\left(a_{ij} * [\sqrt{M} g_1] \right) v_i \partial_{v_j} g_2, \tag{4.8}$$

$$\Gamma_4(g_1, g_2) = \frac{1}{4} \left(a_{ij} * [\sqrt{M} g_1] \right) v_i v_j g_2, \tag{4.9}$$

$$\Gamma_5(g_1, g_2) = -\frac{1}{2} \left(a_{ii} * [\sqrt{M} g_1] \right) g_2. \tag{4.10}$$

Proposition 4.2. *Let g_1, g_2 and g_3 be smooth enough functions. For any $\alpha \in \mathbf{R}$, there holds for any $i = 0, \dots, 3$,*

$$\left\langle \langle v\rangle^\alpha \nabla_x^i \Gamma(g_1, g_2), \nabla_x^i g_3 \right\rangle_{L^2_{x,v}} \lesssim \|g_1\|_{H^3_x L^2_v} \|\langle v\rangle^\alpha g_2\|_{H^3_x(H^1_{v,*})} \|g_3\|_{H^3_x(H^1_{v,*})} \tag{4.11}$$

and

$$\langle \langle v\rangle^\alpha \Gamma(g_1, g_2), g_3 \rangle_{\mathscr{X}} \lesssim \|g_1\|_{\mathscr{X}} \|\langle v\rangle^\alpha g_2\|_{\mathscr{Y}_1} \|g_3\|_{\mathscr{Y}_1}. \tag{4.12}$$

As a consequence, one has by duality

$$\|\Gamma(g_1, g_2)\|_{\mathscr{Y}_1'} \lesssim \|g_1\|_{\mathscr{X}} \|g_2\|_{\mathscr{Y}_1}. \tag{4.13}$$

Proof of Proposition 4.2. We shall first prove that for any $\alpha \in \mathbf{R}$, there holds

$$\langle \langle v\rangle^\alpha \Gamma(g_1, g_2), g_3 \rangle_{L^2_v} \lesssim \|M^{\frac14} g_1\|_{L^2_v} \|\langle v\rangle^\alpha g_2\|_{H^1_{v,*}} \|g_3\|_{H^1_{v,*}}. \tag{4.14}$$

Once this estimate is established, we shall prove (4.12) in the final step of the proof by integrating it in x and using Sobolev embeddings. Notice that we only prove (4.12) because the proof of (4.11) is similar and simpler. Estimate (4.13) is then a direct consequence of (4.12). We thus write $\Gamma(g_1, g_2) = \Gamma_1(g_1, g_2) + \cdots + \Gamma_5(g_1, g_2)$ as in (4.6)–(4.10), and we estimate each term separately in the sequel.

Step 1. We write, from (4.6) and making an integration by parts,

$$
\begin{aligned}
\langle \langle v\rangle^\alpha \Gamma_1(g_1,g_2), g_3\rangle_{L^2_v} &= -\left\langle \left(a_{ij} * [\sqrt{M} g_1]\right) \partial_{v_j} g_2, \partial_{v_i}(\langle v\rangle^\alpha g_3)\right\rangle_{L^2_v} \\
&= -\left\langle \left(a_{ij} * [\sqrt{M} g_1]\right) \partial_{v_j} g_2, \langle v\rangle^\alpha \partial_{v_i} g_3)\right\rangle_{L^2_v} - \left\langle \left(a_{ij} * [\sqrt{M} g_1]\right) \partial_{v_j} g_2, (\partial_{v_i}\langle v\rangle^\alpha) g_3\right\rangle_{L^2_v} \\
&=: \mathrm{I}_1 + \mathrm{I}_2.
\end{aligned}
$$

For the term I_1, we use Lemma 4.1, which yields

$$
\begin{aligned}
\mathrm{I}_1 &\lesssim \|M^{\frac14} g_1\|_{L^2_v} \left\langle \langle v\rangle^\alpha |\widetilde\nabla_v g_2|, |\widetilde\nabla_v g_3|\right\rangle_{L^2_v} \\
&\lesssim \|M^{\frac14} g_1\|_{L^2_v} \|\langle v\rangle^\alpha \widetilde\nabla_v g_2\|_{L^2_v} \|\widetilde\nabla_v g_3\|_{L^2_v}.
\end{aligned}
$$

In a similar way, thanks to Lemma 4.1 and using that $|\widetilde\nabla_v \langle v\rangle^\alpha| \lesssim \langle v\rangle^{\frac{\gamma}{2}-1+\alpha}$, we obtain

$$
\begin{aligned}
\mathrm{I}_2 &\lesssim \|M^{\frac14} g_1\|_{L^2_v} \left\langle \langle v\rangle^{\frac{\gamma}{2}-1} \langle v\rangle^\alpha |\widetilde\nabla_v g_2|, |g_3|\right\rangle_{L^2_v} \\
&\lesssim \|M^{\frac14} g_1\|_{L^2_v} \|\langle v\rangle^\alpha \widetilde\nabla_v g_2\|_{L^2_v} \|\langle v\rangle^{\frac{\gamma}{2}-1} g_3\|_{L^2_v}.
\end{aligned}
$$

We therefore obtain

$$
\langle \langle v\rangle^\alpha \Gamma_1(g_1,g_2), g_3\rangle_{L^2_v} \lesssim \|M^{\frac14} g_1\|_{L^2_v} \|\langle v\rangle^\alpha \widetilde\nabla_v g_2\|_{L^2_v} \|g_3\|_{H^1_{v,*}}. \tag{4.15}
$$

Step 2. Starting from (4.7) and making an integration by parts, we get

$$
\begin{aligned}
\langle \langle v\rangle^\alpha \Gamma_2(g_1,g_2), g_3\rangle_{L^2_v} &= \left\langle \left(b_i * [\sqrt{M} g_1]\right) g_2, \partial_{v_i}(\langle v\rangle^\alpha g_3)\right\rangle_{L^2_v} \\
&= \left\langle \left(b_i * [\sqrt{M} g_1]\right) g_2, \langle v\rangle^\alpha \partial_{v_i} g_3\right\rangle_{L^2_v} + \left\langle \left(b_i * [\sqrt{M} g_1]\right) g_2, (\partial_{v_i}\langle v\rangle^\alpha) g_3\right\rangle_{L^2_v} \\
&=: \mathrm{II}_1 + \mathrm{II}_2.
\end{aligned}
$$

For the term II_1, we use Lemma 2.4 to obtain

$$
\begin{aligned}
\mathrm{II}_1 &\lesssim \|M^{\frac14} g_1\|_{L^2_v} \left\langle \langle v\rangle^{\gamma+1} \langle v\rangle^\alpha |g_2|, |\nabla_v g_3|\right\rangle_{L^2_v} \\
&\lesssim \|M^{\frac14} g_1\|_{L^2_v} \|\langle v\rangle^\alpha \langle v\rangle^{\frac{\gamma}{2}+1} g_2\|_{L^2_v} \|\langle v\rangle^{\frac{\gamma}{2}} \nabla_v g_3\|_{L^2_v}.
\end{aligned}
$$

In a similar fashion, Lemma 2.4 yields

$$
\begin{aligned}
\mathrm{II}_2 &\lesssim \|M^{\frac14} g_1\|_{L^2_v} \left\langle \langle v\rangle^{\gamma+1} \, \langle v\rangle^{\alpha-1} \, |g_2|, |g_3|\right\rangle_{L^2_v} \\
&\lesssim \|M^{\frac14} g_1\|_{L^2_v} \|\langle v\rangle^\alpha \langle v\rangle^{\frac{\gamma}{2}+1} g_2\|_{L^2_v} \|\langle v\rangle^{\frac{\gamma}{2}-1} g_3\|_{L^2_v}.
\end{aligned}
$$

We thus get

$$
\langle \langle v\rangle^\alpha \Gamma_2(g_1,g_2), g_3\rangle_{L^2_v} \lesssim \|M^{\frac14} g_1\|_{L^2_v} \|\langle v\rangle^\alpha \langle v\rangle^{\frac{\gamma}{2}+1} g_2\|_{L^2_v} \|g_3\|_{H^1_{v,*}}. \tag{4.16}
$$

Step 3. All the remainder terms associated to Γ_3, Γ_4 and Γ_5 can be estimated directly thanks to Lemma 2.4 or Lemma 4.1 and Cauchy-Schwarz inequality. Indeed we first have, using Lemma 4.1 and $|\mathbf{B}(v)v| \lesssim \langle v\rangle^{\frac{\gamma}{2}+1}$,

$$
\begin{aligned}
\langle \langle v\rangle^\alpha \Gamma_3(g_1,g_2), g_3\rangle_{L^2_v} &= -\left\langle \left(a_{ij} * [\sqrt{M} g_1]\right) v_i \partial_{v_j} g_2, \langle v\rangle^\alpha g_3\right\rangle_{L^2_v} \\
&\lesssim \|M^{\frac14} g_1\|_{L^2_v} \left\langle \langle v\rangle^\alpha \langle v\rangle^{\frac{\gamma}{2}+1} |\widetilde\nabla_v g_2|, |g_3|\right\rangle_{L^2_v} \\
&\lesssim \|M^{\frac14} g_1\|_{L^2_v} \|\langle v\rangle^\alpha \widetilde\nabla_v g_2\|_{L^2_v} \|\langle v\rangle^{\frac{\gamma}{2}+1} g_3\|_{L^2_v}.
\end{aligned} \tag{4.17}
$$

In a similar way, we also get

$$
\begin{aligned}
\langle \langle v\rangle^\alpha \Gamma_4(g_1,g_2), g_3\rangle_{L^2_v} &= \frac14 \left\langle \left(a_{ij} * [\sqrt{M} g_1]\right) v_i v_j g_2, \langle v\rangle^\alpha g_3\right\rangle_{L^2_v} \\
&\lesssim \|M^{\frac14} g_1\|_{L^2_v} \left\langle \langle v\rangle^{\gamma+2} \langle v\rangle^\alpha |g_2|, |g_3|\right\rangle_{L^2_v} \\
&\lesssim \|M^{\frac14} g_1\|_{L^2_v} \|\langle v\rangle^\alpha \langle v\rangle^{\frac{\gamma}{2}+1} g_2\|_{L^2_v} \|\langle v\rangle^{\frac{\gamma}{2}+1} g_3\|_{L^2_v}.
\end{aligned} \tag{4.18}
$$

Finally, now using Lemma 2.4, we obtain

$$
\begin{aligned}
\langle \langle v\rangle^\alpha \Gamma_5(g_1,g_2),g_3\rangle_{L^2_v} &= -\frac{1}{2}\left\langle \langle v\rangle^\alpha \left(a_{ii}*[\sqrt{M}g_1]\right)g_2,g_3\right\rangle_{L^2_v} \\
&\lesssim \|M^{\frac14}g_1\|_{L^2_v}\left\langle \langle v\rangle^\alpha\langle v\rangle^{\gamma+2}|g_2|,|g_3|\right\rangle_{L^2_v} \\
&\lesssim \|M^{\frac14}g_1\|_{L^2_v}\|\langle v\rangle^\alpha\langle v\rangle^{\frac{\gamma}{2}+1}g_2\|_{L^2_v}\|\langle v\rangle^{\frac{\gamma}{2}+1}g_3\|_{L^2_v}.
\end{aligned}
\tag{4.19}
$$

We thus conclude the proof of (4.14) by gathering estimates (4.15)–(4.16)–(4.17)–(4.18)–(4.19) and observing that

$$
\|\langle v\rangle^\alpha g_2\|^2_{H^1_{v,*}} \lesssim \|\langle v\rangle^\alpha\widetilde\nabla_v g_2\|^2_{L^2_v}+\|\langle v\rangle^\alpha\langle v\rangle^{\frac{\gamma}{2}+1}g_2\|^2_{L^2_v}\lesssim \|\langle v\rangle^\alpha g_2\|^2_{H^1_{v,*}}.
$$

Step 4. Let us now prove (4.12), (4.11) is proven in a similar and simpler way. Recalling the definition of $\langle\cdot,\cdot\rangle_{\mathscr X}$ in (1.19), we have

$$
\begin{aligned}
\langle\langle v\rangle^\alpha\Gamma(g_1,g_2),g_3\rangle_{\mathscr X} &= \sum_{i=0}^3\left\langle \langle v\rangle^\alpha\langle v\rangle^{(3-i)(\frac{\gamma}{2}+1)}\nabla^i_x\Gamma(g_1,g_2),\langle v\rangle^{(3-i)(\frac{\gamma}{2}+1)}\nabla^i_x g_3\right\rangle_{L^2_{x,v}}. \\
&=: T_0+T_1+T_2+T_3.
\end{aligned}
$$

Thanks to (4.14) and the fact that $\|\cdot\|_{L^\infty_x}\lesssim\|\cdot\|_{H^2_x}$, we get

$$
\begin{aligned}
T_0 &\lesssim \int_{\mathbf T^3}\|M^{\frac14}g_1\|_{L^2_v}\|\langle v\rangle^\alpha\langle v\rangle^{3(\frac{\gamma}{2}+1)}g_2\|_{H^1_{v,*}}\|\langle v\rangle^{3(\frac{\gamma}{2}+1)}g_3\|_{H^1_{v,*}}\,\mathrm dx \\
&\lesssim \|M^{\frac14}g_1\|_{H^2_xL^2_v}\|\langle v\rangle^\alpha\langle v\rangle^{3(\frac{\gamma}{2}+1)}g_2\|_{L^2_x(H^1_{v,*})}\|\langle v\rangle^{3(\frac{\gamma}{2}+1)}g_3\|_{L^2_x(H^1_{v,*})} \\
&\lesssim \|g_1\|_{\mathscr X}\|\langle v\rangle^\alpha g_2\|_{\mathscr Y_1}\|g_3\|_{\mathscr Y_1}.
\end{aligned}
$$

Using that $\partial_{x_k}\Gamma(g_1,g_2)=\Gamma(\partial_{x_k}g_1,g_2)+\Gamma(g_1,\partial_{x_k}g_2)$, Hölder inequality and the fact that from classical Sobolev embeddings, $\|\cdot\|_{L^6_x}+\|\cdot\|_{L^3_x}\lesssim\|\cdot\|_{H^1_x}$, estimate (4.14) yields

$$
\begin{aligned}
T_1 &\lesssim \int_{\mathbf T^3}\|M^{\frac14}\nabla_x g_1\|_{L^2_v}\|\langle v\rangle^{2(\frac{\gamma}{2}+1)}\langle v\rangle^\alpha g_2\|_{H^1_{v,*}}\|\langle v\rangle^{2(\frac{\gamma}{2}+1)}\nabla_x g_3\|_{H^1_{v,*}}\,\mathrm dx \\
&\quad+\int_{\mathbf T^3}\|M^{\frac14}g_1\|_{L^2_v}\|\langle v\rangle^{2(\frac{\gamma}{2}+1)}\langle v\rangle^\alpha\nabla_x g_2\|_{H^1_{v,*}}\|\langle v\rangle^{2(\frac{\gamma}{2}+1)}\nabla_x g_3\|_{H^1_{v,*}}\,\mathrm dx \\
&\lesssim \|M^{\frac14}\nabla_x g_1\|_{H^1_xL^2_v}\|\langle v\rangle^{2(\frac{\gamma}{2}+1)}\langle v\rangle^\alpha g_2\|_{H^1_x(H^1_{v,*})}\|\langle v\rangle^{2(\frac{\gamma}{2}+1)}\nabla_x g_3\|_{L^2_x(H^1_{v,*})} \\
&\quad+\|M^{\frac14}g_1\|_{H^2_xL^2_v}\|\langle v\rangle^{2(\frac{\gamma}{2}+1)}\langle v\rangle^\alpha\nabla_x g_2\|_{L^2_x(H^1_{v,*})}\|\langle v\rangle^{2(\frac{\gamma}{2}+1)}\nabla_x g_3\|_{L^2_x(H^1_{v,*})} \\
&\lesssim \|g_1\|_{\mathscr X}\|\langle v\rangle^\alpha g_2\|_{\mathscr Y_1}\|g_3\|_{\mathscr Y_1}.
\end{aligned}
$$

Moreover, for the term T_2, similarly, we have

$$
\begin{aligned}
T_2 &\lesssim \int_{\mathbf T^3}\Big(\|M^{\frac14}\nabla^2_x g_1\|_{L^2_v}\|\langle v\rangle^{\frac{\gamma}{2}+1}\langle v\rangle^\alpha g_2\|_{H^1_{v,*}}+\|M^{\frac14}\nabla_x g_1\|_{L^2_v}\|\langle v\rangle^{\frac{\gamma}{2}+1}\langle v\rangle^\alpha\nabla_x g_2\|_{H^1_{v,*}} \\
&\qquad\qquad+\|M^{\frac14}g_1\|_{L^2_v}\|\langle v\rangle^{\frac{\gamma}{2}+1}\langle v\rangle^\alpha\nabla^2_x g_2\|_{H^1_{v,*}}\Big)\|\langle v\rangle^{\frac{\gamma}{2}+1}\nabla^2_x g_3\|_{H^1_{v,*}}\,\mathrm dx \\
&\lesssim \Big(\|M^{\frac14}\nabla^2_x g_1\|_{L^2_xL^2_v}\|\langle v\rangle^{\frac{\gamma}{2}+1}\langle v\rangle^\alpha g_2\|_{H^2_x(H^1_{v,*})} \\
&\qquad\qquad+\|M^{\frac14}\nabla_x g_1\|_{H^1_xL^2_v}\|\langle v\rangle^{\frac{\gamma}{2}+1}\langle v\rangle^\alpha\nabla_x g_2\|_{H^1_x(H^1_{v,*})} \\
&\qquad\qquad+\|M^{\frac14}g_1\|_{H^2_xL^2_v}\|\langle v\rangle^{\frac{\gamma}{2}+1}\langle v\rangle^\alpha\nabla^2_x g_2\|_{L^2_x(H^1_{v,*})}\Big)\|\langle v\rangle^{\frac{\gamma}{2}+1}\nabla^2_x g_3\|_{L^2_x(H^1_{v,*})} \\
&\lesssim \|g_1\|_{\mathscr X}\|\langle v\rangle^\alpha g_2\|_{\mathscr Y_1}\|g_3\|_{\mathscr Y_1}.
\end{aligned}
$$

Finally, for the term T_3, we have

$$
\begin{aligned}
T_3 &\lesssim \int_{\mathbf{T}^3} \Big(\|M^{\frac14}\nabla_x^3 g_1\|_{L^2_v} \|\langle v\rangle^\alpha g_2\|_{H^1_{v,*}} + \|M^{\frac14}\nabla_x^2 g_1\|_{L^2_v} \|\langle v\rangle^\alpha \nabla_x g_2\|_{H^1_{v,*}} \\
&\qquad + \|M^{\frac14}\nabla_x g_1\|_{L^2_v} \|\langle v\rangle^\alpha \nabla_x^2 g_2\|_{H^1_{v,*}} + \|M^{\frac14} g_1\|_{L^2_v} \|\langle v\rangle^\alpha \nabla_x^3 g_2\|_{H^1_{v,*}} \Big) \|\nabla_x^3 g_3\|_{H^1_{v,*}} \, \mathrm{d}x \\
&\lesssim \Big(\|M^{\frac14}\nabla_x^3 g_1\|_{L^2_x L^2_v} \|\langle v\rangle^\alpha g_2\|_{H^2_x(H^1_{v,*})} + \|M^{\frac14}\nabla_x^2 g_1\|_{H^1_x L^2_v} \|\langle v\rangle^\alpha \nabla_x g_2\|_{H^1_x(H^1_{v,*})} \\
&\quad + \|M^{\frac14}\nabla_x g_1\|_{H^2_x L^2_v} \|\langle v\rangle^\alpha \nabla_x^2 g_2\|_{L^2_x(H^1_{v,*})} + \|M^{\frac14} g_1\|_{H^2_x L^2_v} \|\langle v\rangle^\alpha \nabla_x^3 g_2\|_{L^2_x(H^1_{v,*})} \Big) \|\nabla_x^2 g_3\|_{L^2_x(H^1_{v,*})} \\
&\lesssim \|g_1\|_{\mathscr{X}} \|\langle v\rangle^\alpha g_2\|_{\mathscr{Y}_1} \|g_3\|_{\mathscr{Y}_1},
\end{aligned}
$$

which concludes the proof of (4.12).

Step 5. Proof of (4.13). The result is immediate using the definition of the norm of $\mathscr{Y}_1'$ given in (2.14) and (4.12). ☐

Proposition 4.3. *Let g_1, g_2 be smooth enough functions and G_3 a smooth enough vector field, then*

$$
\Big\langle \widetilde\nabla_v \Gamma(g_1,g_2), G_3 \Big\rangle_{\mathscr{X}} \lesssim \Big(\|g_1\|_{\mathscr{X}} \|g_2\|_{\mathscr{Y}_2} + \|g_1\|_{\mathscr{Y}_1} \|g_2\|_{\mathscr{X}} \Big) \|G_3\|_{\mathscr{Y}_1}. \tag{4.20}
$$

Proof of Proposition 4.3. We shall only prove that for any $\alpha \in \mathbf{R}$, there holds

$$
\begin{aligned}
&\Big\langle \langle v\rangle^\alpha \widetilde\nabla_v \Gamma(g_1,g_2), G_3 \Big\rangle_{L^2_v} \\
&\qquad \lesssim \Big(\|M^{\frac14} g_1\|_{L^2_v} \|\langle v\rangle^\alpha g_2\|_{H^2_{v,*}} + \|M^{\frac14} g_1\|_{H^1_v} \|\langle v\rangle^\alpha g_2\|_{L^2_v} \Big) \|G_3\|_{H^1_{v,*}},
\end{aligned} \tag{4.21}
$$

from which we obtain the desired result by integrating in x and arguing as in Step 4 of the proof of Proposition 4.2. We thus write $\Gamma(g_1,g_2) = \Gamma_1(g_1,g_2) + \cdots + \Gamma_5(g_1,g_2)$ as in (4.6)–(4.10), and we estimate each term separately in the sequel. We shall use during the proof the following equivalence:

$$
\begin{aligned}
\|\langle v\rangle^\alpha g_2\|_{H^2_{v,*}} \lesssim \|\langle v\rangle^\alpha \langle v\rangle^{\gamma+2} g_2\|_{L^2_v} + \|\langle v\rangle^\alpha \langle v\rangle^{\frac{\gamma}{2}+1} \widetilde\nabla_v g_2\|_{L^2_v} \\
+ \|\langle v\rangle^\alpha \widetilde\nabla_v (\widetilde\nabla_v g_2)\|_{L^2_v} \lesssim \|\langle v\rangle^\alpha g_2\|_{H^2_{v,*}}.
\end{aligned}
$$

Step 1. Term associated to Γ_1. Writing $\widetilde\nabla_{v_k} = B_{k\ell}\partial_{v_\ell}$ and observing that $[\widetilde\nabla_{v_k}, \partial_{v_i}] = -(\partial_{v_i} B_{k\ell})\partial_{v_\ell}$, we first get that for any $k \in \{1,2,3\}$,

$$
\begin{aligned}
\widetilde\nabla_{v_k} \Gamma_1(g_1,g_2) &= \partial_{v_i} \Big\{ \Big(a_{ij} * [\sqrt{M} g_1]\Big) \partial_{v_j} (\widetilde\nabla_{v_k} g_2) \Big\} + \partial_{v_i} \Big\{ \widetilde\nabla_{v_k} \Big(a_{ij} * [\sqrt{M} g_1]\Big) \partial_{v_j} g_2 \Big\} \\
&\quad - \partial_{v_i} \Big\{ \Big(a_{ij} * [\sqrt{M} g_1]\Big) (\partial_{v_j} B_{k\ell}) \partial_{v_\ell} g_2 \Big\} - (\partial_{v_i} B_{k\ell}) \partial_{v_\ell} \Big\{ \Big(a_{ij} * [\sqrt{M} g_1]\Big) \partial_{v_j} g_2 \Big\}
\end{aligned}
$$

whence

$$
\Big\langle \langle v\rangle^\alpha \widetilde\nabla_{v_k} \Gamma_1(g_1,g_2), G_{3,k} \Big\rangle_{L^2_v} = \mathrm{I}_1 + \mathrm{I}_2 + \mathrm{I}_3 + \mathrm{I}_4,
$$

where $G_{3,k}$ denotes the k-th component of G_3.

For the term I_1, we first make an integration by parts, then we use Lemma 4.1 and the fact that $|\widetilde\nabla_v \langle v\rangle^\alpha| \lesssim \langle v\rangle^{\frac{\gamma}{2}-1} \langle v\rangle^\alpha$ together with Cauchy-Schwarz inequality to obtain

$$
\begin{aligned}
\mathrm{I}_1 &= - \Big\langle \langle v\rangle^\alpha \Big(a_{ij} * [\sqrt{M} g_1]\Big) \partial_{v_j} (\widetilde\nabla_{v_k} g_2), \partial_{v_i} G_{3,k} \Big\rangle_{L^2_v} \\
&\qquad - \Big\langle \Big(a_{ij} * [\sqrt{M} g_1]\Big) \partial_{v_j} (\widetilde\nabla_{v_k} g_2) \partial_{v_i} \langle v\rangle^\alpha, G_{3,k} \Big\rangle_{L^2_v} \\
&\lesssim \|M^{\frac14} g_1\|_{L^2_v} \Big(\Big\langle \langle v\rangle^\alpha |\widetilde\nabla_v (\widetilde\nabla_{v_k} g_2)|, |\widetilde\nabla_v G_{3,k}| \Big\rangle_{L^2_v} + \Big\langle \langle v\rangle^{\frac{\gamma}{2}-1} \langle v\rangle^\alpha |\widetilde\nabla_v (\widetilde\nabla_{v_k} g_2)|, |G_{3,k}| \Big\rangle_{L^2_v} \Big) \\
&\lesssim \|M^{\frac14} g_1\|_{L^2_v} \|\langle v\rangle^\alpha \widetilde\nabla_v (\widetilde\nabla_v g_2)\|_{L^2_v} \Big(\|\widetilde\nabla_v G_3\|_{L^2_v} + \|\langle v\rangle^{\frac{\gamma}{2}-1} G_3\|_{L^2_v} \Big).
\end{aligned}
$$

We argue in a similar fashion for the term I_2. We first make an integration by parts and write that $\widetilde{\nabla}_{v_k}\left(a_{ij}*[\sqrt{M}g_1]\right)=B_{k\ell}\left(\partial_{v_\ell}a_{ij}*[\sqrt{M}g_1]\right)$, then we use Lemma 4.1 together with $|B_{k\ell}|\lesssim\langle v\rangle^{\frac{\gamma}{2}+1}$, thus we obtain

$$\begin{aligned}
\mathrm{I}_2 &= -\left\langle\langle v\rangle^{\alpha}B_{k\ell}\left(\partial_{v_\ell}a_{ij}*[\sqrt{M}g_1]\right)\partial_{v_j}g_2,\partial_{v_i}G_{3,k}\right\rangle_{L^2_v}\\
&\qquad -\left\langle B_{k\ell}\left(\partial_{v_\ell}a_{ij}*[\sqrt{M}g_1]\right)\partial_{v_j}g_2\partial_{v_i}\langle v\rangle^{\alpha},G_{3,k}\right\rangle_{L^2_v}\\
&\lesssim \|M^{\frac14}g_1\|_{L^2_v}\Big(\left\langle\langle v\rangle^{\alpha}\langle v\rangle^{\frac{\gamma}{2}+1}|\widetilde{\nabla}_v g_2|,|\widetilde{\nabla}_v G_{3,k}|\right\rangle_{L^2_v}+\left\langle\langle v\rangle^{\frac{\gamma}{2}+1}\langle v\rangle^{\frac{\gamma}{2}-1}\langle v\rangle^{\alpha}|\widetilde{\nabla}_v g_2|,|G_{3,k}|\right\rangle_{L^2_v}\Big)\\
&\lesssim \|M^{\frac14}g_1\|_{L^2_v}\|\langle v\rangle^{\alpha}\langle v\rangle^{\frac{\gamma}{2}+1}\widetilde{\nabla}_v g_2\|_{L^2_v}\Big(\|\widetilde{\nabla}_v G_3\|_{L^2_v}+\|\langle v\rangle^{\frac{\gamma}{2}-1}G_3\|_{L^2_v}\Big).
\end{aligned}$$

For the term I_3, arguing similarly as above using also that $|\widetilde{\nabla}_v B_{k\ell}|\lesssim\langle v\rangle^{\frac{\gamma}{2}+1}\langle v\rangle^{\frac{\gamma}{2}}$, we get

$$\begin{aligned}
\mathrm{I}_3 &= \left\langle\langle v\rangle^{\alpha}\left(a_{ij}*[\sqrt{M}g_1]\right)(\partial_{v_j}B_{k\ell})\partial_{v_\ell}g_2,\partial_{v_i}G_{3,k}\right\rangle_{L^2_v}\\
&\qquad +\left\langle\left(a_{ij}*[\sqrt{M}g_1]\right)(\partial_{v_j}B_{k\ell})\partial_{v_\ell}g_2\partial_{v_i}\langle v\rangle^{\alpha},G_{3,k}\right\rangle_{L^2_v}\\
&\lesssim \|M^{\frac14}g_1\|_{L^2_v}\Big(\left\langle\langle v\rangle^{\alpha}\langle v\rangle^{\frac{\gamma}{2}+1}\langle v\rangle^{\frac{\gamma}{2}}|\nabla_v g_2|,|\widetilde{\nabla}_v G_3|\right\rangle_{L^2_v}\\
&\qquad +\left\langle\langle v\rangle^{\frac{\gamma}{2}+1}\langle v\rangle^{\frac{\gamma}{2}}\langle v\rangle^{\frac{\gamma}{2}-1}\langle v\rangle^{\alpha}|\nabla_v g_2|,|G_3|\right\rangle_{L^2_v}\Big)\\
&\lesssim \|M^{\frac14}g_1\|_{L^2_v}\|\langle v\rangle^{\alpha}\langle v\rangle^{\frac{\gamma}{2}+1}\widetilde{\nabla}_v g_2\|_{L^2_v}\Big(\|\widetilde{\nabla}_v G_3\|_{L^2_v}+\|\langle v\rangle^{\frac{\gamma}{2}-1}G_3\|_{L^2_v}\Big).
\end{aligned}$$

We treat the term I_4 in the same way, first performing an integration by parts and using also that $|\partial_{v_i}\partial_{v_\ell}B_{k\ell}|\lesssim\langle v\rangle^{\frac{\gamma}{2}-1}$ and $\langle v\rangle^{\frac{\gamma}{2}}|\nabla_v G_3|\lesssim|\widetilde{\nabla}_v G_3|$, it gives us

$$\begin{aligned}
\mathrm{I}_4 &= \left\langle\langle v\rangle^{\alpha}\left(a_{ij}*[\sqrt{M}g_1]\right)\partial_{v_j}g_2(\partial_{v_i}B_{k\ell}),\partial_{v_\ell}G_{3,k}\right\rangle_{L^2_v}\\
&\quad +\left\langle\left(a_{ij}*[\sqrt{M}g_1]\right)\partial_{v_j}g_2(\partial_{v_i}B_{k\ell})\partial_{v_\ell}\langle v\rangle^{\alpha},G_{3,k}\right\rangle_{L^2_v}\\
&\quad +\left\langle\langle v\rangle^{\alpha}\left(a_{ij}*[\sqrt{M}g_1]\right)\partial_{v_j}g_2,(\partial_{v_i}\partial_{v_\ell}B_{k\ell})G_{3,k}\right\rangle_{L^2_v}\\
&\lesssim \|M^{\frac14}g_1\|_{L^2_v}\Big(\left\langle\langle v\rangle^{\alpha}\langle v\rangle^{\frac{\gamma}{2}+1}\langle v\rangle^{\frac{\gamma}{2}}|\widetilde{\nabla}_v g_2|,|\nabla_v G_3|\right\rangle_{L^2_v}+\left\langle\langle v\rangle^{\alpha}\langle v\rangle^{\frac{\gamma}{2}+1}\langle v\rangle^{\frac{\gamma}{2}-1}|\widetilde{\nabla}_v g_2|,|G_3|\right\rangle_{L^2_v}\Big)\\
&\lesssim \|M^{\frac14}g_1\|_{L^2_v}\|\langle v\rangle^{\alpha}\langle v\rangle^{\frac{\gamma}{2}+1}\widetilde{\nabla}_v g_2\|_{L^2_v}\Big(\|\widetilde{\nabla}_v G_3\|_{L^2_v}+\|\langle v\rangle^{\frac{\gamma}{2}-1}G_3\|_{L^2_v}\Big).
\end{aligned}$$

Finally, gathering previous estimates, we get

$$\left\langle\langle v\rangle^{\alpha}\widetilde{\nabla}_v\Gamma_1(g_1,g_2),G_3\right\rangle_{L^2_v}\lesssim\|M^{\frac14}g_1\|_{L^2_v}\|\langle v\rangle^{\alpha}g_2\|_{H^2_{v,*}}\|G_3\|_{H^1_{v,*}}. \tag{4.22}$$

Step 2. Term associated to Γ_2. We write $\widetilde{\nabla}_{v_k}=B_{k\ell}\partial_{v_\ell}$ so that $[\widetilde{\nabla}_{v_k},\partial_{v_i}]=-(\partial_{v_i}B_{k\ell})\partial_{v_\ell}$, thus we get, for any $k\in\{1,2,3\}$,

$$\begin{aligned}
\widetilde{\nabla}_{v_k}\Gamma_2 &= -\widetilde{\nabla}_{v_k}\partial_{v_i}\left\{\left(b_i*[\sqrt{M}g_1]\right)g_2\right\}\\
&= -\partial_{v_i}\left\{\left(b_i*[\sqrt{M}g_1]\right)\widetilde{\nabla}_{v_k}g_2\right\}-\partial_{v_i}\left\{\widetilde{\nabla}_{v_k}\left(b_i*[\sqrt{M}g_1]\right)g_2\right\}\\
&\quad +(\partial_{v_i}B_{k\ell})\partial_{v_\ell}\left\{\left(b_i*[\sqrt{M}g_1]\right)g_2\right\},
\end{aligned}$$

whence

$$\left\langle\langle v\rangle^{\alpha}\widetilde{\nabla}_{v_k}\Gamma_2(g_1,g_2),G_{3,k}\right\rangle=\mathrm{II}_1+\mathrm{II}_2+\mathrm{II}_3.$$

For the term II_1, we make an integration by parts and then use Lemma 2.4 together with $\langle v\rangle^{\frac{\gamma}{2}}|\nabla_v G_3| \lesssim |\widetilde{\nabla}_v G_3|$, which yields

$$\begin{aligned}
\mathrm{II}_1 &= \left\langle \langle v\rangle^\alpha \left(b_i * [\sqrt{M} g_1]\right) \widetilde{\nabla}_{v_k} g_2, \partial_{v_i} G_{3,k} \right\rangle_{L^2_v} + \left\langle \left(b_i * [\sqrt{M} g_1]\right) \widetilde{\nabla}_{v_k} g_2 \partial_{v_i} \langle v\rangle^\alpha, G_{3,k} \right\rangle_{L^2_v} \\
&\lesssim \|M^{\frac14} g_1\|_{L^2_v} \left(\left\langle \langle v\rangle^\alpha \langle v\rangle^{\gamma+1} |\widetilde{\nabla}_{v_k} g_2|, |\nabla_v G_{3,k}| \right\rangle_{L^2_v} + \left\langle \langle v\rangle^{\gamma+1} |\widetilde{\nabla}_{v_k} g_2| \langle v\rangle^{\alpha-1}, |G_{3,k}| \right\rangle_{L^2_v} \right) \\
&\lesssim \|M^{\frac14} g_1\|_{L^2_v} \|\langle v\rangle^\alpha \langle v\rangle^{\frac{\gamma}{2}+1} \widetilde{\nabla}_v g_2\|_{L^2_v} \left(\|\widetilde{\nabla}_v G_3\|_{L^2_v} + \|\langle v\rangle^{\frac{\gamma}{2}-1} G_3\|_{L^2_v} \right).
\end{aligned}$$

For the term II_2, we first make an integration by parts using that $\widetilde{\nabla}_{v_k}\left(b_i * [\sqrt{M} g_1]\right) = B_{k\ell}\left(\partial_{v_\ell} b_i * [\sqrt{M} g_1]\right)$, which yields

$$\begin{aligned}
\mathrm{II}_2 &= \left\langle \langle v\rangle^\alpha B_{k\ell} \left(\partial_{v_\ell} b_i * [\sqrt{M} g_1]\right) g_2, \partial_{v_i} G_{3,k} \right\rangle_{L^2_v} \\
&\quad + \left\langle B_{k\ell} \left(\partial_{v_\ell} b_i * [\sqrt{M} g_1]\right) g_2 \partial_{v_i} \langle v\rangle^\alpha, G_{3,k} \right\rangle_{L^2_v}.
\end{aligned}$$

We now split into two cases according to the estimates of Lemma 4.1: If $0 \leqslant \gamma \leqslant 1$, using that $|\widetilde{\nabla}_v \langle v\rangle^\alpha| \lesssim \langle v\rangle^{\frac{\gamma}{2}-1+\alpha}$ we get

$$\begin{aligned}
\mathrm{II}_2 &\lesssim \|M^{\frac14} g_1\|_{L^2_v} \left(\left\langle \langle v\rangle^\alpha \langle v\rangle^{\frac{\gamma}{2}+1} \langle v\rangle^{\frac{\gamma}{2}} |g_2|, |\widetilde{\nabla}_v G_3| \right\rangle_{L^2_v} + \left\langle \langle v\rangle^{\frac{\gamma}{2}+1} \langle v\rangle^{\frac{\gamma}{2}} |g_2| \langle v\rangle^{\frac{\gamma}{2}-1+\alpha}, |G_3| \right\rangle_{L^2_v} \right) \\
&\lesssim \|M^{\frac14} g_1\|_{L^2_v} \|\langle v\rangle^\alpha \langle v\rangle^{\gamma+1} g_2\|_{L^2_v} \left(\|\widetilde{\nabla}_v G_3\|_{L^2_v} + \|\langle v\rangle^{\frac{\gamma}{2}-1} G_3\|_{L^2_v} \right),
\end{aligned}$$

and if $-2 \leqslant \gamma < 0$ we get, using also that $\langle v\rangle^\gamma \lesssim 1$,

$$\begin{aligned}
\mathrm{II}_2 &\lesssim \|M^{\frac14} g_1\|_{L^2_v} \left\langle \langle v\rangle^\alpha \langle v\rangle^{\frac{\gamma}{2}+1} \langle v\rangle^{\frac{\gamma}{2}} |g_2|, |\widetilde{\nabla}_v G_3| \right\rangle_{L^2_v} \\
&\quad + \|M^{\frac14} g_1\|_{H^1_v} \left\langle \langle v\rangle^\alpha \langle v\rangle^{\frac{\gamma}{2}+1} \langle v\rangle^{\frac{\gamma}{2}-1} |g_2|, |\widetilde{\nabla}_v G_3| \right\rangle_{L^2_v} \\
&\quad + \|M^{\frac14} g_1\|_{L^2_v} \left\langle \langle v\rangle^{\frac{\gamma}{2}+1} \langle v\rangle^{\frac{\gamma}{2}} \langle v\rangle^{\frac{\gamma}{2}-1+\alpha} |g_2|, |G_3| \right\rangle_{L^2_v} \\
&\quad + \|M^{\frac14} g_1\|_{H^1_v} \left\langle \langle v\rangle^{\frac{\gamma}{2}+1} \langle v\rangle^{\frac{\gamma}{2}-1} \langle v\rangle^{\frac{\gamma}{2}-1+\alpha} |g_2|, |G_3| \right\rangle_{L^2_v} \\
&\lesssim \|M^{\frac14} g_1\|_{L^2_v} \|\langle v\rangle^\alpha \langle v\rangle^{\gamma+1} g_2\|_{L^2_v} \left(\|\widetilde{\nabla}_v G_3\|_{L^2_v} + \|\langle v\rangle^{\frac{\gamma}{2}-1} G_3\|_{L^2_v} \right) \\
&\quad + \|M^{\frac14} g_1\|_{H^1_v} \|\langle v\rangle^\alpha g_2\|_{L^2_v} \left(\|\widetilde{\nabla}_v G_3\|_{L^2_v} + \|\langle v\rangle^{\frac{\gamma}{2}-1} G_3\|_{L^2_v} \right).
\end{aligned}$$

We deal with the term II_3 by first making an integration by parts and then using Lemma 2.4 together with $|\partial_{v_i} B_{k\ell}| \lesssim \langle v\rangle^{\frac{\gamma}{2}}$ and $|\partial_{v_i}\partial_{v_\ell} B_{k\ell}| \lesssim \langle v\rangle^{\frac{\gamma}{2}-1}$, which yields

$$\begin{aligned}
\mathrm{II}_3 &= -\left\langle \langle v\rangle^\alpha \left(b_i * [\sqrt{M} g_1]\right) g_2 (\partial_{v_i} B_{k\ell}), \partial_{v_\ell} G_{3,k} \right\rangle_{L^2_v} \\
&\quad - \left\langle \left(b_i * [\sqrt{M} g_1]\right) g_2 (\partial_{v_i} B_{k\ell}) \partial_{v_\ell} \langle v\rangle^\alpha, G_{3,k} \right\rangle_{L^2_v} \\
&\quad - \left\langle \langle v\rangle^\alpha \left(b_i * [\sqrt{M} g_1]\right) g_2 (\partial_{v_i}\partial_{v_\ell} B_{k\ell}), G_{3,k} \right\rangle_{L^2_v} \\
&\lesssim \|M^{\frac14} g_1\|_{L^2_v} \left(\left\langle \langle v\rangle^\alpha \langle v\rangle^{\gamma+1} |g_2| \langle v\rangle^{\frac{\gamma}{2}}, |\nabla_v G_3| \right\rangle_{L^2_v} + \left\langle \langle v\rangle^\alpha \langle v\rangle^{\gamma+1} |g_2| \langle v\rangle^{\frac{\gamma}{2}-1}, |G_3| \right\rangle_{L^2_v} \right) \\
&\lesssim \|M^{\frac14} g_1\|_{L^2_v} \|\langle v\rangle^\alpha \langle v\rangle^{\gamma+1} g_2\|_{L^2_v} \left(\|\widetilde{\nabla}_v G_3\|_{L^2_v} + \|\langle v\rangle^{\frac{\gamma}{2}-1} G_3\|_{L^2_v} \right).
\end{aligned}$$

Finally, gathering previous estimates, we get

$$\begin{aligned}
\left\langle \langle v\rangle^\alpha \widetilde{\nabla}_v \Gamma_2(g_1, g_2), G_3 \right\rangle_{L^2_v} &\lesssim \|M^{\frac14} g_1\|_{L^2_v} \|\langle v\rangle^\alpha g_2\|_{H^2_{v,*}} \|G_3\|_{H^1_{v,*}} \\
&\quad + \|M^{\frac14} g_1\|_{H^1_v} \|\langle v\rangle^\alpha g_2\|_{L^2_v} \|G_3\|_{H^1_{v,*}}.
\end{aligned} \tag{4.23}$$

Step 3. Term associated to Γ_3. We write $\widetilde\nabla_{v_k} = B_{k\ell}\partial_{v_\ell}$ so that $[\widetilde\nabla_{v_k}, \partial_{v_i}] = -(\partial_{v_i} B_{k\ell})\partial_{v_\ell}$, thus we get, for any $k \in \{1,2,3\}$,

$$\begin{aligned}\widetilde\nabla_{v_k}\Gamma_3(g_1,g_2) &= -\frac12 \widetilde\nabla_{v_k}\left\{\left(a_{ij} * [\sqrt{M} g_1]\right) v_i \partial_{v_j} g_2\right\}\\ &= -\frac12 \left(a_{ij} * [\sqrt{M} g_1]\right) v_i \partial_{v_j}(\widetilde\nabla_{v_k} g_2) \; - \frac12 \widetilde\nabla_{v_k}\left(a_{ij} * [\sqrt{M} g_1]\right) v_i \partial_{v_j} g_2\\ &\quad - \frac12 \left(a_{ij} * [\sqrt{M} g_1]\right) (\widetilde\nabla_{v_k} v_i)\partial_{v_j} g_2 + \frac12 \left(a_{ij} * [\sqrt{M} g_1]\right) v_i (\partial_{v_j} B_{k\ell})\partial_{v_\ell} g_2\end{aligned}$$

whence

$$\left\langle \langle v\rangle^\alpha \widetilde\nabla_{v_k}\Gamma_3(g_1,g_2), G_{3,k}\right\rangle = \mathrm{III}_1 + \mathrm{III}_2 + \mathrm{III}_3 + \mathrm{III}_4.$$

We now bound each term separately using Lemma 4.1. For the term III_1, using that $|\mathbf{B}(v)v| \lesssim \langle v\rangle^{\frac\gamma2+1}$, we first obtain,

$$\begin{aligned}\mathrm{III}_1 &= -\frac12 \left\langle \langle v\rangle^\alpha \left(a_{ij} * [\sqrt{M} g_1]\right) v_i \partial_{v_j}(\widetilde\nabla_{v_k} g_2), G_{3,k}\right\rangle_{L^2_v}\\ &\lesssim \|M^{\frac14} g_1\|_{L^2_v} \left\langle \langle v\rangle^\alpha \langle v\rangle^{\frac\gamma2+1} |\widetilde\nabla_v(\widetilde\nabla_{v_k} g_2)|, |G_{3,k}|\right\rangle_{L^2_v}\\ &\lesssim \|M^{\frac14} g_1\|_{L^2_v} \|\langle v\rangle^\alpha \widetilde\nabla_v(\widetilde\nabla_v g_2)\|_{L^2_v} \|\langle v\rangle^{\frac\gamma2+1} G_3\|_{L^2_v}.\end{aligned}$$

Using also that $|B_{k\ell}| \lesssim \langle v\rangle^{\frac\gamma2+1}$, we then get

$$\begin{aligned}\mathrm{III}_2 &= -\frac12 \left\langle \langle v\rangle^\alpha B_{k\ell} \left(\partial_{v_\ell} a_{ij} * [\sqrt{M} g_1]\right) v_i \partial_{v_j} g_2, G_{3,k}\right\rangle_{L^2_v}\\ &\lesssim \|M^{\frac14} g_1\|_{L^2_v} \left\langle \langle v\rangle^\alpha \langle v\rangle^{\frac\gamma2+1} \langle v\rangle^{\frac\gamma2+1} |\widetilde\nabla_v g_2|, |G_3|\right\rangle_{L^2_v}\\ &\lesssim \|M^{\frac14} g_1\|_{L^2_v} \|\langle v\rangle^\alpha \langle v\rangle^{\frac\gamma2+1} \widetilde\nabla_v g_2\|_{L^2_v} \|\langle v\rangle^{\frac\gamma2+1} G_3\|_{L^2_v}.\end{aligned}$$

Since $\widetilde\nabla_{v_k} v_i = B_{ki}$, we obtain

$$\begin{aligned}\mathrm{III}_3 &= -\frac12 \left\langle \langle v\rangle^\alpha \left(a_{ij} * [\sqrt{M} g_1]\right) B_{ki} \partial_{v_j} g_2, G_{3,k}\right\rangle_{L^2_v}\\ &\lesssim \|M^{\frac14} g_1\|_{L^2_v} \left\langle \langle v\rangle^\alpha \langle v\rangle^{\gamma+2} |\widetilde\nabla_v g_2|, |G_3|\right\rangle_{L^2_v}\\ &\lesssim \|M^{\frac14} g_1\|_{L^2_v} \|\langle v\rangle^\alpha \langle v\rangle^{\frac\gamma2+1} \widetilde\nabla_v g_2\|_{L^2_v} \|\langle v\rangle^{\frac\gamma2+1} G_3\|_{L^2_v}.\end{aligned}$$

Using now that $|\widetilde\nabla_v B_{k\ell}| \lesssim \langle v\rangle^{\frac\gamma2+1}\langle v\rangle^{\frac\gamma2}$ and $\langle v\rangle^{\frac\gamma2}|\nabla_v g_2| \lesssim |\widetilde\nabla_v g_2|$, we thus get

$$\begin{aligned}\mathrm{III}_4 &= \frac12 \left\langle \langle v\rangle^\alpha \left(a_{ij} * [\sqrt{M} g_1]\right) v_i \partial_{v_j} B_{k\ell} \partial_{v_\ell} g_2, G_{3,k}\right\rangle_{L^2_v}\\ &\lesssim \|M^{\frac14} g_1\|_{L^2_v} \left\langle \langle v\rangle^\alpha \langle v\rangle^{\frac\gamma2+1} \langle v\rangle^{\frac\gamma2+1} \langle v\rangle^{\frac\gamma2} |\nabla_v g_2|, |G_3|\right\rangle_{L^2_v}\\ &\lesssim \|M^{\frac14} g_1\|_{L^2_v} \|\langle v\rangle^\alpha \langle v\rangle^{\frac\gamma2+1} \widetilde\nabla_v g_2\|_{L^2_v} \|\langle v\rangle^{\frac\gamma2+1} G_3\|_{L^2_v}.\end{aligned}$$

Finally, gathering previous estimates, we get

$$\left\langle \langle v\rangle^\alpha \widetilde\nabla_v \Gamma_3(g_1,g_2), G_3\right\rangle_{L^2_v} \lesssim \|M^{\frac14} g_1\|_{L^2_v} \|\langle v\rangle^\alpha g_2\|_{H^2_{v,*}} \|G_3\|_{H^1_{v,*}}. \tag{4.24}$$

Step 4. Term associated to Γ_4. Writing $\widetilde\nabla_{v_k} = B_{k\ell}\partial_{v_\ell}$ and observing that $[\widetilde\nabla_{v_k}, \partial_{v_i}] = -(\partial_{v_i} B_{k\ell})\partial_{v_\ell}$ we first get, for any $k \in \{1,2,3\}$,

$$\begin{aligned}\widetilde\nabla_{v_k}\Gamma_4 &= \frac14 \widetilde\nabla_{v_k}\left\{\left(a_{ij} * [\sqrt{M} g_1]\right) v_i v_j g_2\right\}\\ &= \frac14 \left(a_{ij} * [\sqrt{M} g_1]\right) v_i v_j \widetilde\nabla_{v_k} g_2 + \frac14 \widetilde\nabla_{v_k}\left(a_{ij} * [\sqrt{M} g_1]\right) v_i v_j g_2\\ &\quad + \frac14 \left(a_{ij} * [\sqrt{M} g_1]\right) \widetilde\nabla_{v_k}(v_i v_j) g_2\end{aligned}$$

whence

$$\left\langle \langle v\rangle^\alpha \widetilde\nabla_{v_k}\Gamma_4(g_1,g_2), G_{3,k}\right\rangle = \mathrm{IV}_1 + \mathrm{IV}_2 + \mathrm{IV}_3.$$

We now bound each term separately using Lemma 2.4. For the term IV_1, we first obtain

$$\begin{aligned}
\mathrm{IV}_1 &= \frac{1}{4}\left\langle \langle v\rangle^\alpha \left(a_{ij} * [\sqrt{M} g_1]\right) v_i v_j \widetilde{\nabla}_{v_k} g_2, G_{3,k} \right\rangle_{L^2_v} \\
&\lesssim \|M^{\frac14} g_1\|_{L^2_v} \left\langle \langle v\rangle^\alpha \langle v\rangle^{\gamma+2} |\widetilde{\nabla}_{v_k} g_2|, |G_{3,k}| \right\rangle_{L^2_v} \\
&\lesssim \|M^{\frac14} g_1\|_{L^2_v} \|\langle v\rangle^\alpha \langle v\rangle^{\frac{\gamma}{2}+1} \widetilde{\nabla}_v g_2\|_{L^2_v} \|\langle v\rangle^{\frac{\gamma}{2}+1} G_3\|_{L^2_v}.
\end{aligned}$$

For the term IV_2, we also use that $|B_{k\ell}| \lesssim \langle v\rangle^{\frac{\gamma}{2}+1}$ to obtain

$$\begin{aligned}
\mathrm{IV}_2 &= \frac{1}{4}\left\langle \langle v\rangle^\alpha B_{k\ell} \left(\partial_{v_\ell} a_{ij} * [\sqrt{M} g_1]\right) v_i v_j g_2, G_{3,k} \right\rangle_{L^2_v} \\
&\lesssim \|M^{\frac14} g_1\|_{L^2_v} \left\langle \langle v\rangle^\alpha \langle v\rangle^{\frac{\gamma}{2}+1} \langle v\rangle^{\gamma+2} |g_2|, |G_3| \right\rangle_{L^2_v} \\
&\lesssim \|M^{\frac14} g_1\|_{L^2_v} \|\langle v\rangle^\alpha \langle v\rangle^{\gamma+2} g_2\|_{L^2_v} \|\langle v\rangle^{\frac{\gamma}{2}+1} G_3\|_{L^2_v}.
\end{aligned}$$

For the last term IV_3, we write $\widetilde{\nabla}_{v_k}(v_i v_j) = v_i B_{kj} + v_j B_{ki}$, and thus we get

$$\begin{aligned}
\mathrm{IV}_3 &= \frac{1}{4}\left\langle \langle v\rangle^\alpha \left(a_{ij} * [\sqrt{M} g_1]\right) [v_i B_{kj} + v_j B_{ki}] g_2, G_{3,k} \right\rangle_{L^2_v} \\
&\lesssim \|M^{\frac14} g_1\|_{L^2_v} \left\langle \langle v\rangle^\alpha \langle v\rangle^{\gamma+2} \langle v\rangle^{\frac{\gamma}{2}+1} |g_2|, |G_3| \right\rangle_{L^2_v} \\
&\lesssim \|M^{\frac14} g_1\|_{L^2_v} \|\langle v\rangle^\alpha \langle v\rangle^{\gamma+2} g_2\|_{L^2_v} \|\langle v\rangle^{\frac{\gamma}{2}+1} G_3\|_{L^2_v}.
\end{aligned}$$

Finally, gathering previous estimates we get

$$\left\langle \langle v\rangle^\alpha \widetilde{\nabla}_v \Gamma_4(g_1, g_2), G_3 \right\rangle_{L^2_v} \lesssim \|M^{\frac14} g_1\|_{L^2_v} \|\langle v\rangle^\alpha g_2\|_{H^2_{v,*}} \|G_3\|_{H^1_{v,*}}. \tag{4.25}$$

Step 5. Term associated to Γ_5. We write $\widetilde{\nabla}_{v_k} = B_{k\ell}\partial_{v_\ell}$ so that $[\widetilde{\nabla}_{v_k}, \partial_{v_i}] = -(\partial_{v_i} B_{k\ell})\partial_{v_\ell}$, thus we get, for any $k \in \{1,2,3\}$,

$$\begin{aligned}
\widetilde{\nabla}_{v_k} \Gamma_5(g_1, g_2) &= -\frac{1}{2}\widetilde{\nabla}_{v_k}\left\{\left(a_{ii} * [\sqrt{M} g_1]\right) g_2\right\} \\
&= -\frac{1}{2}\left(a_{ii} * [\sqrt{M} g_1]\right) \widetilde{\nabla}_{v_k} g_2 - \frac{1}{2}\widetilde{\nabla}_{v_k}\left(a_{ii} * [\sqrt{M} g_1]\right) g_2
\end{aligned}$$

whence

$$\left\langle \langle v\rangle^\alpha \widetilde{\nabla}_{v_k} \Gamma_5(g_1, g_2), G_{3,k} \right\rangle_{L^2_v} = \mathrm{V}_1 + \mathrm{V}_2.$$

We now bound each term separately using Lemma 2.4. For the term V_1, we get

$$\begin{aligned}
\mathrm{V}_1 &= -\frac{1}{2}\left\langle \langle v\rangle^\alpha \left(a_{ii} * [\sqrt{M} g_1]\right) \widetilde{\nabla}_{v_k} g_2, G_{3,k} \right\rangle_{L^2_v} \\
&\lesssim \|M^{\frac14} g_1\|_{L^2_v} \left\langle \langle v\rangle^\alpha \langle v\rangle^{\gamma+2} |\widetilde{\nabla}_{v_k} g_2|, |G_{3,k}| \right\rangle_{L^2_v} \\
&\lesssim \|M^{\frac14} g_1\|_{L^2_v} \|\langle v\rangle^\alpha \langle v\rangle^{\frac{\gamma}{2}+1} \widetilde{\nabla}_v g_2\|_{L^2_v} \|\langle v\rangle^{\frac{\gamma}{2}+1} G_3\|_{L^2_v},
\end{aligned}$$

and, for the term V_2, we use $|B_{k\ell}| \lesssim \langle v\rangle^{\frac{\gamma}{2}+1}$ to deduce

$$\begin{aligned}
\mathrm{V}_2 &= -\frac{1}{2}\left\langle \langle v\rangle^\alpha B_{k\ell} \left(\partial_{v_\ell} a_{ii} * [\sqrt{M} g_1]\right) g_2, G_{3,k} \right\rangle_{L^2_v} \\
&\lesssim \|M^{\frac14} g_1\|_{L^2_v} \left\langle \langle v\rangle^\alpha \langle v\rangle^{\frac{\gamma}{2}+1} \langle v\rangle^{\gamma+1} |g_2|, |G_3| \right\rangle_{L^2_v} \\
&\lesssim \|M^{\frac14} g_1\|_{L^2_v} \|\langle v\rangle^\alpha \langle v\rangle^{\gamma+1} g_2\|_{L^2_v} \|\langle v\rangle^{\frac{\gamma}{2}+1} G_3\|_{L^2_v}.
\end{aligned}$$

Finally, gathering previous estimates we get

$$\left\langle \langle v\rangle^\alpha \widetilde{\nabla}_v \Gamma_5(g_1, g_2), G_3 \right\rangle_{L^2_v} \lesssim \|M^{\frac14} g_1\|_{L^2_v} \|\langle v\rangle^\alpha g_2\|_{H^2_{v,*}} \|G_3\|_{H^1_{v,*}}. \tag{4.26}$$

Step 6. Proof of (4.21). We gather estimates (4.22)–(4.23)–(4.24)–(4.25)–(4.26). ∎

Proposition 4.4. *Let g_1, g_2 be smooth enough functions and G_3 a smooth enough vector field, then*

$$\begin{aligned}(4.27)\qquad &\left\langle \widetilde\nabla_x \Gamma(g_1,g_2), G_3\right\rangle_{\mathscr X}\\ &\lesssim \left(\|g_1\|_{\mathscr X}\|\widetilde\nabla_x g_2\|_{\mathscr Y_1}+\|\widetilde\nabla_x g_1\|_{\mathscr X}\|g_2\|_{\mathscr Y_1}+\|g_1\|_{\mathscr X}\|g_2\|_{\mathscr Y_2}\right)\|G_3\|_{\mathscr Y_1}.\end{aligned}$$

Proof of Proposition 4.4. We shall first prove that for any $\alpha\in\mathbf R$, there holds

$$\begin{aligned}(4.28)\qquad &\left\langle \langle v\rangle^\alpha \widetilde\nabla_x \Gamma(g_1,g_2), G_3\right\rangle_{L^2_v}\\ &\lesssim \left(\|M^{\frac14} g_1\|_{L^2_v}\|\langle v\rangle^\alpha\widetilde\nabla_x g_2\|_{H^1_{v,*}}+\|M^{\frac14}\nabla_x g_1\|_{L^2_v}\|\langle v\rangle^\alpha\langle v\rangle^{\frac\gamma2+1} g_2\|_{H^1_{v,*}}\right)\|G_3\|_{H^1_{v,*}}.\end{aligned}$$

We thus write $\Gamma(g_1,g_2)=\Gamma_1(g_1,g_2)+\cdots+\Gamma_5(g_1,g_2)$ as in (4.6)–(4.10), and we estimate each term separately in the sequel. The desired result (4.27) is then obtained by integrating in x, the proof is given in Step 7 because it differs from Step 4 of the proof of Proposition 4.2.

Step 1. Term associated to Γ_1. Writing $\widetilde\nabla_{x_k}=B_{k\ell}\partial_{x_\ell}$, observing that $[\widetilde\nabla_{x_k},\partial_{v_i}]=-(\partial_{v_i}B_{k\ell})\partial_{x_\ell}$ and using that $\widetilde\nabla_{x_k}\left(a_{ij}*[\sqrt M g_1]\right)=B_{k\ell}\left(a_{ij}*[\sqrt M\partial_{x_\ell} g_1]\right)$, we first get that for any $k\in\{1,2,3\}$,

$$\begin{aligned}\widetilde\nabla_{x_k}\Gamma_1(g_1,g_2)&=\partial_{v_i}\left\{\left(a_{ij}*[\sqrt M g_1]\right)\partial_{v_j}(\widetilde\nabla_{x_k} g_2)\right\}+\partial_{v_i}\left\{B_{k\ell}\left(a_{ij}*[\sqrt M\partial_{x_\ell} g_1]\right)\partial_{v_j}g_2\right\}\\ &\quad-\partial_{v_i}\left\{\left(a_{ij}*[\sqrt M g_1]\right)(\partial_{v_j}B_{k\ell})\partial_{x_\ell}g_2\right\}-(\partial_{v_i}B_{k\ell})\left(a_{ij}*[\sqrt M\partial_{x_\ell} g_1]\right)\partial_{v_j}g_2\\ &\quad-(\partial_{v_i}B_{k\ell})\left(a_{ij}*[\sqrt M g_1]\right)\partial_{v_j}(\partial_{x_\ell}g_2).\end{aligned}$$

whence

$$\left\langle \langle v\rangle^\alpha\widetilde\nabla_{x_k}\Gamma_1(g_1,g_2),G_{3,k}\right\rangle_{L^2_v}=\mathrm I_1+\mathrm I_2+\mathrm I_3+\mathrm I_4+\mathrm I_5.$$

We can then estimate each term separately using Lemma 4.1 and arguing similarly as in Step 1 of the proof of Proposition 4.3, which we brief explain below.

For the term $\mathrm I_1$, we make an integration by parts and use the fact that $|\widetilde\nabla_v\langle v\rangle^\alpha|\lesssim\langle v\rangle^{\frac\gamma2-1}\langle v\rangle^\alpha$ to obtain

$$\begin{aligned}\mathrm I_1&=-\left\langle\langle v\rangle^\alpha\left(a_{ij}*[\sqrt M g_1]\right)\partial_{v_j}(\widetilde\nabla_{x_k}g_2),\partial_{v_i}G_{3,k}\right\rangle_{L^2_v}\\ &\quad-\left\langle\left(a_{ij}*[\sqrt M g_1]\right)\partial_{v_j}(\widetilde\nabla_{x_k}g_2)\partial_{v_i}\langle v\rangle^\alpha,G_{3,k}\right\rangle_{L^2_v}\\ &\lesssim\|M^{\frac14}g_1\|_{L^2_v}\|\langle v\rangle^\alpha\widetilde\nabla_v(\widetilde\nabla_x g_2)\|_{L^2_v}\left(\|\widetilde\nabla_v G_3\|_{L^2_v}+\|\langle v\rangle^{\frac\gamma2+1}G_3\|_{L^2_v}\right).\end{aligned}$$

In a similar way, for the term $\mathrm I_2$, we make an integration by parts and use that $|B_{k\ell}|\lesssim\langle v\rangle^{\frac\gamma2+1}$, which yields

$$\begin{aligned}\mathrm I_2&=-\left\langle\langle v\rangle^\alpha B_{k\ell}\left(a_{ij}*[\sqrt M\partial_{x_\ell}g_1]\right)\partial_{v_j}g_2,\partial_{v_i}G_{3,k}\right\rangle_{L^2_v}\\ &\quad-\left\langle\langle v\rangle^\alpha B_{k\ell}\left(a_{ij}*[\sqrt M\partial_{x_\ell}g_1]\right)\partial_{v_j}g_2\partial_{v_i}\langle v\rangle^\alpha,G_{3,k}\right\rangle_{L^2_v}\\ &\lesssim\|M^{\frac14}\nabla_x g_1\|_{L^2_v}\|\langle v\rangle^\alpha\langle v\rangle^{\frac\gamma2+1}\widetilde\nabla_v g_2\|_{L^2_v}\left(\|\widetilde\nabla_v G_3\|_{L^2_v}+\|\langle v\rangle^{\frac\gamma2+1}G_3\|_{L^2_v}\right).\end{aligned}$$

For the term $\mathrm I_3$, arguing as above and using also that $|\widetilde\nabla_v B_{k\ell}|\lesssim\langle v\rangle^{\frac\gamma2+1}\langle v\rangle^{\frac\gamma2}$, we get

$$\begin{aligned}\mathrm I_3&=\left\langle\langle v\rangle^\alpha\left(a_{ij}*[\sqrt M g_1]\right)(\partial_{v_j}B_{k\ell})\partial_{x_\ell}g_2,\partial_{v_i}G_{3,k}\right\rangle_{L^2_v}\\ &\quad+\left\langle\left(a_{ij}*[\sqrt M g_1]\right)(\partial_{v_j}B_{k\ell})\partial_{x_\ell}g_2\partial_{v_i}\langle v\rangle^\alpha,G_{3,k}\right\rangle_{L^2_v}\\ &\lesssim\|M^{\frac14}g_1\|_{L^2_v}\|\langle v\rangle^\alpha\langle v\rangle^{\frac\gamma2+1}\widetilde\nabla_x g_2\|_{L^2_v}\left(\|\widetilde\nabla_v G_3\|_{L^2_v}+\|\langle v\rangle^{\frac\gamma2+1}G_3\|_{L^2_v}\right).\end{aligned}$$

For the term I_4, we use that $|\widetilde{\nabla}_v B_{k\ell}| \lesssim \langle v\rangle^{\gamma+1}$ to get

$$\begin{aligned}\mathrm{I}_4 &= -\left\langle \langle v\rangle^{\alpha} \left(a_{ij} * [\sqrt{M}\partial_{x_\ell} g_1]\right)(\partial_{v_i} B_{k\ell})\partial_{v_j} g_2, G_{3,k}\right\rangle_{L^2_v} \\ &\lesssim \|M^{\frac14}\nabla_x g_1\|_{L^2_v} \|\langle v\rangle^{\alpha}\langle v\rangle^{\frac{\gamma}{2}}\widetilde{\nabla}_v g_2\|_{L^2_v} \|\langle v\rangle^{\frac{\gamma}{2}+1} G_3\|_{L^2_v}.\end{aligned}$$

Performing an integration by parts in the term I_5, we first obtain

$$\begin{aligned}\mathrm{I}_5 &= -\left\langle \langle v\rangle^{\alpha} \left(a_{ij} * [\sqrt{M} g_1]\right)(\partial_{v_i} B_{k\ell})\partial_{v_j}(\partial_{x_\ell} g_2), G_{3,k}\right\rangle_{L^2_v} \\ &= \left\langle \langle v\rangle^{\alpha} \left(a_{ij} * [\sqrt{M} g_1]\right)(\partial_{v_i} B_{k\ell})\partial_{x_\ell} g_2, \partial_{v_j} G_{3,k}\right\rangle_{L^2_v} \\ &\quad + \left\langle \langle v\rangle^{\alpha} \left(a_{ij} * [\sqrt{M} g_1]\right)(\partial_{v_j}\partial_{v_i} B_{k\ell})\partial_{x_\ell} g_2, G_{3,k}\right\rangle_{L^2_v} \\ &\quad + \left\langle \left(a_{ij} * [\sqrt{M} g_1]\right)(\partial_{v_i} B_{k\ell})(\partial_{x_\ell} g_2)\partial_{v_j}\langle v\rangle^{\alpha}, G_{3,k}\right\rangle_{L^2_v} \\ &\quad + \left\langle \langle v\rangle^{\alpha} \left(b_i * [\sqrt{M} g_1]\right)(\partial_{v_i} B_{k\ell})\partial_{x_\ell} g_2, G_{3,k}\right\rangle_{L^2_v}.\end{aligned}$$

Using Lemmas 2.4 and 4.1 together with $|\partial_{v_j} B_{k\ell}| \lesssim \langle v\rangle^{\frac{\gamma}{2}}$ and $|\partial_{v_\ell}\partial_{v_j} B_{k\ell}| \lesssim \langle v\rangle^{\frac{\gamma}{2}-1}$, we hence obtain

$$\begin{aligned}\mathrm{I}_5 &\lesssim \|M^{\frac14} g_1\|_{L^2_v} \left\langle \langle v\rangle^{\alpha}\langle v\rangle^{\gamma+1} |\nabla_x g_2| |\widetilde{\nabla}_v G_3|\right\rangle_{L^2_v} \\ &\quad + \|M^{\frac14} g_1\|_{L^2_v} \left\langle \langle v\rangle^{\alpha}\langle v\rangle^{\gamma+2}\langle v\rangle^{\frac{\gamma}{2}-1} |\nabla_x g_2| |G_3|\right\rangle_{L^2_v} \\ &\quad + \|M^{\frac14} g_1\|_{L^2_v} \left\langle \langle v\rangle^{\gamma+1}\langle v\rangle^{\frac{\gamma}{2}-1}\langle v\rangle^{\alpha} |\nabla_x g_2| |G_3|\right\rangle_{L^2_v} \\ &\quad + \|M^{\frac14} g_1\|_{L^2_v} \left\langle \langle v\rangle^{\alpha}\langle v\rangle^{\gamma+1}\langle v\rangle^{\frac{\gamma}{2}} |\nabla_x g_2| |G_3|\right\rangle_{L^2_v} \\ &\lesssim \|M^{\frac14} g_1\|_{L^2_v} \|\langle v\rangle^{\alpha}\langle v\rangle^{\frac{\gamma}{2}+1}\widetilde{\nabla}_x g_2\|_{L^2_v} \Big(\|\widetilde{\nabla}_v G_3\|_{L^2_v} + \|\langle v\rangle^{\frac{\gamma}{2}+1} G_3\|_{L^2_v}\Big).\end{aligned}$$

Finally, gathering previous estimates, we get

$$\begin{aligned}&\left\langle \langle v\rangle^{\alpha}\widetilde{\nabla}_x \Gamma_1(g_1,g_2), G_3\right\rangle_{L^2_v} \\ &\quad\lesssim \Big(\|M^{\frac14} g_1\|_{L^2_v} \|\langle v\rangle^{\alpha}\widetilde{\nabla}_x g_2\|_{H^1_{v,*}} + \|M^{\frac14}\nabla_x g_1\|_{L^2_v} \|\langle v\rangle^{\alpha}\langle v\rangle^{\frac{\gamma}{2}+1}\widetilde{\nabla}_v g_2\|_{L^2_v}\Big) \|G_3\|_{H^1_{v,*}}.\end{aligned} \tag{4.29}$$

Step 2. Term associated to Γ_2. Similarly as for Γ_1, we have that for any $k \in \{1,2,3\}$,

$$\begin{aligned}\widetilde{\nabla}_{x_k}\Gamma_2 &= -\widetilde{\nabla}_{x_k}\partial_{v_i}\left\{ \left(b_i * [\sqrt{M} g_1]\right) g_2\right\} \\ &= -\partial_{v_i}\left\{ \left(b_i * [\sqrt{M} g_1]\right) \widetilde{\nabla}_{x_k} g_2\right\} - \partial_{v_i}\left\{ B_{k\ell}\left(b_i * [\sqrt{M}\partial_{x_\ell} g_1]\right) g_2\right\} \\ &\quad + (\partial_{v_i} B_{k\ell})\left(b_i * [\sqrt{M} g_1]\right)\partial_{x_\ell} g_2 + (\partial_{v_i} B_{k\ell})\left(b_i * [\sqrt{M}\partial_{x_\ell} g_1]\right) g_2.\end{aligned}$$

We can then estimate each term separately using Lemma 4.1 and arguing as in Step 2 of the proof of Proposition 4.3, which yields

$$\begin{aligned}&\left\langle \langle v\rangle^{\alpha}\widetilde{\nabla}_x \Gamma_2(g_1,g_2), G_3\right\rangle_{L^2_v} \\ &\quad\lesssim \Big(\|M^{\frac14} g_1\|_{L^2_v} \|\langle v\rangle^{\alpha}\widetilde{\nabla}_x g_2\|_{H^1_{v,*}} + \|M^{\frac14}\nabla_x g_1\|_{L^2_v} \|\langle v\rangle^{\alpha}\langle v\rangle^{\gamma+2} g_2\|_{L^2_v}\Big) \|G_3\|_{H^1_{v,*}}.\end{aligned} \tag{4.30}$$

Step 3. Term associated to Γ_3. As previously, we first get that for any $k \in \{1,2,3\}$,

$$\begin{aligned}\widetilde{\nabla}_{x_k}\Gamma_3 &= -\widetilde{\nabla}_{x_k}\left\{ \left(a_{ij} * [\sqrt{M} g_1]\right) v_i\partial_{v_j} g_2\right\} \\ &= -\left(a_{ij} * [\sqrt{M} g_1]\right) v_i\partial_{v_j}(\widetilde{\nabla}_{x_k} g_2) - B_{k\ell}\left(a_{ij} * [\sqrt{M}\partial_{x_\ell} g_1]\right) v_i\partial_{v_j} g_2 \\ &\quad + \left(a_{ij} * [\sqrt{M} g_1]\right) v_i(\partial_{v_j} B_{k\ell})\partial_{x_\ell} g_2.\end{aligned}$$

We can then estimate each term separately using Lemma 4.1 and arguing as in Step 3 of the proof of Proposition 4.3, which yields

$$
\begin{aligned}
&\left\langle \langle v\rangle^\alpha \widetilde\nabla_x \Gamma_3(g_1,g_2), G_3\right\rangle_{L^2_v} \\
&\quad\lesssim \left(\|M^{\frac14} g_1\|_{L^2_v} \|\langle v\rangle^\alpha \widetilde\nabla_x g_2\|_{H^1_{v,*}} + \|M^{\frac14}\nabla_x g_1\|_{L^2_v} \|\langle v\rangle^\alpha \langle v\rangle^{\frac{\gamma}{2}+1} \widetilde\nabla_v g_2\|_{L^2_v}\right) \|G_3\|_{H^1_{v,*}}.
\end{aligned} \tag{4.31}
$$

Step 4. Term associated to Γ_4. As previously, we first get that for any $k\in\{1,2,3\}$,

$$
\begin{aligned}
\widetilde\nabla_{x_k}\Gamma_4 &= \frac14 \widetilde\nabla_{x_k}\left\{ \left(a_{ij} * [\sqrt{M} g_1]\right) v_i v_j g_2\right\} \\
&= \frac14 \left(a_{ij} * [\sqrt{M} g_1]\right) v_i v_j \widetilde\nabla_{x_k} g_2 + \frac14 B_{k\ell} \left(a_{ij} * [\sqrt{M}\partial_{x_\ell} g_1]\right) v_i v_j g_2.
\end{aligned}
$$

We can then estimate each term separately using Lemma 2.4 and arguing as in Step 4 of the proof of Proposition 4.3, which yields

$$
\begin{aligned}
&\left\langle \langle v\rangle^\alpha \widetilde\nabla_x \Gamma_4(g_1,g_2), G_3\right\rangle_{L^2_v} \\
&\quad\lesssim \left(\|M^{\frac14} g_1\|_{L^2_v} \|\langle v\rangle^\alpha \widetilde\nabla_x g_2\|_{H^1_{v,*}} + \|M^{\frac14}\nabla_x g_1\|_{L^2_v} \|\langle v\rangle^\alpha \langle v\rangle^{\gamma+2} g_2\|_{L^2_v}\right) \|G_3\|_{H^1_{v,*}}.
\end{aligned} \tag{4.32}
$$

Step 5. Term associated to Γ_5. As previously, we first get that for any $k\in\{1,2,3\}$,

$$
\begin{aligned}
\widetilde\nabla_{x_k}\Gamma_5 &= -\frac12 \widetilde\nabla_{x_k}\left\{ \left(a_{ii} * [\sqrt{M} g_1]\right) g_2\right\} \\
&= -\frac12 \left(a_{ii} * [\sqrt{M} g_1]\right) \widetilde\nabla_{x_k} g_2 - \frac12 B_{k\ell} \left(a_{ii} * [\sqrt{M}\partial_{x_\ell} g_1]\right) g_2.
\end{aligned}
$$

We can then estimate each term separately using Lemma 2.4 and arguing as in Step 5 of the proof of Proposition 4.3, which yields

$$
\begin{aligned}
&\left\langle \langle v\rangle^\alpha \widetilde\nabla_x \Gamma_5(g_1,g_2), G_3\right\rangle_{L^2_v} \\
&\quad\lesssim \left(\|M^{\frac14} g_1\|_{L^2_v} \|\langle v\rangle^\alpha \widetilde\nabla_x g_2\|_{H^1_{v,*}} + \|M^{\frac14}\nabla_x g_1\|_{L^2_v} \|\langle v\rangle^\alpha \langle v\rangle^{\gamma+2} g_2\|_{L^2_v}\right) \|G_3\|_{H^1_{v,*}}.
\end{aligned} \tag{4.33}
$$

Step 6. Proof of (4.28). We gather estimates (4.29)–(4.30)–(4.31)–(4.32)–(4.33).

Step 7. Proof of (4.27). We first write

$$
\begin{aligned}
\left\langle \widetilde\nabla_x \Gamma(g_1,g_2), G_3\right\rangle_{\mathscr{X}} &= \sum_{i=0}^{3} \left\langle \langle v\rangle^{(3-i)(\frac{\gamma}{2}+1)} \nabla_x^i \widetilde\nabla_x \Gamma(g_1,g_2), \langle v\rangle^{(3-i)(\frac{\gamma}{2}+1)} \nabla_x^i G_3\right\rangle_{L^2_{x,v}} \\
&=: T_0 + T_1 + T_2 + T_3
\end{aligned}
$$

and recall that $\partial_{x_\ell}\Gamma(g_1,g_2) = \Gamma(\partial_{x_\ell} g_1, g_2) + \Gamma(g_1, \partial_{x_\ell} g_2)$ and that ∇_x and $\widetilde\nabla_x$ commute. In the remainder of the proof, we shall use (4.28) as well as some classical Sobolev embeddings as in Step 4 of the proof of Proposition 4.2 without no further mention. We here point out that in estimate (4.28), it is important to keep $\|\langle v\rangle^\alpha \langle v\rangle^{\frac{\gamma}{2}+1} g_2\|_{H^1_{v,*}}$ in our estimate instead of bounding this term by $\|\langle v\rangle^\alpha g_2\|_{H^2_{v,*}}$ and also that to close our estimate, we widely use that the weights in our functional spaces depend on the order of the derivatives in x.

For T_0, we have:

$$
\begin{aligned}
T_0 &\lesssim \Big(\|M^{\frac14} g_1\|_{H^2_x L^2_v} \|\langle v\rangle^{3(\frac{\gamma}{2}+1)} \widetilde\nabla_x g_2\|_{L^2_x(H^1_{v,*})} \\
&\qquad + \|M^{\frac14}\nabla_x g_1\|_{H^2_x L^2_v} \|\langle v\rangle^{3(\frac{\gamma}{2}+1)} \langle v\rangle^{\frac{\gamma}{2}+1} g_2\|_{L^2_x(H^1_{v,*})}\Big) \|\langle v\rangle^{3(\frac{\gamma}{2}+1)} G_3\|_{L^2_x(H^1_{v,*})} \\
&\lesssim \left(\|g_1\|_{\mathscr{X}} \|\widetilde\nabla_x g_2\|_{\mathscr{Y}_1} + \|g_1\|_{\mathscr{X}} \|g_2\|_{\mathscr{Y}_2}\right) \|G_3\|_{\mathscr{Y}_1}.
\end{aligned}
$$

For T_1, we have:

$$\begin{aligned}
T_1 \lesssim \Big(& \|M^{\frac14} g_1\|_{H^2_x L^2_v} \|\langle v\rangle^{2(\frac{\gamma}{2}+1)} \nabla_x(\widetilde\nabla_x g_2)\|_{L^2_x(H^1_{v,*})} \\
& + \|M^{\frac14}\nabla_x g_1\|_{H^1_x L^2_v} \|\langle v\rangle^{2(\frac{\gamma}{2}+1)} \widetilde\nabla_x g_2\|_{H^1_x(H^1_{v,*})}\Big) \|\langle v\rangle^{2(\frac{\gamma}{2}+1)}\nabla_x G_3\|_{L^2_x(H^1_{v,*})} \\
& + \Big(\|M^{\frac14}\nabla_x g_1\|_{H^2_x L^2_v} \|\langle v\rangle^{2(\frac{\gamma}{2}+1)} \nabla_x(\langle v\rangle^{\frac{\gamma}{2}+1} g_2)\|_{L^2_x(H^1_{v,*})} \\
& + \|M^{\frac14}\nabla^2_x g_1\|_{H^1_x L^2_v} \|\langle v\rangle^{2(\frac{\gamma}{2}+1)} \langle v\rangle^{\frac{\gamma}{2}+1} g_2\|_{H^1_x(H^1_{v,*})}\Big) \|\langle v\rangle^{2(\frac{\gamma}{2}+1)}\nabla_x G_3\|_{L^2_x(H^1_{v,*})} \\
\lesssim & \Big(\|g_1\|_{\mathscr X} \|\widetilde\nabla_x g_2\|_{\mathscr Y_1} + \|g_1\|_{\mathscr X}\|g_2\|_{\mathscr Y_2}\Big) \|G_3\|_{\mathscr Y_1}.
\end{aligned}$$

For T_2, we have:

$$\begin{aligned}
T_2 \lesssim \Big(& \|M^{\frac14} g_1\|_{H^2_x L^2_v} \|\langle v\rangle^{\frac{\gamma}{2}+1} \nabla^2_x(\widetilde\nabla_x g_2)\|_{L^2_x(H^1_{v,*})} \\
& + \|M^{\frac14}\nabla_x g_1\|_{H^1_x L^2_v} \|\langle v\rangle^{\frac{\gamma}{2}+1} \nabla_x(\widetilde\nabla_x g_2)\|_{H^1_x(H^1_{v,*})} \\
& + \|M^{\frac14}\nabla^2_x g_1\|_{L^2_x L^2_v} \|\langle v\rangle^{\frac{\gamma}{2}+1} \widetilde\nabla_x g_2\|_{H^2_x(H^1_{v,*})}\Big) \|\langle v\rangle^{\frac{\gamma}{2}+1}\nabla^2_x G_3\|_{L^2_x(H^1_{v,*})} \\
& + \Big(\|M^{\frac14}\nabla_x g_1\|_{H^2_x L^2_v} \|\langle v\rangle^{\frac{\gamma}{2}+1} \nabla^2_x(\langle v\rangle^{\frac{\gamma}{2}+1} g_2)\|_{L^2_x(H^1_{v,*})} \\
& + \|M^{\frac14}\nabla^2_x g_1\|_{H^1_x L^2_v} \|\langle v\rangle^{\frac{\gamma}{2}+1} \nabla_x(\langle v\rangle^{\frac{\gamma}{2}+1} g_2)\|_{H^1_x(H^1_{v,*})} \\
& + \|M^{\frac14}\nabla^3_x g_1\|_{L^2_x L^2_v} \|\langle v\rangle^{\frac{\gamma}{2}+1} \langle v\rangle^{\frac{\gamma}{2}+1} g_2\|_{H^2_x(H^1_{v,*})}\Big) \|\langle v\rangle^{\frac{\gamma}{2}+1}\nabla^2_x G_3\|_{L^2_x(H^1_{v,*})} \\
\lesssim & \Big(\|g_1\|_{\mathscr X} \|\widetilde\nabla_x g_2\|_{\mathscr Y_1} + \|g_1\|_{\mathscr X}\|g_2\|_{\mathscr Y_2}\Big) \|G_3\|_{\mathscr Y_1}.
\end{aligned}$$

For T_3, we have:

$$\begin{aligned}
T_3 \lesssim \Big(& \|M^{\frac14} g_1\|_{H^2_x L^2_v} \|\nabla^3_x(\widetilde\nabla_x g_2)\|_{L^2_x(H^1_{v,*})} + \|M^{\frac14}\nabla_x g_1\|_{H^1_x L^2_v} \|\nabla^2_x(\widetilde\nabla_x g_2)\|_{H^1_x(H^1_{v,*})} \\
& + \|M^{\frac14}\nabla^2_x g_1\|_{L^2_x L^2_v} \|\nabla_x(\widetilde\nabla_x g_2)\|_{H^2_x(H^1_{v,*})} \\
& + \|M^{\frac14}\nabla^3_x g_1\|_{L^2_x L^2_v} \|\widetilde\nabla_x g_2\|_{H^2_x(H^1_{v,*})}\Big) \|\nabla^3_x G_3\|_{L^2_x(H^1_{v,*})} \\
+ \Big(& \|M^{\frac14}\nabla_x g_1\|_{H^2_x L^2_v} \|\nabla^3_x(\langle v\rangle^{\frac{\gamma}{2}+1} g_2)\|_{L^2_x(H^1_{v,*})} + \|M^{\frac14}\nabla^2_x g_1\|_{H^1_x L^2_v} \|\nabla^2_x(\langle v\rangle^{\frac{\gamma}{2}+1} g_2)\|_{H^1_x(H^1_{v,*})} \\
& + \|M^{\frac14}\nabla^3_x g_1\|_{L^2_x L^2_v} \|\nabla_x(\langle v\rangle^{\frac{\gamma}{2}+1} g_2)\|_{H^2_x(H^1_{v,*})} \\
& + \|M^{\frac14}\nabla^4_x g_1\|_{L^2_x L^2_v} \|\langle v\rangle^{\frac{\gamma}{2}+1} g_2\|_{H^2_x(H^1_{v,*})}\Big) \|\nabla^3_x G_3\|_{L^2_x(H^1_{v,*})} \\
\lesssim & \Big(\|g_1\|_{\mathscr X} \|\widetilde\nabla_x g_2\|_{\mathscr Y_1} + \|g_1\|_{\mathscr X}\|g_2\|_{\mathscr Y_2} + \|M^{\frac14}\nabla^4_x g_1\|_{L^2_x L^2_v} \|\langle v\rangle^{\frac{\gamma}{2}+1} g_2\|_{H^2_x(H^1_{v,*})}\Big) \|G_3\|_{\mathscr Y_1}.
\end{aligned}$$

We conclude the proof by gathering previous estimates and observing that

$$\|M^{\frac14}\nabla^4_x g_1\|_{L^2_x L^2_v} \| \langle v\rangle^{\frac{\gamma}{2}+1} g_2\|_{H^2_x(H^1_{v,*})} \lesssim \|\widetilde\nabla_x g_1\|_{\mathscr X} \|g_2\|_{\mathscr Y_1}.$$

□

Proposition 4.5. *For any smooth enough functions g_1 and g_2 there holds*

$$\|\Gamma(g_1,g_2)\|_{\mathscr X} \lesssim \|g_1\|_{\mathscr X}\|g_2\|_{\mathscr Y_2} + \|g_1\|_{\mathscr Y_1}\|g_2\|_{\mathscr X}.$$

Proof. We shall only prove that for any $\alpha\in\mathbf R$, there holds

$$\|\langle v\rangle^\alpha \Gamma(g_1,g_2)\|_{L^2_v} \lesssim \|M^{\frac14} g_1\|_{L^2_v}\|\langle v\rangle^\alpha g_2\|_{H^2_{v,*}} + \|M^{\frac18} g_1\|_{H^1_{v,*}}\|\langle v\rangle^\alpha g_2\|_{L^2_v}, \tag{4.34}$$

from which we obtain the desired result by integrating in x and arguing as in Step 4 of the proof of Proposition 4.2.

Starting from the formulation (2.2) of $\Gamma(g_1,g_2)$, we perform the ∂_{v_i} derivative in the first two terms, which gives

$$\begin{aligned}\Gamma(g_1,g_2) &= \left(a_{ij} * [\sqrt{M}g_1]\right)\left\{\partial_{v_i}\partial_{v_j}g_2 - v_i\partial_{v_j}g_2 + \frac{1}{4}v_iv_jg_2\right\} - \left(\left[\frac{1}{2}a_{ii}+c\right] * [\sqrt{M}g_1]\right)g_2\\ &=: \widetilde{\Gamma}_1(g_1,g_2)+\widetilde{\Gamma}_2(g_1,g_2)+\widetilde{\Gamma}_3(g_1,g_2)+\widetilde{\Gamma}_4(g_1,g_2),\end{aligned}$$

and we estimate each term separately in the sequel.

The terms $\widetilde{\Gamma}_2(g_1,g_2)$ and $\widetilde{\Gamma}_3(g_1,g_2)$ can be easily estimated thanks to Lemma 4.1. Indeed, we have

$$\begin{aligned}\|\langle v\rangle^\alpha\widetilde{\Gamma}_2(g_1,g_2)\|_{L^2_v} &= \|\langle v\rangle^\alpha\left(a_{ij} * [\sqrt{M}g_1]\right)v_i\partial_{v_j}g_2\|_{L^2_v}\\ &\lesssim \|M^{\frac14}g_1\|_{L^2_v}\|\langle v\rangle^\alpha\langle v\rangle^{\frac{\gamma}{2}+1}\widetilde{\nabla}_v g_2\|_{L^2_v} \lesssim \|M^{\frac14}g_1\|_{L^2_v}\|\langle v\rangle^\alpha g_2\|_{H^2_{v,*}},\end{aligned}$$

and also

$$\begin{aligned}\|\langle v\rangle^\alpha\widetilde{\Gamma}_3(g_1,g_2)\|_{L^2_v} &= \frac14\|\langle v\rangle^\alpha\left(a_{ij} * [\sqrt{M}g_1]\right)v_iv_jg_2\|_{L^2_v}\\ &\lesssim \|M^{\frac14}g_1\|_{L^2_v}\|\langle v\rangle^\alpha\langle v\rangle^{\gamma+2}g_2\|_{L^2_v} \lesssim \|M^{\frac14}g_1\|_{L^2_v}\|\langle v\rangle^\alpha g_2\|_{H^2_{v,*}}.\end{aligned}$$

Moreover, we can easily estimate $\widetilde{\Gamma}_4(g_1,g_2)$ thanks to Lemma 2.4. We have

$$\begin{aligned}\|\langle v\rangle^\alpha\left(a_{ii} * [\sqrt{M}g_1]\right)g_2\|_{L^2_v} &\lesssim \|M^{\frac14}g_1\|_{L^2_v}\|\langle v\rangle^\alpha\langle v\rangle^{\gamma+2}g_2\|_{L^2_v}\\ &\lesssim \|M^{\frac14}g_1\|_{L^2_v}\|\langle v\rangle^\alpha g_2\|_{H^2_{v,*}},\end{aligned}$$

as well as, if $\gamma \geqslant 0$,

$$\begin{aligned}\|\langle v\rangle^\alpha\left(c * [\sqrt{M}g_1]\right)g_2\|_{L^2_v} &\lesssim \|M^{\frac14}g_1\|_{L^2_v}\|\langle v\rangle^\alpha\langle v\rangle^{\gamma}g_2\|_{L^2_v}\\ &\lesssim \|M^{\frac14}g_1\|_{L^2_v}\|\langle v\rangle^\alpha g_2\|_{H^2_{v,*}},\end{aligned}$$

and, if $-2 \leqslant \gamma < 0$, since $\|\cdot\|_{L^4_v} \lesssim \|\cdot\|_{H^1_v}$, we have

$$\begin{aligned}\|\langle v\rangle^\alpha\left(c * [\sqrt{M}g_1]\right)g_2\|_{L^2_v} &\lesssim \|M^{\frac14}g_1\|_{H^1_v}\|\langle v\rangle^\alpha\langle v\rangle^{\gamma}g_2\|_{L^2_v}\\ &\lesssim \|M^{\frac18}g_1\|_{H^1_{v,*}}\|\langle v\rangle^\alpha g_2\|_{L^2_v}.\end{aligned}$$

We now prove (4.34) for $\widetilde{\Gamma}_1(g_1,g_2)$. When $|v| \leqslant 1$, the result is straightforward using Lemma 2.4. Consider now $|v| \geqslant 1$. We first write

$$\partial_{v_j}g_2 = \frac{v_j}{|v|}\frac{v_\ell}{|v|}\partial_{v_\ell}g_2 + (\mathrm{Id}-P_v)_j\nabla_v g_2$$

and

$$\partial_{v_i}\partial_{v_j}g_2 = \frac{v_i}{|v|}\frac{v_k}{|v|}\partial_{v_k}\partial_{v_j}g_2 + (\mathrm{Id}-P_v)_i\nabla_v(\partial_{v_j}g_2),$$

thus we obtain

$$\begin{aligned}\partial_{v_i}\partial_{v_j}g_2 = &\frac{v_i}{|v|}\frac{v_k}{|v|}\partial_{v_k}\left(\frac{v_j}{|v|}\frac{v_\ell}{|v|}\partial_{v_\ell}g_2\right) + \frac{v_i}{|v|}\frac{v_k}{|v|}\partial_{v_k}\left((\mathrm{Id}-P_v)_j\nabla_v g_2\right)\\ &+ (\mathrm{Id}-P_v)_i\nabla_v\left(\frac{v_j}{|v|}\frac{v_\ell}{|v|}\partial_{v_\ell}g_2\right) + (\mathrm{Id}-P_v)_i\nabla_v\left((\mathrm{Id}-P_v)_j\nabla_v g_2\right).\end{aligned}$$

Thanks to a straightforward computation, we remark that

$$\begin{aligned}&(\mathrm{Id}-P_v)_i\nabla_v\left(\frac{v_j}{|v|}\frac{v_\ell}{|v|}\partial_{v_\ell}g_2\right)\\ &\quad= \frac{v_j}{|v|}\frac{v_\ell}{|v|}(\mathrm{Id}-P_v)_i\nabla_v(\partial_{v_\ell}g_2) + \partial_{v_k}\left(\frac{v_j}{|v|}\frac{v_\ell}{|v|}\right)\left(1-\frac{v_i}{|v|}\frac{v_k}{|v|}\right)\partial_{v_\ell}g_2.\end{aligned}$$

and

$$\frac{v_i}{|v|}\frac{v_k}{|v|}\partial_{v_k}\left(\frac{v_j}{|v|}\frac{v_\ell}{|v|}\partial_{v_\ell}g_2\right) = \frac{v_i}{|v|}\frac{v_j}{|v|}\left(\frac{v_k}{|v|}\frac{v_\ell}{|v|}\partial_{v_k}\partial_{v_\ell}g_2\right).$$

Therefore, we get

$$
\begin{aligned}
\langle v\rangle^{\alpha}\widetilde{\Gamma}_1(g_1,g_2)(v) &= \langle v\rangle^{\alpha}\left(a_{ij}*[\sqrt{M}g_1]\right)\frac{v_i}{|v|}\frac{v_j}{|v|}\left(\frac{v_k}{|v|}\frac{v_\ell}{|v|}\partial_{v_k}\partial_{v_\ell}g_2\right)\\
&+\langle v\rangle^{\alpha}\left(a_{ij}*[\sqrt{M}g_1]\right)\frac{v_i}{|v|}\frac{v_k}{|v|}\partial_{v_k}[(\mathrm{Id}-P_v)_j\nabla_v g_2]\\
&+\langle v\rangle^{\alpha}\left(a_{ij}*[\sqrt{M}g_1]\right)\frac{v_j}{|v|}\frac{v_\ell}{|v|}(\mathrm{Id}-P_v)_i\nabla_v(\partial_{v_\ell}g_2)\\
&+\langle v\rangle^{\alpha}\left(a_{ij}*[\sqrt{M}g_1]\right)\partial_{v_k}\left(\frac{v_j}{|v|}\frac{v_\ell}{|v|}\right)\left(1-\frac{v_i}{|v|}\frac{v_k}{|v|}\right)\partial_{v_\ell}g_2\\
&+\langle v\rangle^{\alpha}\left(a_{ij}*[\sqrt{M}g_1]\right)(\mathrm{Id}-P_v)_i\nabla_v\left((\mathrm{Id}-P_v)_j\nabla_v g_2\right)\\
&=: I_1+I_2+I_3+I_4+I_5.
\end{aligned}
$$

We can now estimate each of these terms using Lemma 2.4 and Lemma 4.1. We obtain

$$
\begin{aligned}
\|I_1\|_{L^2_v} &\lesssim \|M^{\frac14}g_1\|_{L^2_v}\|\langle v\rangle^{\alpha}\langle v\rangle^{\gamma}\nabla_v\nabla_v g_2\|_{L^2_v} \lesssim \|M^{\frac14}g_1\|_{L^2_v}\|\langle v\rangle^{\alpha}g_2\|_{H^2_{v,*}},\\
\|I_2\|_{L^2_v} &\lesssim \|M^{\frac14}g_1\|_{L^2_v}\|\langle v\rangle^{\alpha}\langle v\rangle^{\gamma+1}\nabla_v[(\mathrm{Id}-P_v)\nabla_v g_2]\|_{L^2_v} \lesssim \|M^{\frac14}g_1\|_{L^2_v}\|\langle v\rangle^{\alpha}g_2\|_{H^2_{v,*}},\\
\|I_3\|_{L^2_v} &\lesssim \|M^{\frac14}g_1\|_{L^2_v}\|\langle v\rangle^{\alpha}\langle v\rangle^{\gamma+1}(\mathrm{Id}-P_v)\nabla_v(\nabla_v g_2)\|_{L^2_v} \lesssim \|M^{\frac14}g_1\|_{L^2_v}\|\langle v\rangle^{\alpha}g_2\|_{H^2_{v,*}},\\
\|I_4\|_{L^2_v} &\lesssim \|M^{\frac14}g_1\|_{L^2_v}\|\langle v\rangle^{\alpha}\langle v\rangle^{\gamma+1}\nabla_v g_2\|_{L^2_v} \lesssim \|M^{\frac14}g_1\|_{L^2_v}\|\langle v\rangle^{\alpha}g_2\|_{H^2_{v,*}},
\end{aligned}
$$

and finally

$$
\|I_5\|_{L^2_v} \lesssim \|M^{\frac14}g_1\|_{L^2_v}\|\langle v\rangle^{\alpha}\langle v\rangle^{\gamma+2}(\mathrm{Id}-P_v)\nabla_v[(\mathrm{Id}-P_v)\nabla_v g_2]\|_{L^2_v} \lesssim \|M^{\frac14}g_1\|_{L^2_v}\|\langle v\rangle^{\alpha}g_2\|_{H^2_{v,*}}.
$$

We conclude the proof of (4.34) by gathering previous estimates. □

4.2. **Well-posedness for the Landau equation.** In this section, we shall prove the well-posedness part of Theorem 1.1.

Proof of Theorem 1.1-(i). Let g^ε be a solution to (1.7) associated to $g^\varepsilon_{\rm in}\in\mathscr{X}$ satisfying (1.21). Notice that it implies that $g^\varepsilon_{\rm in}\in(\operatorname{Ker}\Lambda_\varepsilon)^\perp$ and thus $g^\varepsilon(t)\in(\operatorname{Ker}\Lambda_\varepsilon)^\perp$ for all $t>0$ from the conservation laws (1.4). We shall use the norm $|||\cdot|||_{\mathscr{X}}$ (and the associated inner product $\langle\langle\cdot,\cdot\rangle\rangle_{\mathscr{X}}$) defined in (3.4) during the proof of Proposition 3.3, in order to establish below an a priori estimate for g^ε.

Let $\sigma\in(0,\sigma_0)$ be fixed and compute

$$
\frac12\frac{\mathrm d}{\mathrm dt}|||g^\varepsilon|||^2_{\mathscr{X}} = \langle\langle\Lambda_\varepsilon g^\varepsilon,g^\varepsilon\rangle\rangle_{\mathscr{X}}+\frac1\varepsilon\langle\langle\Gamma(g^\varepsilon,g^\varepsilon),g^\varepsilon\rangle\rangle_{\mathscr{X}}.
$$

For the linear part, estimate (3.1) in Proposition 3.3 already gives us

$$
\langle\langle\Lambda_\varepsilon g^\varepsilon,g^\varepsilon\rangle\rangle_{\mathscr{X}} \leqslant -\sigma|||g^\varepsilon|||^2_{\mathscr{X}}-\kappa\|g^\varepsilon\|^2_{\mathscr{Y}_1}-\frac{\kappa}{\varepsilon^2}\|(g^\varepsilon)^\perp\|^2_{\mathscr{Y}_1}.
$$

For the nonlinear part, from (4.1), we get

$$
\begin{aligned}
\langle\langle\Gamma(g^\varepsilon,g^\varepsilon),g^\varepsilon\rangle\rangle_{\mathscr{X}} &= \delta\left\langle\Gamma(g^\varepsilon,g^\varepsilon),(g^\varepsilon)^\perp\right\rangle_{\mathscr{X}}+\sum_{i=0}^{2}\left\langle\nabla^i_x\Gamma(g^\varepsilon,g^\varepsilon),\nabla^i_x(g^\varepsilon)^\perp\right\rangle_{L^2_{x,v}}\\
&\quad+(1-\delta)\left\langle\nabla^3_x\Gamma(g^\varepsilon,g^\varepsilon),\nabla^3_x(g^\varepsilon)^\perp\right\rangle_{L^2_{x,v}}.
\end{aligned}
$$

Thanks to (4.12) in Proposition 4.2, we have

$$
\left\langle\Gamma(g^\varepsilon,g^\varepsilon),(g^\varepsilon)^\perp\right\rangle_{\mathscr{X}} \lesssim \|g^\varepsilon\|_{\mathscr{X}}\|g^\varepsilon\|_{\mathscr{Y}_1}\|(g^\varepsilon)^\perp\|_{\mathscr{Y}_1}
$$

and also from (4.11), since the additional weights involved in the spaces $\mathscr{X}$ and $\mathscr{Y}_1$ are greater than 1,

$$
\left\langle\nabla^3_x\Gamma(g^\varepsilon,g^\varepsilon),\nabla^3_x(g^\varepsilon)^\perp\right\rangle_{L^2_{x,v}} \lesssim \|g^\varepsilon\|_{\mathscr{X}}\|g^\varepsilon\|_{\mathscr{Y}_1}\|(g^\varepsilon)^\perp\|_{\mathscr{Y}_1}.
$$

Similarly, (4.11) also implies that

$$\sum_{i=0}^{2}\left\langle \nabla_x^i\Gamma(g^\varepsilon,g^\varepsilon),\nabla_x^i(g^\varepsilon)^\perp\right\rangle_{L^2_{x,v}} \lesssim \|g^\varepsilon\|_{\mathscr{X}}\|g^\varepsilon\|_{\mathscr{Y}_1}\|(g^\varepsilon)^\perp\|_{\mathscr{Y}_1}.$$

In summary, and recalling that $\|\cdot\|_{\mathscr{X}}$ and $|||\cdot|||_{\mathscr{X}}$ are equivalent, we then have

$$\frac{1}{\varepsilon}\langle\langle\Gamma(g^\varepsilon,g^\varepsilon),g^\varepsilon\rangle\rangle_{\mathscr{X}} \leqslant \frac{C}{\varepsilon}|||g^\varepsilon|||_{\mathscr{X}}\|g^\varepsilon\|_{\mathscr{Y}_1}\|(g^\varepsilon)^\perp\|_{\mathscr{Y}_1} \tag{4.35}$$

for some constant $C>0$.

Denoting $\mathrm{e}_\sigma : t\mapsto e^{\sigma t}$ and $g^\varepsilon_\sigma = \mathrm{e}_\sigma g^\varepsilon$, we therefore obtain

$$\frac12\frac{\mathrm{d}}{\mathrm{d}t}|||g^\varepsilon_\sigma|||^2_{\mathscr{X}} \leqslant -\kappa\|g^\varepsilon_\sigma\|^2_{\mathscr{Y}_1} - \frac{\kappa}{\varepsilon^2}\|(g^\varepsilon_\sigma)^\perp\|^2_{\mathscr{Y}_1} + \frac{C}{\varepsilon}\|g^\varepsilon_\sigma\|_{\mathscr{X}}\|g^\varepsilon_\sigma\|_{\mathscr{Y}_1}\|(g^\varepsilon_\sigma)^\perp\|_{\mathscr{Y}_1}.$$

Thanks to Young's inequality we write

$$\frac{C}{\varepsilon}\|g^\varepsilon_\sigma\|_{\mathscr{X}}\|g^\varepsilon_\sigma\|_{\mathscr{Y}_1}\|(g^\varepsilon_\sigma)^\perp\|_{\mathscr{Y}_1} \leqslant C|||g^\varepsilon_\sigma|||^2_{\mathscr{X}}\|g^\varepsilon_\sigma\|^2_{\mathscr{Y}_1} + \frac{\kappa}{2\varepsilon^2}\|(g^\varepsilon_\sigma)^\perp\|^2_{\mathscr{Y}_1},$$

which then gives the following a priori estimate

$$\frac12\frac{\mathrm{d}}{\mathrm{d}t}|||g^\varepsilon_\sigma|||^2_{\mathscr{X}} \leqslant -\left(\kappa - C|||g^\varepsilon_\sigma|||^2_{\mathscr{X}}\right)\|g^\varepsilon_\sigma\|^2_{\mathscr{Y}_1} - \frac{\kappa}{2\varepsilon^2}\|(g^\varepsilon_\sigma)^\perp\|^2_{\mathscr{Y}_1}. \tag{4.36}$$

At least formally, from this differential inequality we easily obtain that if $|||g^\varepsilon_{\rm in}|||_{\mathscr{X}}$ is small enough then g^ε satisfies the uniform in time estimate (1.22). The proof of existence and uniqueness of a solution g^ε to (1.7) satisfying (1.22) for small data $\|g^\varepsilon_{\rm in}\|_{\mathscr{X}} \leqslant \eta_0$ follows a standard iterative scheme that uses estimate (4.36) (see for example [13]). ☐

4.3. **Regularity for the Landau equation.** In this part, we provide a result of regularization for the solutions to the nonlinear Landau equation which is quantified in time, namely the regularization estimate of Theorem 1.1. Notice here that if we only wanted to handle the case $\varepsilon=1$, we could have used the triple norm introduced in [31] (see [13] for the Landau equation) which is dissipative for the whole linearized operator and equivalent to the usual one. Here, to handle the ε-dependencies, we have to use our hypocoercive norm defined in Proposition 3.3 and separate carefully the behaviors of microscopic and macroscopic parts of the solution. Some additional remainder terms coming from the fact that the transport operator and the projector π onto the kernel of L (see (1.12)) do not commute have to be treated. The computations are thus much more intricate.

Proof of Theorem 1.1-(ii). Let g^ε be a global solution to (1.7) associated to the initial data $g^\varepsilon_{\rm in}\in\mathscr{X}$ satisfying (1.21), with $\|g^\varepsilon_{\rm in}\|_{\mathscr{X}}\leqslant\eta_0$, provided by Theorem 1.1-(i). As in the proof of Theorem 1.1-(i), we shall only obtain an a priori estimate implying the desired regularization estimates.

We recall that the spaces $\mathscr{Y}_1$ and $\mathscr{Z}_1^\varepsilon$ are defined in (1.20)-(2.12) and we shall prove that for any $t\in(0,1]$, one has

$$\|g^\varepsilon(t)\|_{\mathscr{Y}_1} \lesssim \frac{1}{\sqrt{t}}\|g^\varepsilon_{\rm in}\|_{\mathscr{X}} \quad\text{and}\quad \|g^\varepsilon(t)\|_{\mathscr{Z}_1^\varepsilon} \lesssim \frac{1}{t^{3/2}}\|g^\varepsilon_{\rm in}\|_{\mathscr{X}}, \tag{4.37}$$

which readily implies that, for all $t>0$, there holds

$$\|g^\varepsilon(t)\|_{\mathscr{Y}_1} \lesssim \frac{e^{-\sigma t}}{\min(1,\sqrt{t})}\|g^\varepsilon_{\rm in}\|_{\mathscr{X}} \quad\text{and}\quad \|g^\varepsilon(t)\|_{\mathscr{Z}_1^\varepsilon} \lesssim \frac{e^{-\sigma t}}{\min(1,t^{3/2})}\|g^\varepsilon_{\rm in}\|_{\mathscr{X}}$$

by using the exponential decay in $\mathscr{X}$ given by Theorem 1.1-(i) and hence concludes the proof of Theorem 1.1-(ii). We split the proof of (4.37) into several steps. We shall use without no further mention that since $\pi\in\mathscr{B}(\mathscr{X},\mathscr{Y}_i)$, we have $\|g^\varepsilon\|_{\mathscr{Y}_i}\lesssim\|g^\varepsilon\|_{\mathscr{X}}+\|(g^\varepsilon)^\perp\|_{\mathscr{Y}_i}$ for $i=1,2$.

Step 1. We consider the same functional $\mathscr{U}_\varepsilon$ as defined in Step 1 of Proposition 3.4, which we recall is given by

$$\mathscr{U}_\varepsilon(t,g^\varepsilon) = |||g^\varepsilon|||^2_{\mathscr{X}} + \alpha_1 t\left(\|\widetilde\nabla_v(g^\varepsilon)^\perp\|^2_{\mathscr{X}} + K\|\langle v\rangle^{\frac{\gamma}{2}+1}(g^\varepsilon)^\perp\|^2_{\mathscr{X}}\right) \tag{4.38}$$

$$+\,\varepsilon\alpha_2 t^2 \left\langle \widetilde\nabla_v g^\varepsilon, \widetilde\nabla_x g^\varepsilon \right\rangle_{\mathscr X} + \varepsilon^2\alpha_3 t^3 \left(\|\widetilde\nabla_x g^\varepsilon\|_{\mathscr X}^2 + K \|\langle v\rangle^{\frac{\gamma}{2}} \nabla_x g^\varepsilon\|_{\mathscr X}^2 \right),$$

with constants $K>0$ and $0<\alpha_3 \ll \alpha_2 \ll \alpha_1 \ll 1$ so that $\alpha_2 \leqslant \sqrt{\alpha_1\alpha_3}$. We recall that the constant $K>0$ is chosen large enough in the proof of Proposition 3.4. The constants α_i will be chosen small enough during the proof here. We also recall that for any $t\in(0,1]$, one has the lower bounds

$$t\|g^\varepsilon\|_{\mathscr Y_1}^2 \lesssim \mathscr U_\varepsilon(t,g^\varepsilon) \quad \text{and} \quad t^3\|g^\varepsilon\|_{\mathscr X_1^\varepsilon}^2 \lesssim \mathscr U_\varepsilon(t,g^\varepsilon). \tag{4.39}$$

Step 2. Thanks to the proof of Theorem 1.1-(i) we already have

$$\begin{aligned} \frac12 \frac{\mathrm d}{\mathrm dt} |\!|\!| g^\varepsilon |\!|\!|_{\mathscr X}^2 &= \langle\!\langle \Lambda_\varepsilon g^\varepsilon, g^\varepsilon \rangle\!\rangle_{\mathscr X} + \frac1\varepsilon \langle\!\langle \Gamma(g^\varepsilon,g^\varepsilon), g^\varepsilon \rangle\!\rangle_{\mathscr X} \\ &\leqslant -\left(\kappa - C |\!|\!| g^\varepsilon |\!|\!|_{\mathscr X}^2\right) \|g^\varepsilon\|_{\mathscr Y_1}^2 - \frac{\kappa}{2\varepsilon^2} \|(g^\varepsilon)^\perp\|_{\mathscr Y_1}^2 \end{aligned} \tag{4.40}$$

for some constants $\kappa, C>0$.

Step 3. We first observe that since $\pi\Gamma(g^\varepsilon,g^\varepsilon)=0$, $(g^\varepsilon)^\perp$ satisfies the equation

$$\partial_t (g^\varepsilon)^\perp = (\mathrm{Id}-\pi)\Lambda_\varepsilon g^\varepsilon + \frac1\varepsilon \Gamma(g^\varepsilon,g^\varepsilon).$$

We then compute

$$\begin{aligned} &\frac12\frac{\mathrm d}{\mathrm dt}\left\{ K\|\langle v\rangle^{\frac\gamma2+1}(g^\varepsilon)^\perp\|_{\mathscr X}^2 + \|\widetilde\nabla_v (g^\varepsilon)^\perp\|_{\mathscr X}^2 \right\} \\ &\quad = K\left\langle \langle v\rangle^{\frac\gamma2+1}(\mathrm{Id}-\pi)\Lambda_\varepsilon g^\varepsilon, \langle v\rangle^{\frac\gamma2+1}(g^\varepsilon)^\perp \right\rangle_{\mathscr X} + \left\langle \widetilde\nabla_v(\mathrm{Id}-\pi)\Lambda_\varepsilon g^\varepsilon, \widetilde\nabla_v (g^\varepsilon)^\perp \right\rangle_{\mathscr X} \\ &\qquad + \frac K\varepsilon \left\langle \langle v\rangle^{\frac\gamma2+1}\Gamma(g^\varepsilon,g^\varepsilon), \langle v\rangle^{\frac\gamma2+1}(g^\varepsilon)^\perp \right\rangle_{\mathscr X} + \frac1\varepsilon \left\langle \widetilde\nabla_v \Gamma(g^\varepsilon,g^\varepsilon), \widetilde\nabla_v (g^\varepsilon)^\perp \right\rangle_{\mathscr X} \\ &\quad =: I_1+I_2+I_3+I_4. \end{aligned}$$

From (3.17) in the proof of Proposition 3.4, we already know that

$$I_1+I_2 \leqslant -\frac{\kappa_1}{\varepsilon^2}\|(g^\varepsilon)^\perp\|_{\mathscr Y_2}^2 + \frac{C}{\varepsilon^2}\|(g^\varepsilon)^\perp\|_{\mathscr Y_1}^2 + \frac C\varepsilon \|(g^\varepsilon)^\perp\|_{\mathscr Y_1}\|\widetilde\nabla_x g^\varepsilon\|_{\mathscr X}$$

for some constants $\kappa_1, C>0$. Thanks to (4.12) in Proposition 4.2, we have

$$I_3 \lesssim \frac1\varepsilon \|g^\varepsilon\|_{\mathscr X}\|\langle v\rangle^{\frac\gamma2+1} g^\varepsilon\|_{\mathscr Y_1}\|\langle v\rangle^{\frac\gamma2+1}(g^\varepsilon)^\perp\|_{\mathscr Y_1} \lesssim \frac1\varepsilon\|g^\varepsilon\|_{\mathscr X}\|g^\varepsilon\|_{\mathscr Y_2}\|(g^\varepsilon)^\perp\|_{\mathscr Y_2}.$$

Moreover, thanks to (4.20) in Proposition 4.3, we have

$$\begin{aligned} I_4 &\lesssim \frac1\varepsilon\|g^\varepsilon\|_{\mathscr X}\|g^\varepsilon\|_{\mathscr Y_2}\|\widetilde\nabla_v (g^\varepsilon)^\perp\|_{\mathscr Y_1} \lesssim \frac1\varepsilon\|g^\varepsilon\|_{\mathscr X}\|g^\varepsilon\|_{\mathscr Y_2}\|(g^\varepsilon)^\perp\|_{\mathscr Y_2} \\ &\lesssim \frac1\varepsilon\|g^\varepsilon\|_{\mathscr X}^2\|(g^\varepsilon)^\perp\|_{\mathscr Y_2} + \frac1\varepsilon\|g^\varepsilon\|_{\mathscr X}\|(g^\varepsilon)^\perp\|_{\mathscr Y_2}^2. \end{aligned}$$

We therefore obtain, using that $\|h\|_{\mathscr X}\lesssim\|h\|_{\mathscr Y_1}$,

$$\begin{aligned} &\frac12\frac{\mathrm d}{\mathrm dt}\left\{ K\|\langle v\rangle^{\frac\gamma2+1}(g^\varepsilon)^\perp\|_{\mathscr X}^2 + \|\widetilde\nabla_v (g^\varepsilon)^\perp\|_{\mathscr X}^2 \right\} \\ &\leqslant -\frac{\kappa_1}{\varepsilon^2}\|(g^\varepsilon)^\perp\|_{\mathscr Y_2}^2 + \frac{C}{\varepsilon^2}\|(g^\varepsilon)^\perp\|_{\mathscr Y_1}^2 + \frac C\varepsilon \|(g^\varepsilon)^\perp\|_{\mathscr Y_1}\|\widetilde\nabla_x g^\varepsilon\|_{\mathscr X} \\ &\quad + \frac C\varepsilon\|g^\varepsilon\|_{\mathscr X}\|g^\varepsilon\|_{\mathscr Y_1}\|(g^\varepsilon)^\perp\|_{\mathscr Y_2} + \frac C\varepsilon\|g^\varepsilon\|_{\mathscr X}\|(g^\varepsilon)^\perp\|_{\mathscr Y_2}^2. \end{aligned} \tag{4.41}$$

Step 4. We compute

$$\begin{aligned} \frac{\mathrm d}{\mathrm dt}\left\langle \widetilde\nabla_v g^\varepsilon, \widetilde\nabla_x g^\varepsilon\right\rangle_{\mathscr X} &= \left\langle \widetilde\nabla_v(\Lambda_\varepsilon g^\varepsilon), \widetilde\nabla_x g^\varepsilon\right\rangle_{\mathscr X} + \left\langle \widetilde\nabla_x(\Lambda_\varepsilon g^\varepsilon), \widetilde\nabla_v g^\varepsilon\right\rangle_{\mathscr X} \\ &\quad + \frac1\varepsilon\left\langle \widetilde\nabla_v \Gamma(g^\varepsilon,g^\varepsilon), \widetilde\nabla_x g^\varepsilon\right\rangle_{\mathscr X} + \frac1\varepsilon\left\langle \widetilde\nabla_x \Gamma(g^\varepsilon,g^\varepsilon), \widetilde\nabla_v g^\varepsilon\right\rangle_{\mathscr X} \\ &=: J_1+J_2+J_3+J_4. \end{aligned}$$

Thanks to (3.20) in Proposition 3.4, we already have

$$J_1 + J_2 \leqslant \frac{C}{\varepsilon^2} \|(g^\varepsilon)^\perp\|_{\mathscr{Y}_2} \|\widetilde{\nabla}_x g^\varepsilon\|_{\mathscr{Y}_1} - \frac{1}{\varepsilon} \|\widetilde{\nabla}_x g^\varepsilon\|^2_{\mathscr{X}}$$

fo some constant $C > 0$. For the term J_3, estimate (4.20) in Proposition 4.3 yields

$$\begin{aligned} J_3 &\lesssim \frac{1}{\varepsilon} \|g^\varepsilon\|_{\mathscr{X}} \|g^\varepsilon\|_{\mathscr{Y}_2} \|\widetilde{\nabla}_x g^\varepsilon\|_{\mathscr{Y}_1} \\ &\lesssim \frac{1}{\varepsilon} \|g^\varepsilon\|^2_{\mathscr{X}} \|\widetilde{\nabla}_x g^\varepsilon\|_{\mathscr{Y}_1} + \frac{1}{\varepsilon} \|g^\varepsilon\|_{\mathscr{X}} \|(g^\varepsilon)^\perp\|_{\mathscr{Y}_2} \|\widetilde{\nabla}_x g^\varepsilon\|_{\mathscr{Y}_1}. \end{aligned}$$

Moreover, for the term J_4, estimate (4.27) in Proposition 4.4 gives us

$$\begin{aligned} J_4 &\lesssim \frac{1}{\varepsilon} \left(\|g^\varepsilon\|_{\mathscr{X}} \|\widetilde{\nabla}_x g^\varepsilon\|_{\mathscr{Y}_1} + \|\widetilde{\nabla}_x g^\varepsilon\|_{\mathscr{X}} \|g^\varepsilon\|_{\mathscr{Y}_1} + \|g^\varepsilon\|_{\mathscr{X}} \|g^\varepsilon\|_{\mathscr{Y}_2} \right) \|\widetilde{\nabla}_v g^\varepsilon\|_{\mathscr{Y}_1} \\ &\lesssim \frac{1}{\varepsilon} \left(\|g^\varepsilon\|_{\mathscr{X}} \|\widetilde{\nabla}_x g^\varepsilon\|_{\mathscr{Y}_1} + \|\widetilde{\nabla}_x g^\varepsilon\|_{\mathscr{X}} \|g^\varepsilon\|_{\mathscr{Y}_1} + \|g^\varepsilon\|_{\mathscr{X}} \|g^\varepsilon\|_{\mathscr{Y}_2} \right) \|g^\varepsilon\|_{\mathscr{Y}_2} \\ &\lesssim \frac{1}{\varepsilon} \Big(\|g^\varepsilon\|^2_{\mathscr{X}} \|\widetilde{\nabla}_x g^\varepsilon\|_{\mathscr{Y}_1} + \|g^\varepsilon\|_{\mathscr{X}} \|\widetilde{\nabla}_x g^\varepsilon\|_{\mathscr{Y}_1} \|(g^\varepsilon)^\perp\|_{\mathscr{Y}_2} + \|g^\varepsilon\|_{\mathscr{X}} \|\widetilde{\nabla}_x g^\varepsilon\|_{\mathscr{X}} \|g^\varepsilon\|_{\mathscr{Y}_1} \\ &\quad + \|g^\varepsilon\|_{\mathscr{X}} \|\widetilde{\nabla}_x g^\varepsilon\|_{\mathscr{X}} \|(g^\varepsilon)^\perp\|_{\mathscr{Y}_2} + \|(g^\varepsilon)^\perp\|_{\mathscr{Y}_1} \|\widetilde{\nabla}_x g^\varepsilon\|_{\mathscr{X}} \|(g^\varepsilon)^\perp\|_{\mathscr{Y}_2} \\ &\quad + \|g^\varepsilon\|^3_{\mathscr{X}} + \|g^\varepsilon\|^2_{\mathscr{X}} \|(g^\varepsilon)^\perp\|_{\mathscr{Y}_2} + \|g^\varepsilon\|_{\mathscr{X}} \|(g^\varepsilon)^\perp\|^2_{\mathscr{Y}_2} \Big). \end{aligned}$$

Putting together previous estimates, we get

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t} \left\langle \widetilde{\nabla}_v g^\varepsilon, \widetilde{\nabla}_x g^\varepsilon \right\rangle_{\mathscr{X}} \\ &\quad \leqslant -\frac{1}{\varepsilon} \|\widetilde{\nabla}_x g^\varepsilon\|^2_{\mathscr{X}} + \frac{C}{\varepsilon^2} \|\widetilde{\nabla}_x g^\varepsilon\|_{\mathscr{Y}_1} \|(g^\varepsilon)^\perp\|_{\mathscr{Y}_2} + \frac{C}{\varepsilon} \|(g^\varepsilon)^\perp\|_{\mathscr{Y}_1} \|\widetilde{\nabla}_x g^\varepsilon\|_{\mathscr{X}} \|(g^\varepsilon)^\perp\|_{\mathscr{Y}_2} \\ &\qquad + \frac{C}{\varepsilon} \|g^\varepsilon\|_{\mathscr{X}} \Big(\|\widetilde{\nabla}_x g^\varepsilon\|_{\mathscr{X}} \|(g^\varepsilon)^\perp\|_{\mathscr{Y}_2} + \|g^\varepsilon\|^2_{\mathscr{Y}_1} + \|g^\varepsilon\|_{\mathscr{Y}_1} \|\widetilde{\nabla}_x g^\varepsilon\|_{\mathscr{Y}_1} \\ &\qquad + \|g^\varepsilon\|_{\mathscr{Y}_1} \|(g^\varepsilon)^\perp\|_{\mathscr{Y}_2} + \|\widetilde{\nabla}_x g^\varepsilon\|_{\mathscr{Y}_1} \|(g^\varepsilon)^\perp\|_{\mathscr{Y}_2} + \|(g^\varepsilon)^\perp\|^2_{\mathscr{Y}_2} \Big). \end{aligned} \tag{4.42}$$

Step 5. We compute

$$\begin{aligned} &\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \left\{ \|\widetilde{\nabla}_x g^\varepsilon\|^2_{\mathscr{X}} + K \|\langle v \rangle^{\frac{\gamma}{2}} \nabla_x g^\varepsilon\|^2_{\mathscr{X}} \right\} \\ &\quad = \left\langle \widetilde{\nabla}_x (\Lambda_\varepsilon g^\varepsilon), \widetilde{\nabla}_x g^\varepsilon \right\rangle_{\mathscr{X}} + K \left\langle \langle v \rangle^{\frac{\gamma}{2}} \nabla_x (\Lambda_\varepsilon g^\varepsilon), \langle v \rangle^{\frac{\gamma}{2}} \nabla_x g^\varepsilon \right\rangle_{\mathscr{X}} \\ &\qquad + \frac{1}{\varepsilon} \left\langle \widetilde{\nabla}_x \Gamma(g^\varepsilon, g^\varepsilon), \widetilde{\nabla}_x g^\varepsilon \right\rangle_{\mathscr{X}} + \frac{K}{\varepsilon} \left\langle \langle v \rangle^{\frac{\gamma}{2}} \nabla_x \Gamma(g^\varepsilon, g^\varepsilon), \langle v \rangle^{\frac{\gamma}{2}} \nabla_x g^\varepsilon \right\rangle_{\mathscr{X}} \\ &\quad =: R_1 + R_2 + R_3 + R_4. \end{aligned}$$

From (3.23) in the proof of Proposition 3.4, we already know that

$$R_1 + R_2 \leqslant -\frac{\kappa_2}{\varepsilon^2} \|\widetilde{\nabla}_x g^\varepsilon\|^2_{\mathscr{Y}_1} + \frac{C}{\varepsilon^2} \|\widetilde{\nabla}_x g^\varepsilon\|^2_{\mathscr{X}}$$

for some constants $\kappa_2, C > 0$. Thanks to (4.27) in Proposition 4.4 for the term R_3 (and a slight adaptation of it for the term R_4), we obtain

$$\begin{aligned} R_3 + R_4 &\lesssim \frac{1}{\varepsilon} \left(\|g^\varepsilon\|_{\mathscr{X}} \|\widetilde{\nabla}_x g^\varepsilon\|_{\mathscr{Y}_1} + \|\widetilde{\nabla}_x g^\varepsilon\|_{\mathscr{X}} \|g^\varepsilon\|_{\mathscr{Y}_1} + \|g^\varepsilon\|_{\mathscr{X}} \|g^\varepsilon\|_{\mathscr{Y}_2} \right) \|\widetilde{\nabla}_x g^\varepsilon\|_{\mathscr{Y}_1} \\ &\lesssim \frac{1}{\varepsilon} \Big(\|g^\varepsilon\|_{\mathscr{X}} \|\widetilde{\nabla}_x g^\varepsilon\|_{\mathscr{Y}_1} + \|\widetilde{\nabla}_x g^\varepsilon\|_{\mathscr{X}} \|g^\varepsilon\|_{\mathscr{X}} + \|\widetilde{\nabla}_x g^\varepsilon\|_{\mathscr{X}} \|(g^\varepsilon)^\perp\|_{\mathscr{Y}_1} \\ &\qquad + \|g^\varepsilon\|^2_{\mathscr{X}} + \|g^\varepsilon\|_{\mathscr{X}} \|(g^\varepsilon)^\perp\|_{\mathscr{Y}_2} \Big) \|\widetilde{\nabla}_x g^\varepsilon\|_{\mathscr{Y}_1}. \end{aligned}$$

We therefore obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\{\|\widetilde{\nabla}_x g^\varepsilon\|_{\mathscr{X}}^2 + K\|\langle v\rangle^{\frac{\gamma}{2}}\nabla_x g^\varepsilon\|_{\mathscr{X}}^2\right\}\\
&\leqslant -\frac{\kappa_2}{\varepsilon^2}\|\widetilde{\nabla}_x g^\varepsilon\|_{\mathscr{Y}_1}^2 + \frac{C}{\varepsilon^2}\|\widetilde{\nabla}_x g^\varepsilon\|_{\mathscr{X}}^2 + \frac{C}{\varepsilon}\|(g^\varepsilon)^\perp\|_{\mathscr{Y}_1}\|\widetilde{\nabla}_x g^\varepsilon\|_{\mathscr{X}}\|\widetilde{\nabla}_x g^\varepsilon\|_{\mathscr{Y}_1}\\
&\quad + \frac{C}{\varepsilon}\|g^\varepsilon\|_{\mathscr{X}}\left(\|\widetilde{\nabla}_x g^\varepsilon\|_{\mathscr{X}}\|\widetilde{\nabla}_x g^\varepsilon\|_{\mathscr{Y}_1} + \|\widetilde{\nabla}_x g^\varepsilon\|_{\mathscr{Y}_1}^2 + \|\widetilde{\nabla}_x g^\varepsilon\|_{\mathscr{Y}_1}\|(g^\varepsilon)^\perp\|_{\mathscr{Y}_2}\right).
\end{aligned}
\tag{4.43}
$$

Step 6. Conclusion. Gathering estimates (4.40)–(4.41)–(4.42)–(4.43) and using that $\sup_{t\geqslant 0}\|g^\varepsilon(t)\|_{\mathscr{X}} \lesssim \eta_0$ from Theorem 1.1-(i), we then obtain

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{U}_\varepsilon(t,g^\varepsilon) &\leqslant -\left(\kappa - C\eta_0^2 - C\alpha_2 t^2\eta_0\right)\|g^\varepsilon\|_{\mathscr{Y}_1}^2 - \frac{1}{\varepsilon^2}\left(\kappa - C\alpha_1 t - C\alpha_1\right)\|(g^\varepsilon)^\perp\|_{\mathscr{Y}_1}^2\\
&\quad - \frac{t}{\varepsilon^2}\left(\kappa_1\alpha_1 - C\varepsilon^2\alpha_2 t\eta_0 - C\varepsilon\alpha_1\eta_0\right)\|(g^\varepsilon)^\perp\|_{\mathscr{Y}_2}^2 + \frac{C\alpha_1 t}{\varepsilon}\|(g^\varepsilon)^\perp\|_{\mathscr{Y}_1}\|\widetilde{\nabla}_x g^\varepsilon\|_{\mathscr{X}}\\
&\quad + \frac{C\alpha_1 t}{\varepsilon}\eta_0\|g^\varepsilon\|_{\mathscr{Y}_1}\|(g^\varepsilon)^\perp\|_{\mathscr{Y}_2} + 2\varepsilon\alpha_2 t\left\langle\widetilde{\nabla}_v g^\varepsilon, \widetilde{\nabla}_x g^\varepsilon\right\rangle_{\mathscr{X}}\\
&\quad - t^2\left(\alpha_2 - C\varepsilon^2\alpha_3 - C\alpha_3 t\right)\|\widetilde{\nabla}_x g^\varepsilon\|_{\mathscr{X}}^2 + \frac{C\alpha_2 t^2}{\varepsilon}\|\widetilde{\nabla}_x g^\varepsilon\|_{\mathscr{Y}_1}\|(g^\varepsilon)^\perp\|_{\mathscr{Y}_2}\\
&\quad + C\alpha_2 t^2\|(g^\varepsilon)^\perp\|_{\mathscr{Y}_1}\|\widetilde{\nabla}_x g^\varepsilon\|_{\mathscr{X}}\|(g^\varepsilon)^\perp\|_{\mathscr{Y}_2}\\
&\quad + C\alpha_2 t^2\eta_0\Big(\|\widetilde{\nabla}_x g^\varepsilon\|_{\mathscr{X}}\|(g^\varepsilon)^\perp\|_{\mathscr{Y}_2} + \|g^\varepsilon\|_{\mathscr{Y}_1}\|\widetilde{\nabla}_x g^\varepsilon\|_{\mathscr{Y}_1}\\
&\qquad\qquad + \|g^\varepsilon\|_{\mathscr{Y}_1}\|(g^\varepsilon)^\perp\|_{\mathscr{Y}_2} + \|\widetilde{\nabla}_x g^\varepsilon\|_{\mathscr{Y}_1}\|(g^\varepsilon)^\perp\|_{\mathscr{Y}_2}\Big)\\
&\quad - t^3\left(\kappa_2\alpha_3 - C\varepsilon\alpha_3\eta_0\right)\|\widetilde{\nabla}_x g^\varepsilon\|_{\mathscr{Y}_1}^2 + C\varepsilon\alpha_3 t^3\|(g^\varepsilon)^\perp\|_{\mathscr{Y}_1}\|\widetilde{\nabla}_x g^\varepsilon\|_{\mathscr{X}}\|\widetilde{\nabla}_x g^\varepsilon\|_{\mathscr{Y}_1}\\
&\quad + C\varepsilon\alpha_3 t^3\eta_0\left(\|\widetilde{\nabla}_x g^\varepsilon\|_{\mathscr{X}}\|\widetilde{\nabla}_x g^\varepsilon\|_{\mathscr{Y}_1} + \|g^\varepsilon\|_{\mathscr{Y}_1}\|\widetilde{\nabla}_x g^\varepsilon\|_{\mathscr{Y}_1} + \|\widetilde{\nabla}_x g^\varepsilon\|_{\mathscr{Y}_1}\|(g^\varepsilon)^\perp\|_{\mathscr{Y}_2}\right).
\end{aligned}
$$

We now use Young's inequality to write

$$
\begin{aligned}
\frac{C\alpha_1 t}{\varepsilon}\|\widetilde{\nabla}_x g^\varepsilon\|_{\mathscr{X}}\|(g^\varepsilon)^\perp\|_{\mathscr{Y}_1} &\leqslant \frac{\alpha_2}{4}t^2\|\widetilde{\nabla}_x g^\varepsilon\|_{\mathscr{X}}^2 + C\frac{\alpha_1^2}{\alpha_2}\frac{1}{\varepsilon^2}\|(g^\varepsilon)^\perp\|_{\mathscr{Y}_1}^2\\
\frac{C\alpha_1 t}{\varepsilon}\eta_0\|g^\varepsilon\|_{\mathscr{Y}_1}\|(g^\varepsilon)^\perp\|_{\mathscr{Y}_2} &\leqslant \frac{\kappa_1\alpha_1 t}{4\varepsilon^2}\|(g^\varepsilon)^\perp\|_{\mathscr{Y}_2}^2 + C\alpha_1 t\eta_0^2\|g^\varepsilon\|_{\mathscr{Y}_1}^2\\
\frac{C\alpha_2 t^2}{\varepsilon}\|\widetilde{\nabla}_x g^\varepsilon\|_{\mathscr{Y}_1}\|(g^\varepsilon)^\perp\|_{\mathscr{Y}_2} &\leqslant \frac{\kappa_2\alpha_3}{12}t^3\|\widetilde{\nabla}_x g^\varepsilon\|_{\mathscr{Y}_1}^2 + C\frac{\alpha_2^2}{\alpha_3}\frac{t}{\varepsilon^2}\|(g^\varepsilon)^\perp\|_{\mathscr{Y}_2}^2\\
C\eta_0\alpha_2 t^2\|g^\varepsilon\|_{\mathscr{Y}_1}\|\widetilde{\nabla}_x g^\varepsilon\|_{\mathscr{Y}_1} &\leqslant \frac{\kappa_2\alpha_3}{12}t^3\|\widetilde{\nabla}_x g^\varepsilon\|_{\mathscr{Y}_1}^2 + C\frac{\alpha_2^2}{\alpha_3}\eta_0^2 t\|g^\varepsilon\|_{\mathscr{Y}_1}^2\\
C\eta_0\alpha_2 t^2\|g^\varepsilon\|_{\mathscr{Y}_1}\|(g^\varepsilon)^\perp\|_{\mathscr{Y}_2} &\leqslant \frac{\kappa_1\alpha_1 t}{4\varepsilon^2}\|(g^\varepsilon)^\perp\|_{\mathscr{Y}_2}^2 + C\varepsilon\eta_0^2 t^{3/2}\frac{\alpha_2^2}{\alpha_1}\|g^\varepsilon\|_{\mathscr{Y}_1}^2\\
C\eta_0\alpha_2 t^2\|\widetilde{\nabla}_x g^\varepsilon\|_{\mathscr{Y}_1}\|(g^\varepsilon)^\perp\|_{\mathscr{Y}_2} &\leqslant \frac{\kappa_2\alpha_3 t^3}{12}\|\widetilde{\nabla}_x g^\varepsilon\|_{\mathscr{Y}_1}^2 + C\eta_0^2 t\frac{\alpha_2^2}{\alpha_3}\|(g^\varepsilon)^\perp\|_{\mathscr{Y}_2}^2\\
C\varepsilon\eta_0\alpha_3 t^3\|g^\varepsilon\|_{\mathscr{Y}_1}\|\widetilde{\nabla}_x g^\varepsilon\|_{\mathscr{Y}_1} &\leqslant \frac{\kappa_2\alpha_3}{12}t^3\|\widetilde{\nabla}_x g^\varepsilon\|_{\mathscr{Y}_1}^2 + C\varepsilon^2\eta_0^2\alpha_3 t^3\|g^\varepsilon\|_{\mathscr{Y}_1}^2\\
C\varepsilon\eta_0\alpha_3 t^3\|\widetilde{\nabla}_x g^\varepsilon\|_{\mathscr{Y}_1}\|(g^\varepsilon)^\perp\|_{\mathscr{Y}_2} &\leqslant \frac{\kappa_2\alpha_3 t^3}{12}\|\widetilde{\nabla}_x g^\varepsilon\|_{\mathscr{Y}_1}^2 + C\varepsilon^2\eta_0^2\alpha_3 t^3\|(g^\varepsilon)^\perp\|_{\mathscr{Y}_2}^2\\
C\varepsilon\alpha_3\eta_0 t^3\|\widetilde{\nabla}_x g^\varepsilon\|_{\mathscr{X}}\|\widetilde{\nabla}_x g^\varepsilon\|_{\mathscr{Y}_1} &\leqslant C\alpha_3\eta_0 t^3\|\widetilde{\nabla}_x g^\varepsilon\|_{\mathscr{X}}^2 + \alpha_3\eta_0 t^3\|\widetilde{\nabla}_x g^\varepsilon\|_{\mathscr{Y}_1}^2\\
2\varepsilon\alpha_2 t\left|\left\langle\widetilde{\nabla}_x g^\varepsilon, \widetilde{\nabla}_v g^\varepsilon\right\rangle_{\mathscr{X}}\right| &\leqslant \frac{\alpha_2}{4}t^2\|\widetilde{\nabla}_x g^\varepsilon\|_{\mathscr{X}}^2 + C\alpha_2\varepsilon^2\|g^\varepsilon\|_{\mathscr{Y}_1}^2,
\end{aligned}
$$

as well as

$$
\begin{aligned}
C\alpha_2 t^2\|\widetilde{\nabla}_x g^\varepsilon\|_{\mathscr{X}}\|(g^\varepsilon)^\perp\|_{\mathscr{Y}_1}\|(g^\varepsilon)^\perp\|_{\mathscr{Y}_2} &\leqslant C\alpha_2 t^3\|\widetilde{\nabla}_x g^\varepsilon\|_{\mathscr{X}}^2\|(g^\varepsilon)^\perp\|_{\mathscr{Y}_1}^2 + C\alpha_2 t\|(g^\varepsilon)^\perp\|_{\mathscr{Y}_2}^2\\
C\varepsilon\alpha_3 t^3\|\widetilde{\nabla}_x g^\varepsilon\|_{\mathscr{X}}\|(g^\varepsilon)^\perp\|_{\mathscr{Y}_1}\|\widetilde{\nabla}_x g^\varepsilon\|_{\mathscr{Y}_1} &\leqslant C\varepsilon^2\alpha_3 t^3\|\widetilde{\nabla}_x g^\varepsilon\|_{\mathscr{X}}^2\|(g^\varepsilon)^\perp\|_{\mathscr{Y}_1}^2 + \frac{\kappa_2\alpha_3}{12}t^3\|\widetilde{\nabla}_x g^\varepsilon\|_{\mathscr{Y}_1}^2.
\end{aligned}
$$

We thus obtain

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{U}_\varepsilon(t,g^\varepsilon) \leqslant -\Big(\kappa - C\eta_0^2 - C\alpha_2 t^2\eta_0 - C\alpha_2\varepsilon^2 - C\alpha_1 t\eta_0^2 \\
- C\frac{\alpha_2^2}{\alpha_3}\eta_0^2 t - C\varepsilon\eta_0^2 t^{3/2}\frac{\alpha_2^2}{\alpha_1} - C\varepsilon^2\eta_0^2\alpha_3 t^3\Big)\|g^\varepsilon\|^2_{\mathscr{Y}_1} \\
- \frac{1}{\varepsilon^2}\left(\kappa - C\alpha_1 t - C\frac{\alpha_1^2}{\alpha_2}\right)\|(g^\varepsilon)^\perp\|^2_{\mathscr{Y}_1} \\
- \frac{t}{\varepsilon^2}\Big(\frac{\kappa_1\alpha_1}{2} - C\varepsilon^2\alpha_2 t\eta_0 - C\varepsilon\alpha_1\eta_0 - C\frac{\alpha_2^2}{\alpha_3} \\
- C\eta_0^2\frac{\alpha_2^2}{\alpha_3} - C\varepsilon^4\eta_0^2\alpha_3 t^2 - C\varepsilon^2\alpha_2\Big)\|(g^\varepsilon)^\perp\|^2_{\mathscr{Y}_2} \\
- t^2\left(\frac{\alpha_2}{2} - C\varepsilon^2\alpha_3 - C\alpha_3 t - C\alpha_3\eta_0 t\right)\|\widetilde{\nabla}_x g^\varepsilon\|^2_{\mathscr{X}} \\
- t^3\left(\frac{\kappa_2\alpha_3}{2} - C\varepsilon\alpha_3\eta_0 - C\alpha_3\eta_0\right)\|\widetilde{\nabla}_x g^\varepsilon\|^2_{\mathscr{Y}_1} \\
+ Ct^3\left(\alpha_2 + \varepsilon^2\alpha_3\right)\|(g^\varepsilon)^\perp\|^2_{\mathscr{Y}_1}\|\widetilde{\nabla}_x g^\varepsilon\|^2_{\mathscr{X}}.
\end{aligned}
$$

We now choose $\alpha_1 = \eta$, $\alpha_2 = \eta^{\frac{3}{2}}$, and $\alpha_3 = \eta^{\frac{5}{3}}$, with $\eta \in (0,1)$ small enough as well as η_0 small enough, so that we deduce, for all $t \in [0,1]$,

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{U}_\varepsilon(t,g^\varepsilon) \leqslant -\frac{\kappa'}{\varepsilon^2}\|(g^\varepsilon)^\perp\|^2_{\mathscr{Y}_1} - \kappa'\|g^\varepsilon\|^2_{\mathscr{Y}_1} - \kappa'\frac{t}{\varepsilon^2}\|(g^\varepsilon)^\perp\|^2_{\mathscr{Y}_2} - \kappa' t^2\|\widetilde{\nabla}_x g^\varepsilon\|^2_{\mathscr{X}} - \kappa' t^3\|\widetilde{\nabla}_x g^\varepsilon\|^2_{\mathscr{Y}_1} \\
+ Ct^3\|\widetilde{\nabla}_x g^\varepsilon\|^2_{\mathscr{X}}\|(g^\varepsilon)^\perp\|^2_{\mathscr{Y}_1},
\end{aligned}
$$

for some constants $\kappa', C > 0$. Integrating in time the last inequality, we hence obtain that for any $t \in [0,1]$, there holds

$$
\begin{aligned}
\mathscr{U}_\varepsilon(t,g^\varepsilon) &+ \frac{\kappa'}{\varepsilon^2}\int_0^t \|(g^\varepsilon)^\perp\|^2_{\mathscr{Y}_1}\,\mathrm{d}s + \kappa'\int_0^t \|g^\varepsilon\|^2_{\mathscr{Y}_1}\,\mathrm{d}s \\
&+ \frac{\kappa'}{\varepsilon^2}\int_0^t s\|(g^\varepsilon)^\perp\|^2_{\mathscr{Y}_2}\,\mathrm{d}s + \kappa'\int_0^t s^2\|\widetilde{\nabla}_x g^\varepsilon\|^2_{\mathscr{X}}\,\mathrm{d}s + \kappa'\int_0^t s^3\|\widetilde{\nabla}_x g^\varepsilon\|^2_{\mathscr{Y}_1}\,\mathrm{d}s \\
&\leqslant \mathscr{U}_\varepsilon(0) + C\int_0^t s^3\|\widetilde{\nabla}_x g^\varepsilon\|^2_{\mathscr{X}}\|(g^\varepsilon)^\perp\|^2_{\mathscr{Y}_1}\,\mathrm{d}s \\
&\leqslant \mathscr{U}_\varepsilon(0) + C\varepsilon^2\left(\sup_{t\in[0,1]} t^3\|\widetilde{\nabla}_x g^\varepsilon\|^2_{\mathscr{X}}\right)\frac{1}{\varepsilon^2}\int_0^t \|(g^\varepsilon)^\perp\|^2_{\mathscr{Y}_1}\,\mathrm{d}s \\
&\leqslant \mathscr{U}_\varepsilon(0) + C\eta_0^2\varepsilon^2\left(\sup_{t\in[0,1]} t^3\|\widetilde{\nabla}_x g^\varepsilon\|^2_{\mathscr{X}}\right)
\end{aligned}
$$

where we have used that $\frac{1}{\varepsilon^2}\int_0^t \|(g^\varepsilon)^\perp\|^2_{\mathscr{Y}_1}\,\mathrm{d}s \lesssim \eta_0^2$ from Theorem 1.1-(i) and we have denoted $\mathscr{U}_\varepsilon(0) = \mathscr{U}_\varepsilon(0, g^\varepsilon_{\mathrm{in}})$. Since

$$
\left(\sup_{t\in[0,1]}\|g^\varepsilon(t)\|^2_{\mathscr{X}}\right) + \left(\sup_{t\in[0,1]} t\|(g^\varepsilon(t))^\perp\|^2_{\mathscr{Y}_1}\right) + \varepsilon^2\left(\sup_{t\in[0,1]} t^3\|\widetilde{\nabla}_x g^\varepsilon(t)\|^2_{\mathscr{X}}\right) \lesssim \sup_{t\in[0,1]}\mathscr{U}_\varepsilon(t,g^\varepsilon)
$$

if $\eta_0 > 0$ is small enough (independently of ε), we finally obtain

$$
t\|g^\varepsilon(t)\|^2_{\mathscr{Y}_1} \lesssim \mathscr{U}_\varepsilon(0) = |||g^\varepsilon_{\mathrm{in}}|||^2_{\mathscr{X}} \lesssim \|g^\varepsilon_{\mathrm{in}}\|^2_{\mathscr{X}} \quad \forall\, t \in [0,1]
$$

and

$$
t^3\|g^\varepsilon(t)\|^2_{\mathscr{Z}_1^\varepsilon} \lesssim \mathscr{U}_\varepsilon(0) = |||g^\varepsilon_{\mathrm{in}}|||^2_{\mathscr{X}} \lesssim \|g^\varepsilon_{\mathrm{in}}\|^2_{\mathscr{X}} \quad \forall\, t \in [0,1],
$$

which gives estimates (4.37) and concludes the proof. ∎

5. Refined semigroup estimates on the linearized operator

In this part, we go back to the linearized problem and give some new and refined estimates on it. The first and second subsections are dedicated to the introduction of a nice splitting of the linearized operator Λ_ε coming from [11, 13] and to the proof of dissipativity, regularization and boundedness estimates. Roughly speaking, the linearized operator Λ_ε splits into two parts: $\Lambda_\varepsilon = \mathcal{A}_\varepsilon + \mathcal{B}_\varepsilon$, the first part $\mathcal{A}_\varepsilon$ having some good regularizing properties, the second one $\mathcal{B}_\varepsilon$ having some nice dissipativity (and also regularizing) properties. It is worth mentioning that the regularization estimates on $\mathcal{B}_\varepsilon$ that we develop are sharp (it was not the case in [13] where the authors did not intend to obtain optimal regularization estimates on $\mathcal{B}_\varepsilon$) and only the case $\varepsilon = 1$ was treated in [13]. As already mentioned, from those properties and Duhamel formula, we can give a new proof of Theorem 3.1. More importantly, Duhamel formula applied with this splitting will be used to prove our hydrodynamical limit theorem in Section 6.

In the next three subsections of the present section, we study the semigroup $U^\varepsilon(t)$ associated with Λ_ε from another point of view which is based on a careful spectral analysis carried out in Fourier in x of Λ_1 in [60], it in particular allows us to give a decomposition of the semigroup $U^\varepsilon(t)$, study its limit as ε goes to 0 and give another type of estimates on it.

5.1. **Splitting of the operator.** We now introduce a splitting of the full linearized operator Λ_ε. Let $\chi \in \mathcal{C}_c^\infty(\mathbf{R})$ be a smooth cutoff function such that $0 \leqslant \chi \leqslant 1$, $\chi \equiv 1$ on $[-1/2, 1/2]$ and $\chi \equiv 0$ on $\mathbf{R} \setminus [-1,1]$, consider positive constants $R, \bar{R} > 0$ and define $\chi_{\bar{R}}(v) = \chi(|v|/\bar{R})$ as well as

$$m^2(v) := \frac{1}{4}|\mathbf{B}(v)v|^2 - \frac{1}{2}\nabla_v \cdot \left[\mathbf{B}^\top(v)\mathbf{B}(v)v\right] + R\chi_{\bar{R}}(v) \tag{5.1}$$

where $\mathbf{B}(v)$ is defined in (2.5). Recalling the formulation of L in (2.8), we then decompose $\Lambda_\varepsilon = \frac{1}{\varepsilon^2}L - \frac{1}{\varepsilon}v \cdot \nabla_x$ as

$$\Lambda_\varepsilon = \mathcal{A}_\varepsilon + \mathcal{B}_\varepsilon \quad \text{with} \quad \mathcal{A}_\varepsilon := \frac{1}{\varepsilon^2}\mathcal{A} \quad \text{and} \quad \mathcal{B}_\varepsilon := \frac{1}{\varepsilon^2}\mathcal{B} - \frac{1}{\varepsilon}v \cdot \nabla_x \tag{5.2}$$

where

$$\begin{aligned} \mathcal{A}f &:= -\Big\{-\left(a_{ij} * \sqrt{M}f\right)v_iv_j + \left(a_{ii} * \sqrt{M}f\right) + \left(c * \sqrt{M}f\right)\Big\}\sqrt{M} + R\chi_{\bar{R}}f \\ &= (L_2 + R\chi_{\bar{R}})f \end{aligned} \tag{5.3}$$

and

$$\mathcal{B}f := -\widetilde{\nabla}_v^*\widetilde{\nabla}_v f - m^2(v)f = (L_1 - R\chi_{\bar{R}})f. \tag{5.4}$$

Let us now give a lemma providing estimates on m and its derivatives. We fix

$$\sigma_1 := \begin{cases} +\infty & \text{if} \quad -2 < \gamma \leqslant 1 \\ \frac{1}{2} & \text{if} \quad \gamma = -2. \end{cases} \tag{5.5}$$

Lemma 5.1. *Let $0 < \sigma < \sigma_1$.*

(i) There are R_0 and $\bar{R}_0$ large enough so that for any $R \geqslant R_0$ and $\bar{R} \geqslant \bar{R}_0$, one has

$$m^2(v) \geqslant \sigma + \kappa\langle v\rangle^{\gamma+2}, \quad \forall\, v \in \mathbf{R}^3,$$

for some $0 < \kappa < \sigma_1 - \sigma$.

(ii) For any $\alpha \in \mathbf{R}$, we define

$$m_\alpha^2(v) := m^2(v) - \langle v\rangle^{-2\alpha}|\widetilde{\nabla}_v\langle v\rangle^\alpha|^2.$$

There are R_0 and $\bar{R}_0$ large enough so that for any $R \geqslant R_0$ and $\bar{R} \geqslant \bar{R}_0$, one has

$$m_\alpha^2(v) \geqslant \sigma + \kappa\langle v\rangle^{\gamma+2}, \quad \forall\, v \in \mathbf{R}^3,$$

for some $0 < \kappa < \sigma_1 - \sigma$.

(iii) For any multi-index $\alpha \in \mathbf{N}^3$ we have, for all $v \in \mathbf{R}^3$,

$$|\partial_v^\alpha m(v)| \lesssim \langle v\rangle^{\frac{\gamma}{2}+1-|\alpha|}.$$

Hereafter in the paper, for any $0<\sigma<\sigma_1$ we then fix $R,\bar{R}>0$ large enough so that the results of Lemma 5.1 are in force.

Proof of Lemma 5.1. (i) See [13, Lemma 2.6].

(ii) We have

$$m_\alpha^2(v)=\frac{1}{4}|\mathbf{B}(v)v|^2-\frac{1}{2}\nabla_v\cdot\Big[\mathbf{B}^\top(v)\mathbf{B}(v)v\Big]-\alpha^2|\mathbf{B}(v)v|^2\langle v\rangle^{-4}+R\chi_{\bar{R}}(v)$$

therefore

$$m_\alpha^2(v)=\frac{1}{4}\ell_1(v)|v|^2-\frac{1}{2}\nabla_v\cdot\Big[\ell_1(v)v\Big]-\alpha^2\ell_1(v)|v|^2\langle v\rangle^{-4}+R\chi_{\bar{R}}(v).$$

Now observe that

$$\begin{aligned}\frac{1}{4}\ell_1(v)|v|^2-\frac{1}{2}\nabla_v\cdot\Big[\ell_1(v)v\Big]-\alpha^2\ell_1(v)|v|^2\langle v\rangle^{-4}\\ =\frac{1}{4}\ell_1(v)|v|^2-\frac{1}{2}\nabla_v\ell_1(v)\cdot v-\frac{3}{2}\ell_1(v)-\alpha^2\ell_1(v)|v|^2\langle v\rangle^{-4},\end{aligned}$$

and from (2.4), $\ell_1(v)|v|^2\langle v\rangle^{-4}\lesssim\langle v\rangle^{\gamma-2}$, hence if $-2<\gamma\leqslant 1$ we have

$$\frac{1}{4}\ell_1(v)|v|^2-\frac{1}{2}\nabla_v\cdot\Big[\ell_1(v)v\Big]-\alpha^2\ell_1(v)|v|^2\langle v\rangle^{-4}\xrightarrow[|v|\to\infty]{}+\infty,$$

and if $\gamma=-2$, one has

$$\frac{1}{4}\ell_1(v)|v|^2-\frac{1}{2}\nabla_v\cdot\Big[\ell_1(v)v\Big]-\alpha^2\ell_1(v)|v|^2\langle v\rangle^{-4}\xrightarrow[|v|\to\infty]{}\frac{1}{2}>0.$$

We then conclude the proof by taking $R_0,\bar{R}_0>0$ large enough.

(iii) Direct consequence of (2.9). □

Arguing as in Lemma 2.3 and recalling that $[\widetilde{\nabla}_{v_i},v\cdot\nabla_x]=\widetilde{\nabla}_{x_i}$ and $[\widetilde{\nabla}_{x_i},v\cdot\nabla_x]=0$, we also obtain

Lemma 5.2. *There holds*

(i) For any $1\leqslant i\leqslant 3$, one has

$$\begin{aligned}[\widetilde{\nabla}_{v_i},\mathcal{B}_\varepsilon]f=&-\frac{1}{\varepsilon^2}\widetilde{\nabla}_{v_j}^*[\widetilde{\nabla}_{v_i},\widetilde{\nabla}_{v_j}]f-\frac{1}{\varepsilon^2}\widetilde{\nabla}_{v_j}[\widetilde{\nabla}_{v_i},\widetilde{\nabla}_{v_j}^*]f\\&-\frac{1}{\varepsilon^2}(\widetilde{\nabla}_{v_i}m^2)f-\frac{1}{\varepsilon}\widetilde{\nabla}_{x_i}f-\frac{1}{\varepsilon^2}\Big[[\widetilde{\nabla}_{v_i},\widetilde{\nabla}_{v_j}^*],\widetilde{\nabla}_{v_j}\Big]f.\end{aligned}$$

(ii) For any $1\leqslant i\leqslant 3$, one has

$$[\widetilde{\nabla}_{x_i},\mathcal{B}_\varepsilon]f=-\frac{1}{\varepsilon^2}\widetilde{\nabla}_{v_j}^*[\widetilde{\nabla}_{x_i},\widetilde{\nabla}_{v_j}]f-\frac{1}{\varepsilon^2}\widetilde{\nabla}_{v_j}[\widetilde{\nabla}_{x_i},\widetilde{\nabla}_{v_j}^*]f-\frac{1}{\varepsilon^2}\Big[[\widetilde{\nabla}_{x_i},\widetilde{\nabla}_{v_j}^*],\widetilde{\nabla}_{v_j}\Big]f.$$

Let us recall that the spaces $\mathscr{X}$, $\mathscr{Y}_i$, $(\mathscr{Y}_i)'$, $\mathscr{Z}_i^\varepsilon$ and $(\mathscr{Z}_i^\varepsilon)'$ are respectively defined in (1.19), (1.20), (2.11), (2.14), (2.12), (2.13) and (2.15).

We also have the following bounds on the operator $\mathcal{A}$:

Lemma 5.3. *For any $\alpha\in\mathbf{R}$ one has*

$$\begin{aligned}\|\langle v\rangle^\alpha\mathcal{A}f\|_{L^2_{x,v}}&\lesssim\|M^{\frac14}f\|_{L^2_{x,v}}\\ \|\langle v\rangle^\alpha\widetilde{\nabla}_v\mathcal{A}f\|_{L^2_{x,v}}&\lesssim\|M^{\frac14}f\|_{L^2_{x,v}}+\|M^{\frac14}\widetilde{\nabla}_vf\|_{L^2_{x,v}}\\ \|\langle v\rangle^\alpha\widetilde{\nabla}_v\widetilde{\nabla}_v\mathcal{A}f\|_{L^2_{x,v}}&\lesssim\|M^{\frac14}f\|_{L^2_{x,v}}+\|M^{\frac14}\widetilde{\nabla}_vf\|_{L^2_{x,v}}+\|M^{\frac14}\widetilde{\nabla}_v\widetilde{\nabla}_vf\|_{L^2_{x,v}}\\ \|\langle v\rangle^\alpha\widetilde{\nabla}_x\mathcal{A}f\|_{L^2_{x,v}}&\lesssim\|M^{\frac14}\widetilde{\nabla}_xf\|_{L^2_{x,v}}\\ \|\langle v\rangle^\alpha\widetilde{\nabla}_x\widetilde{\nabla}_x\mathcal{A}f\|_{L^2_{x,v}}&\lesssim\|M^{\frac14}\widetilde{\nabla}_x\widetilde{\nabla}_xf\|_{L^2_{x,v}}.\end{aligned}$$

In particular, one has $\mathcal{A}\in\mathscr{B}(\mathscr{X})$, $\mathcal{A}\in\mathscr{B}(\mathscr{Y}_i)$ and $\mathcal{A}\in\mathscr{B}(\mathscr{Z}_i^\varepsilon)$ for $i=1,2$ and since $\mathcal{A}$ is self-adjoint in $L^2_{x,v}$, we also have $\mathcal{A}\in\mathscr{B}(\mathscr{Y}_i')$ and $\mathcal{A}\in\mathscr{B}((\mathscr{Z}_i^\varepsilon)')$ for $i=1,2$.

Proof. The terms coming from $R\chi_{\bar R}$ are easily treated because $\chi_{\bar R}$ has compact support. The first, second and fourth estimates thus directly come from Lemma 2.6. The proofs of the other estimates are completely similar and rely on Lemmas 2.4 and 2.5. ☐

5.2. **Decay and regularization estimates for $S_{\mathcal{B}_\varepsilon}$.** In this section we provide several results on the dissipation and regularization properties of the operator $\mathcal{B}_\varepsilon$. We start with the dissipative ones.

Lemma 5.4. *Let $\sigma \in (0,\sigma_1)$ (where σ_1 is defined in (5.5)). Then for any $\alpha \in \mathbf{R}$, one has*

$$\langle \langle v\rangle^\alpha f, \langle v\rangle^\alpha \mathcal{B}_\varepsilon f\rangle_{L^2_{x,v}} \leqslant -\frac{\sigma}{\varepsilon^2}\|\langle v\rangle^\alpha f\|^2_{L^2_{x,v}} - \frac{\kappa}{\varepsilon^2}\|\langle v\rangle^\alpha f\|^2_{L^2_x(H^1_{v,*})}, \tag{5.6}$$

for some constant $\kappa > 0$. As a consequence, one has for all $t \geqslant 0$,

$$\|S_{\mathcal{B}_\varepsilon}(t)\|_{\mathscr{X}\to\mathscr{X}} \leqslant e^{-\sigma t/\varepsilon^2}.$$

Proof. We compute:

$$\begin{aligned}
\Big\langle \langle v\rangle^{2\alpha} f, \mathcal{B}_\varepsilon f\Big\rangle_{L^2_{x,v}} &= \Big\langle \langle v\rangle^{2\alpha} f, \Big\{-\frac{1}{\varepsilon^2}\widetilde\nabla_v^*\widetilde\nabla_v f - \frac{1}{\varepsilon^2} m^2 f - \frac{1}{\varepsilon} v\cdot\nabla_x f\Big\}\Big\rangle_{L^2_{x,v}} \\
&= -\frac{1}{\varepsilon^2}\|\langle v\rangle^\alpha m f\|^2_{L^2_{x,v}} - \frac{1}{\varepsilon^2}\Big\langle \widetilde\nabla_v(\langle v\rangle^{2\alpha} f), \widetilde\nabla_v f\Big\rangle_{L^2_{x,v}}.
\end{aligned}$$

Observing that

$$\begin{aligned}
\Big\langle \widetilde\nabla_v(\langle v\rangle^{2\alpha} f), \widetilde\nabla_v f\Big\rangle_{L^2_{x,v}} &= \Big\langle \widetilde\nabla_v(\langle v\rangle^{\alpha} f), \langle v\rangle^\alpha\widetilde\nabla_v f\Big\rangle_{L^2_{x,v}} + \Big\langle (\widetilde\nabla_v\langle v\rangle^\alpha) f, \langle v\rangle^\alpha\widetilde\nabla_v f\Big\rangle_{L^2_{x,v}} \\
&= \|\widetilde\nabla_v(\langle v\rangle^\alpha f)\|^2_{L^2_{x,v}} - \Big\langle \widetilde\nabla_v(\langle v\rangle^{\alpha} f), (\widetilde\nabla_v\langle v\rangle^\alpha) f\Big\rangle_{L^2_{x,v}} + \Big\langle (\widetilde\nabla_v\langle v\rangle^\alpha) f, \langle v\rangle^\alpha\widetilde\nabla_v f\Big\rangle_{L^2_{x,v}} \\
&= \|\widetilde\nabla_v(\langle v\rangle^\alpha f)\|^2_{L^2_{x,v}} - \|(\widetilde\nabla_v\langle v\rangle^\alpha) f\|^2_{L^2_{x,v}}
\end{aligned}$$

and recalling that m_α is defined in Lemma 5.1 so that $\langle v\rangle^{2\alpha} m_\alpha^2 = \langle v\rangle^{2\alpha} m^2 - |\widetilde\nabla_v\langle v\rangle^\alpha|^2$, we obtain the estimate

$$\Big\langle \langle v\rangle^{2\alpha} f, \mathcal{B}_\varepsilon f\Big\rangle_{L^2_{x,v}} = -\frac{1}{\varepsilon^2}\|\langle v\rangle^\alpha m_\alpha f\|^2_{L^2_{x,v}} - \frac{1}{\varepsilon^2}\|\widetilde\nabla_v(\langle v\rangle^\alpha f)\|^2_{L^2_{x,v}}.$$

We then conclude the proof of (5.6) by using the bound by below on m_α of Lemma 5.1.

The exponential decay estimate of $S_{\mathcal{B}_\varepsilon}$ on $\mathscr{X}$ is then a consequence of (5.6) together with the fact that ∇_x commutes with $\mathcal{B}_\varepsilon$. ☐

In what follows, we prove regularization results for the semigroup $S_{\mathcal{B}_\varepsilon}$.

Lemma 5.5. *Let $\sigma \in (0,\sigma_1)$ (where σ_1 is defined in (5.5)). For any $t > 0$, one has*

$$\|S_{\mathcal{B}_\varepsilon}(t)\|_{\mathscr{X}\to\mathscr{Y}_1} \lesssim \frac{\varepsilon}{\sqrt t}\, e^{-\sigma t/\varepsilon^2} \quad \text{and} \quad \|S_{\mathcal{B}_\varepsilon}(t)\|_{\mathscr{X}\to\mathscr{Z}_1^\varepsilon} \lesssim \frac{\varepsilon^3}{t^{3/2}}\, e^{-\sigma t/\varepsilon^2}. \tag{5.7}$$

as well as the dual estimates

$$\|S_{\mathcal{B}_\varepsilon}(t)\|_{\mathscr{Y}_1'\to\mathscr{X}} \lesssim \frac{\varepsilon}{\sqrt t}\, e^{-\sigma t/\varepsilon^2} \quad \text{and} \quad \|S_{\mathcal{B}_\varepsilon}(t)\|_{(\mathscr{Z}_1^\varepsilon)'\to\mathscr{X}} \lesssim \frac{\varepsilon^3}{t^{3/2}}\, e^{-\sigma t/\varepsilon^2}. \tag{5.8}$$

The proof follows similar ideas as the proof of Proposition 3.4 but is somewhat simpler because, the operator $\mathcal{B}_\varepsilon$ provides $1/\varepsilon^2$-dissipativity and regularity on both macroscopic and microscopic parts of the solution whereas the operator Λ_ε only provided a gain of $1/\varepsilon^2$ on the microscopic part. We thus do not need to separate microscopic and macroscopic parts when defining our functional that will be a Lyapunov functional for our equation $\partial_t f = \mathcal{B}_\varepsilon f$ (see (B.3)). For sake of completeness, the proof is given in Appendix B.

Using the same method, we can push our previous result up to the next notch of regularity (we only mention the dual results because they are the only ones that will be used in the sequel) and the proof is also postponed to Appendix B:

Lemma 5.6. *Let $\sigma \in (0,\sigma_1)$ (where σ_1 is defined in (5.5)). For any $t > 0$, one has*

$$\|S_{\mathcal{B}_\varepsilon}(t)\|_{\mathscr{Y}_2'\to\mathscr{X}} \lesssim \frac{\varepsilon^2}{t}\, e^{-\sigma t/\varepsilon^2} \quad \text{and} \quad \|S_{\mathcal{B}_\varepsilon}(t)\|_{(\mathscr{Z}_2^\varepsilon)'\to\mathscr{X}} \lesssim \frac{\varepsilon^6}{t^3}\, e^{-\sigma t/\varepsilon^2}.$$

We also have the following dissipativity properties, the proof of which relies on the same line of proof as the above regularization results. Indeed, the idea is to use the same functionals without the weights in time. We thus skip the proof since the computations are the same. As previously, the result is only given in the dual framework which will be the only one useful in the sequel.

Lemma 5.7. *Let $\sigma \in (0, \sigma_1)$ (where σ_1 is defined in (5.5)). For any $t \geqslant 0$ and $i = 1, 2$, one has*

$$\|S_{\mathcal{B}_\varepsilon}(t)\|_{(\mathscr{X}_i^\varepsilon)' \to (\mathscr{X}_i^\varepsilon)'} \lesssim e^{-\sigma t/\varepsilon^2}.$$

In our forthcoming analysis, we will use an iterated Duhamel formula based on the splitting $\Lambda_\varepsilon = \mathcal{A}_\varepsilon + \mathcal{B}_\varepsilon$. We introduce the following definition of convolution of semigroups: If S_1 and S_2 are two semigroups, their convolution product is defined by

$$S_1 * S_2(t) := \int_0^t S_1(s) S_2(t-s) \, \mathrm{d}s.$$

We also introduce the semigroups $V_j^\varepsilon(t)$ defined through:

$$V_0^\varepsilon(t) := S_{\mathcal{B}_\varepsilon}(t), \quad V_{j+1}^\varepsilon(t) := (S_{\mathcal{B}_\varepsilon} * \mathcal{A}_\varepsilon V_j^\varepsilon)(t) = (V_j^\varepsilon * \mathcal{A}_\varepsilon S_{\mathcal{B}_\varepsilon})(t), \; j \in \mathbf{N}$$

so that for any $n \in \mathbf{N}$ and any $t \geqslant 0$, we have:

$$U^\varepsilon(t) = \sum_{j=0}^n V_j^\varepsilon(t) + (V_n^\varepsilon * \mathcal{A}_\varepsilon U^\varepsilon)(t). \tag{5.9}$$

As a consequence of the previous results on $\mathcal{A}_\varepsilon$ and $\mathcal{B}_\varepsilon$, we obtain:

Corollary 5.8. *Let $\sigma \in (0, \sigma_1)$ (where σ_1 is defined in (5.5)). For any $t > 0$, any $j \in \mathbf{N}$ and $i = 1, 2$, we have:*

$$\|V_j^\varepsilon(t)\|_{\mathscr{X} \to \mathscr{X}} \lesssim e^{-\sigma t/\varepsilon^2} \quad \text{and} \quad \|V_j^\varepsilon(t)\|_{(\mathscr{X}_i^\varepsilon)' \to (\mathscr{X}_i^\varepsilon)'} \lesssim e^{-\sigma t/\varepsilon^2}.$$

Proof. Fix $\sigma \in (0, \sigma_1)$. The first estimate for $j = 0$ is given by Lemma 5.4. We then argue by induction and thus suppose that the property is satisfies for some $j \in \mathbf{N}$. Then, we consider $\sigma' \in (\sigma, \sigma_1)$. From Lemmas 5.3 and 5.4 and the induction hypothesis, we have:

$$\begin{aligned}
\|V_{j+1}^\varepsilon(t)\|_{\mathscr{X} \to \mathscr{X}} &\lesssim \frac{1}{\varepsilon^2} \int_0^t \|S_{\mathcal{B}_\varepsilon}(s)\|_{\mathscr{X} \to \mathscr{X}} \|\mathcal{A}\|_{\mathscr{X} \to \mathscr{X}} \|V_j^\varepsilon(t-s)\|_{\mathscr{X} \to \mathscr{X}} \, \mathrm{d}s \\
&\lesssim \frac{1}{\varepsilon^2} \int_0^t e^{-\sigma' s/\varepsilon^2} e^{-\sigma(t-s)/\varepsilon^2} \, \mathrm{d}s \lesssim \frac{1}{\varepsilon^2} e^{-\sigma t/\varepsilon^2} \int_0^t e^{-(\sigma' - \sigma)s/\varepsilon^2} \, \mathrm{d}s \lesssim e^{-\sigma t/\varepsilon^2}.
\end{aligned}$$

The second estimate can be proven in a similar way by using Lemmas 5.3 and 5.7. □

Corollary 5.9. *Let $\sigma \in (0, \sigma_1)$ (where σ_1 is defined in (5.5)). For any $t > 0$ and any $j \in \mathbf{N}$, we have:*

$$\|V_j^\varepsilon(t)\|_{(\mathscr{X}_1^\varepsilon)' \to \mathscr{X}} \lesssim \frac{\varepsilon^{3-2j}}{t^{\frac{3-2j}{2}}} e^{-\sigma t/\varepsilon^2} \quad \text{and} \quad \|V_j^\varepsilon(t)\|_{(\mathscr{X}_2^\varepsilon)' \to \mathscr{X}} \lesssim \frac{\varepsilon^{6-2j}}{t^{3-j}} e^{-\sigma t/\varepsilon^2}.$$

Proof. Fix $\sigma \in (0, \sigma_1)$. We focus on the proof of the first estimate, the second one is treated in a similar way. We proceed by induction. The case $j = 0$ is given by Lemma 5.5. Suppose then that the estimate holds for some $j \in \mathbf{N}$ and consider $\sigma' \in (\sigma, \sigma_1)$. From Lemmas 5.3-5.4, Corollary 5.8 and the induction hypothesis, we have:

$$\begin{aligned}
\|V_{j+1}^\varepsilon(t)\|_{(\mathscr{X}_1^\varepsilon)' \to \mathscr{X}} &\lesssim \int_0^{t/2} \|S_{\mathcal{B}_\varepsilon}(s)\|_{\mathscr{X} \to \mathscr{X}} \|\mathcal{A}_\varepsilon\|_{\mathscr{X} \to \mathscr{X}} \|V_j^\varepsilon(t-s)\|_{(\mathscr{X}_1^\varepsilon)' \to \mathscr{X}} \, \mathrm{d}s \\
&\quad + \int_{t/2}^t \|V_j^\varepsilon(s)\|_{(\mathscr{X}_1^\varepsilon)' \to \mathscr{X}} \|\mathcal{A}_\varepsilon\|_{(\mathscr{X}_1^\varepsilon)' \to (\mathscr{X}_1^\varepsilon)'} \|S_{\mathcal{B}_\varepsilon}(t-s)\|_{(\mathscr{X}_1^\varepsilon)' \to (\mathscr{X}_1^\varepsilon)'} \, \mathrm{d}s \\
&\lesssim \frac{1}{\varepsilon^2} \int_0^{t/2} e^{-\sigma' s/\varepsilon^2} \frac{\varepsilon^{3-2j}}{(t-s)^{\frac{3-2j}{2}}} e^{-\sigma(t-s)/\varepsilon^2} \, \mathrm{d}s + \frac{1}{\varepsilon^2} \int_{t/2}^t \frac{\varepsilon^{3-2j}}{s^{\frac{3-2j}{2}}} e^{-\sigma' s/\varepsilon^2} e^{-\sigma(t-s)/\varepsilon^2} \, \mathrm{d}s.
\end{aligned}$$

From this, we deduce that

$$\|V^\varepsilon_{j+1}(t)\|_{(\mathscr{X}_1^\varepsilon)'\to\mathscr{X}} \lesssim e^{-\sigma t/\varepsilon^2}\varepsilon^{1-2j}\left(\int_0^{t/2}\frac{\mathrm{d}s}{(t-s)^{\frac{3-2j}{2}}}+\int_{t/2}^t\frac{\mathrm{d}s}{s^{\frac{3-2j}{2}}}\right)\lesssim \frac{\varepsilon^{1-2j}}{t^{\frac{1-2j}{2}}}e^{-\sigma t/\varepsilon^2},$$

which yields the wanted result. □

5.3. **Spectral study in Fourier space.** We denote by $\mathcal{F}_x$ the Fourier transform in $x\in\mathbf{T}^3$ with $\xi\in\mathbf{Z}^3$ its dual variable. Since we will only be working with Fourier transform in x, we will also interchangeably use the classical "hat" notation. Moreover, to lighten the reading, for any operator that acts only on velocity, with a little abuse of notation, we will omit the "hat" in the notation for its x-wise Fourier transform.

In this part, we are going to look at the Fourier transform in $x\in\mathbf{T}^3$ of the operator Λ_1:

$$\widehat{\Lambda}_1(\xi) := -i\,\xi\cdot v + L$$

and study the spectrum of $\widehat{\Lambda}_1(\xi)$ for $\xi\in\mathbf{Z}^3$. This type of analysis was initiated in [52, 15, 22] for the Boltzmann equation for hard spheres and then with hard cutoff potentials (see also [58]). In [60], Yang and Yu were then able to adapt it to more general kinetic equations including the linearized Landau one for hard and moderately soft potentials.

Roughly speaking, for small frequencies, the spectrum of $\widehat{\Lambda}_1(\xi)$ is a perturbation of the one of the homogeneous collision operator L (which acts only on velocity). As already mentioned, in the case of hard and moderately soft potentials ($\gamma\geqslant -2$), the operator L has a spectral gap and in [60], the authors then prove that for small frequencies ξ, the spectrum of $\widehat{\Lambda}_1(\xi)$ is made of "small" eigenvalues around 0 in the right part of the plan. They also provide Taylor expansions of those eigenvalues as well as for their associated projectors. For large frequencies, they prove that the operator $\widehat{\Lambda}_1(\xi)$ has no spectrum in some suitable right part of the plan. All those spectral results provide a decomposition of the semigroup which is given in Lemma 5.10.

In what follows, we write

$$\widehat{U}^\varepsilon(t) = \mathcal{F}_x U^\varepsilon(t)\mathcal{F}_x^{-1}$$

so that $\widehat{U}^\varepsilon$ is the semigroup associated with the operator

$$\widehat{\Lambda}_\varepsilon(\xi) := \frac{1}{\varepsilon^2}(-i\varepsilon\,\xi\cdot v + L).$$

We also introduce the bilinear operator $\Psi^\varepsilon(t)$ defined by

$$\Psi^\varepsilon(t)(f_1,f_2) := \frac{1}{\varepsilon}\int_0^t U^\varepsilon(t-s)\Gamma(f_1(s),f_2(s))\,\mathrm{d}s. \tag{5.10}$$

We recall that χ is a fixed, compactly supported function of the interval $(-1,1)$, equal to one on $[-1/2,1/2]$.

Lemma 5.10. *There exists $\kappa>0$ such that one can write*

$$U^\varepsilon(t) = \sum_{j=1}^4 U_j^\varepsilon(t) + U^{\varepsilon\sharp}(t)$$
$$\textit{with}\quad \widehat{U}_j^\varepsilon(t,\xi) := \widehat{U}_j\Big(\frac{t}{\varepsilon^2},\varepsilon\xi\Big)\quad\textit{and}\quad \widehat{U}^{\varepsilon\sharp}(t,\xi) := \widehat{U}^\sharp\Big(\frac{t}{\varepsilon^2},\varepsilon\xi\Big),$$

where for $1\leqslant j\leqslant 4$,

$$\widehat{U}_j(t,\xi) := \chi\left(\frac{|\xi|}{\kappa}\right)e^{t\lambda_j(\xi)}P_j(\xi)$$

with λ_j satisfying

$$\begin{aligned}
&\lambda_j(\xi) = i\alpha_j|\xi| - \beta_j|\xi|^2 + \gamma_j(|\xi|),\\
&\alpha_1>0\,,\quad \alpha_2>0\,,\quad \alpha_3=\alpha_4=0\,,\quad \beta_j>0,\\
&\gamma_j(|\xi|) =_{|\xi|\to 0} O(|\xi|^3)\quad\textit{and}\quad \gamma_j(|\xi|)\leqslant \beta_j|\xi|^2/2\quad\textit{for}\quad |\xi|\leqslant\kappa,
\end{aligned}\tag{5.11}$$

and
$$P_j(\xi) = P_j^0\left(\frac{\xi}{|\xi|}\right) + |\xi| P_j^1\left(\frac{\xi}{|\xi|}\right) + |\xi|^2 P_j^2(\xi),$$
with P_j^n bounded linear operators on L^2_v with operator norms uniform for $|\xi| \leqslant \kappa$.

We also have that the orthogonal projector π onto $\operatorname{Ker} L$ *(see (1.12)) satisfies*
$$\pi = \sum_{j=1}^4 P_j^0\left(\frac{\xi}{|\xi|}\right)$$
and is independent of $\xi/|\xi|$.

Moreover, $P_j^0(\xi/|\xi|)$, $P_j^1(\xi/|\xi|)$ and $P_j^2(\xi)$ are bounded from L^2_v into $L^2_v(\langle v\rangle^\ell)$ uniformly in $|\xi| \leqslant \kappa$ for any $\ell \geqslant 0$.

Finally, for any $\ell \geqslant 0$, $\widehat{U}^\sharp$ satisfies
$$\|\widehat{U}^\sharp\|_{L^2_v(\langle v\rangle^\ell)\to L^2_v(\langle v\rangle^\ell)} \leqslant C e^{-\alpha t} \tag{5.12}$$
for some positive constants C and α independent of t and ξ.

Proof. The decomposition of $\widehat{U}^\varepsilon(t)$ follows that of $\widehat{U}^1(t)$: We recall that according to [60, Theorem 3.2 and Remark 5.2], one can write
$$\widehat{U}^1(t,\xi) = \sum_{j=1}^4 \widehat{U}_j(t,\xi) + \widehat{U}^\sharp(t,\xi),$$
where for $1 \leqslant j \leqslant 4$,
$$\widehat{U}_j(t,\xi) := \chi\left(\frac{|\xi|}{\kappa}\right) e^{t\lambda_j(\xi)} P_j(\xi)$$
and $\lambda_j(\xi) \in \mathbf{C}$ are the eigenvalues of $\widehat{\Lambda}_1(\xi)$ with associated eigenprojections $P_j(\xi)$ on L^2_v, satisfying the expansions stated in the lemma. The fact that $\pi = \sum_{j=1}^4 P_j^0\left(\frac{\xi}{|\xi|}\right)$ also comes from [60, Theorem 3.2].

Let us now prove that $P_j^0(\xi/|\xi|)$, $P_j^1(\xi/|\xi|)$ and $P_j^2(\xi)$ are bounded from L^2_v into $L^2_v(\langle v\rangle^\ell)$ uniformly in $|\xi| \leqslant \kappa$ for any $\ell \geqslant 0$. We first prove that this property is satisfied for $P_j(\xi)$. Recall that for $|\xi| \leqslant \kappa$,
$$\widehat{\Lambda}_1(\xi) P_j(\xi) = \lambda_j(\xi) P_j(\xi).$$
Thanks to the splitting $\Lambda_1 = \mathcal{A}_1 + \mathcal{B}_1$ introduced in Section 5, denoting $\widehat{\mathcal{B}}_1(\xi) := -i\xi\cdot v + \mathcal{B}_1$, we have for $|\xi| \leqslant \kappa$:
$$P_j(\xi) = (\lambda_j(\xi) - \widehat{\mathcal{B}}_1(\xi))^{-1} \mathcal{A}_1 P_j(\xi).$$
The dissipative properties of $\mathcal{B}_1$ in $L^2_v(\langle v\rangle^\ell)$ and the regularization properties of $\mathcal{A}_1$ (from L^2_v into $L^2_v(\langle v\rangle^\ell)$) established respectively in Lemmas 5.4 and 5.3 and the fact that from [60, Theorem 3.2], we already know that $P_j(\xi)$ is uniformly bounded in $|\xi| \leqslant \kappa$ from L^2_v into itself, imply that $P_j(\xi)$ is bounded from L^2_v into $L^2_v(\langle v\rangle^\ell)$ for any $\ell \geqslant 0$ and uniformly in $|\xi| \leqslant \kappa$. To conclude that the same properties hold for $P_j^0(\xi/|\xi|)$, $P_j^1(\xi/|\xi|)$ and $P_j^2(\xi)$, we notice that $P_j^0(\xi/|\xi|)$, $P_j^1(\xi/|\xi|)$ are given by explicit formula (see the proof of Theorem 3.2 in [60]) that clearly define bounded operators from L^2_v into $L^2_v(\langle v\rangle^\ell)$ for $|\xi| \leqslant \kappa$.

Finally, the estimate (5.12) comes from [60, Remark 5.2] for $\ell = 0$. We can also prove it for any $\ell \geqslant 0$ thanks to Duhamel formula applied with the splitting $\Lambda_1 = \mathcal{A}_1 + \mathcal{B}_1$ introduced in Section 5. We write that
$$\begin{aligned} U^\sharp(t) &= U^1(t) \mathcal{F}_x^{-1}\left(\operatorname{Id} - \chi\left(\frac{|\xi|}{\kappa}\right) e^{t\lambda_j(\xi)} P_j(\xi)\right)\mathcal{F}_x \\ &= \Big(S_{\mathcal{B}_1}(t) + (S_{\mathcal{B}_1} * \mathcal{A}_1 U^1)(t)\Big) \mathcal{F}_x^{-1}\left(\operatorname{Id} - \chi\left(\frac{|\xi|}{\kappa}\right) e^{t\lambda_j(\xi)} P_j(\xi)\right)\mathcal{F}_x. \end{aligned}$$
We are able to get the wanted estimate in $L^2_v(\langle v\rangle^\ell)$ thanks to the uniform boundedness in $|\xi| \leqslant \kappa$ of the projectors $P_j(\xi)$ in L^2_v, the dissipativity properties of $\mathcal{B}_1$ in $L^2_v(\langle v\rangle^\ell)$ and the regularization properties of $\mathcal{A}_1$ (from L^2_v to $L^2_v(\langle v\rangle^\ell)$) established respectively in Lemmas 5.4 and 5.3. ∎

Remark 5.11. Denoting

$$\widetilde{P}_j\left(\xi, \frac{\xi}{|\xi|}\right) := P_j^1\left(\frac{\xi}{|\xi|}\right) + |\xi| P_j^2(\xi) \tag{5.13}$$

for $1 \leqslant j \leqslant 4$, we can further split $\widehat{U}_j^\varepsilon(t)$ into four parts (a main part and three remainder terms):

$$U_j^\varepsilon = U_{j0}^\varepsilon + U_{j0}^{\varepsilon\sharp} + U_{j1}^\varepsilon + U_{j2}^\varepsilon$$

where

$$\begin{aligned}
\widehat{U}_{j0}^\varepsilon(t,\xi) &:= e^{i\alpha_j|\xi|\frac{t}{\varepsilon} - \beta_j t|\xi|^2} P_j^0\left(\frac{\xi}{|\xi|}\right),\\
\widehat{U}_{j0}^{\varepsilon\sharp}(t,\xi) &:= \left(\chi\left(\frac{\varepsilon|\xi|}{\kappa}\right) - 1\right) e^{i\alpha_j|\xi|\frac{t}{\varepsilon} - \beta_j t|\xi|^2} P_j^0\left(\frac{\xi}{|\xi|}\right),\\
\widehat{U}_{j1}^\varepsilon(t,\xi) &:= \chi\left(\frac{\varepsilon|\xi|}{\kappa}\right) e^{i\alpha_j|\xi|\frac{t}{\varepsilon} - \beta_j t|\xi|^2} \left(e^{t\frac{\gamma_j(\varepsilon|\xi|)}{\varepsilon^2}} - 1\right) P_j^0\left(\frac{\xi}{|\xi|}\right),\\
\widehat{U}_{j2}^\varepsilon(t,\xi) &:= \chi\left(\frac{\varepsilon|\xi|}{\kappa}\right) e^{i\alpha_j|\xi|\frac{t}{\varepsilon} - \beta_j t|\xi|^2 + t\frac{\gamma_j(\varepsilon|\xi|)}{\varepsilon^2}} \varepsilon|\xi| \widetilde{P}_j\left(\varepsilon\xi, \frac{\xi}{|\xi|}\right).
\end{aligned}$$

One can notice that $U_{30} := U_{30}^\varepsilon$ and $U_{40} := U_{40}^\varepsilon$ do not depend on ε since $\alpha_3 = \alpha_4 = 0$. We set

$$U := U_{30} + U_{40}.$$

We shall see that the operator $U(t)$ is in some sense the limit of $U^\varepsilon(t)$ (see Lemma 5.14).

The decomposition of the semigroup $U^\varepsilon(t)$ also gives us a decomposition of the operator $\Psi_\varepsilon(t)$ defined in (5.10) (see [24, Lemma A.4] and its proof).

Lemma 5.12. *The following decomposition holds*

$$\Psi^\varepsilon = \sum_{j=1}^4 \Psi_j^\varepsilon + \Psi^{\varepsilon\sharp}$$

with

$$\widehat{\Psi}^{\varepsilon\sharp}(t)(f_1,f_2) := \frac{1}{\varepsilon}\int_0^t \widehat{U}^{\varepsilon\sharp}(t-s)\widehat{\Gamma}(f_1(s),f_2(s))\,\mathrm{d}s$$

and

$$\Psi_j^\varepsilon = \Psi_{j0}^\varepsilon + \Psi_{j0}^{\varepsilon\sharp} + \Psi_{j1}^\varepsilon + \Psi_{j2}^\varepsilon$$

where

$$\mathcal{F}_x\left(\Psi_{j0}^\varepsilon(t)(f,f)\right)(\xi) := \int_0^t e^{i\alpha_j|\xi|\frac{t-s}{\varepsilon} - \beta_j(t-s)|\xi|^2} |\xi| P_j^1\left(\frac{\xi}{|\xi|}\right)\widehat{\Gamma}(f_1,f_2)(s)\,\mathrm{d}s,$$

$$\mathcal{F}_x\left(\Psi_{j0}^{\varepsilon\sharp}(t)(f,f)\right)(\xi) := \left(\chi\left(\frac{\varepsilon|\xi|}{\kappa}\right) - 1\right)\int_0^t e^{i\alpha_j|\xi|\frac{t-s}{\varepsilon} - \beta_j(t-s)|\xi|^2} |\xi| P_j^1\left(\frac{\xi}{|\xi|}\right)\widehat{\Gamma}(f_1,f_2)(s)\,\mathrm{d}s,$$

$$\begin{aligned}
&\mathcal{F}_x\left(\Psi_{j1}^\varepsilon(t)(f,f)\right)(\xi)\\
&\quad := \chi\left(\frac{\varepsilon|\xi|}{\kappa}\right)\int_0^t e^{i\alpha_j|\xi|\frac{t-s}{\varepsilon} - \beta_j(t-s)|\xi|^2}\left(e^{(t-s)\frac{\gamma_j(\varepsilon|\xi|)}{\varepsilon^2}} - 1\right)|\xi| P_j^1\left(\frac{\xi}{|\xi|}\right)\widehat{\Gamma}(f_1,f_2)(s)\,\mathrm{d}s,\\
&\mathcal{F}_x\left(\Psi_{j2}^\varepsilon(t)(f,f)\right)(\xi)\\
&\quad := \chi\left(\frac{\varepsilon|\xi|}{\kappa}\right)\int_0^t e^{i\alpha_j|\xi|\frac{t-s}{\varepsilon} - \beta_j(t-s)|\xi|^2 + (t-s)\frac{\gamma_j(\varepsilon|\xi|)}{\varepsilon^2}} \varepsilon|\xi|^2 P_j^2(\varepsilon\xi)\widehat{\Gamma}(f_1,f_2)(s)\,\mathrm{d}s.
\end{aligned}$$

Remark 5.13. Let us notice that as in Remark 5.11, there holds

$$\Psi_{30}^\varepsilon = \Psi_{30} \quad \text{and} \quad \Psi_{40}^\varepsilon = \Psi_{40}$$

and we set

$$\Psi := \Psi_{30} + \Psi_{40}.$$

5.4. **Limit operators $U(t)$ and $\Psi(t)$.** The following lemma studies the limit of $U^\varepsilon(t)$ as ε goes to 0, its proof is completely similar as the one of [24, Lemma 3.5], the only difference being that we use that the projectors are bounded from L^2_v into $L^2_v\left(\langle v\rangle^{3(\frac{\gamma}{2}+1)}\right)$ (see Lemma 5.10), we thus skip the proof.

Lemma 5.14. *Let f be a well-prepared data as defined in (1.25). Then, we have that*

$$\|(U^\varepsilon(t)-U(t))f\|_{L^\infty_t(\mathscr{X})} \lesssim \|f\|_{H^3_x L^2_v} \tag{5.14}$$

and

$$\|(U^\varepsilon(t)-U(t))f\|_{L^\infty_t(\mathscr{X})} \lesssim \varepsilon\|f\|_{H^4_x L^2_v}. \tag{5.15}$$

In the following lemma, we study the convergence of $\Psi^\varepsilon(t)(f,f)$ towards $\Psi(t)(f,f)$ when f is a well-prepared data and its associated macroscopic quantities solve the limit system (1.24). The proof is similar to the one of [24, Lemma 4.1], we thus omit the proof (notice that this result relies on refined estimates on quantities related to f that can be found in [24, Lemmas B.6 and B.7]).

Lemma 5.15. *Consider f a well-prepared data as defined in (1.25) with associated macroscopic quantities solving the limit system (1.24) on $\mathbf{R}^+$ and with mean free initial data $(\rho_0,u_0,\theta_0)\in H^3_x$ and associated kinetic distribution $f_0\in\mathscr{X}$ (as in (1.27)) satisfying $\|f_0\|_{\mathscr{X}}\leqslant\eta_1$ (so that f is defined globally in time), then*

$$\|\Psi^\varepsilon(t)(f,f)-\Psi(t)(f,f)\|_{L^\infty_t(\mathscr{X})} \lesssim \varepsilon\, C(\|f_0\|_{H^3_x L^2_v}),$$

where $C(\|f_0\|_{H^3_x L^2_v})$ is a constant only depending on $\|f_0\|_{H^3_x L^2_v}$.

5.5. **Decay estimates on the linearized Landau semigroup.** We recall that π is the projector onto the kernel of L and is given in (1.12) and that the spaces $\mathscr{Y}_1$ and $(\mathscr{Z}^\varepsilon_i)'$ are respectively defined in (1.20) and (2.15). From Lemma 5.10, as in [24, Lemma 3.2], we can prove some new decay estimates on the linearized Landau semigroup:

Lemma 5.16. *Let $\sigma_2:=\min(\alpha,\beta_1,\dots,\beta_4)$ (where α and β_j for $j=1,\dots 4$ are defined in Lemma 5.10). Then, for any $\sigma\in(0,\sigma_2)$, we have*

$$\|U^\varepsilon(t)(\mathrm{Id}-\pi)\|_{\mathscr{X}\to\mathscr{X}} \lesssim \varepsilon\,\frac{e^{-\sigma t}}{\sqrt{t}}, \quad \forall\, t>0.$$

Combining this with Corollaries 5.8-5.9, one can deduce the following result which mixes decay and regularization estimates:

Corollary 5.17. *For any $\sigma\in(0,\min(\sigma_1,\sigma_2))$, there holds*

$$\|U^\varepsilon(t)(\mathrm{Id}-\pi)\|_{\mathscr{X}\to\mathscr{Y}_1} \lesssim \varepsilon\,\frac{e^{-\sigma t}}{\sqrt{t}}, \quad \forall\, t>0,$$

where σ_1 is defined in (5.5) and σ_2 in Lemma 5.16.

Proof. Let $\sigma\in(0,\min(\sigma_1,\sigma_2))$. From Duhamel formula, we have:

$$U^\varepsilon(t)(\mathrm{Id}-\pi) = S_{\mathcal{B}_\varepsilon}(t)(\mathrm{Id}-\pi) + (S_{\mathcal{B}_\varepsilon} * \mathcal{A}_\varepsilon U^\varepsilon)(t)(\mathrm{Id}-\pi).$$

From Lemma 5.5, since $\sigma<\sigma_1$, we have:

$$\|S_{\mathcal{B}_\varepsilon}(t)(\mathrm{Id}-\pi)\|_{\mathscr{X}\to\mathscr{Y}_1} \lesssim \varepsilon\frac{e^{-\sigma t}}{\sqrt{t}}\|\mathrm{Id}-\pi\|_{\mathscr{X}\to\mathscr{X}} \lesssim \varepsilon\frac{e^{-\sigma t}}{\sqrt{t}}.$$

For the second term, we use Lemmas 5.3 and 5.5:

$$\begin{aligned}\|(S_{\mathcal{B}_\varepsilon} * \mathcal{A}_\varepsilon U^\varepsilon)(t)(\mathrm{Id}-\pi)\|_{\mathscr{X}\to\mathscr{Y}_1} &\lesssim \int_0^t \|S_{\mathcal{B}_\varepsilon}(t-s)\mathcal{A}_\varepsilon U^\varepsilon(s)(\mathrm{Id}-\pi)\|_{\mathscr{X}\to\mathscr{Y}_1}\,\mathrm{d}s\\ &\lesssim \int_0^t \frac{e^{-\sigma(t-s)}}{\sqrt{t-s}}\,\|U^\varepsilon(s)(\mathrm{Id}-\pi)\|_{\mathscr{X}\to\mathscr{X}}\,\mathrm{d}s.\end{aligned}$$

Finally, from Lemma 5.16,

$$\|(S_{\mathcal{B}_\varepsilon} * \mathcal{A}_\varepsilon U^\varepsilon)(t)(\mathrm{Id}-\pi)\|_{\mathscr{X}\to\mathscr{Y}_1} \lesssim \varepsilon\int_0^t \frac{e^{-\sigma(t-s)}}{\sqrt{t-s}}\frac{e^{-\sigma s}}{\sqrt{s}}\,\mathrm{d}s \lesssim \varepsilon\, e^{-\sigma t},$$

which yields the conclusion. □

By using an interpolation argument, one can deduce the following result:

Lemma 5.18. *For any $\sigma \in (0, \min(\sigma_0, \sigma_1, \sigma_2))$, we have:*

$$\|U^\varepsilon(t)(\mathrm{Id} - \pi)\|_{(\mathscr{X}_1^\varepsilon)' \to (\mathscr{X}_1^\varepsilon)'} \lesssim \sqrt{\varepsilon}\, \frac{e^{-\sigma t}}{t^{1/4}}, \quad \forall\, t > 0,$$

where σ_0, σ_1 and σ_2 are respectively defined in Proposition 3.2, in (5.5) and Lemma 5.16.

Proof. Step 1. First, by using an enlargement argument (from [31]), we prove that

$$\|U^\varepsilon(t)(\mathrm{Id} - \Pi)\|_{(\mathscr{X}_2^\varepsilon)' \to (\mathscr{X}_2^\varepsilon)'} \lesssim e^{-\sigma t}. \tag{5.16}$$

For sake of completeness and in order to carefully handle the ε-dependencies, we write the proof. From Duhamel formula, we have that

$$U^\varepsilon(t) = \sum_{j=0}^{3} V_j^\varepsilon(t) + (U^\varepsilon * V_3^\varepsilon)(t).$$

Moreover, we have $U^\varepsilon(t)(\mathrm{Id} - \Pi) = (\mathrm{Id} - \Pi) U^\varepsilon(t)$. Then, from Corollary 5.8 and the fact that $\Pi \in \mathscr{B}((\mathscr{X}_2^\varepsilon)')$, for any $j = 0, \dots, 3$, we have:

$$\|(\mathrm{Id} - \Pi) V_j^\varepsilon(t)\|_{(\mathscr{X}_2^\varepsilon)' \to (\mathscr{X}_2^\varepsilon)'} \lesssim e^{-\sigma t/\varepsilon^2}. \tag{5.17}$$

For the last term, using that $\mathscr{X} \hookrightarrow (\mathscr{X}_2^\varepsilon)'$ (independently of ε) and Theorem 3.1, we have:

$$\begin{aligned}\|(\mathrm{Id} - \Pi)(U^\varepsilon * V_3^\varepsilon)(t)\|_{(\mathscr{X}_2^\varepsilon)' \to (\mathscr{X}_2^\varepsilon)'} &\lesssim \int_0^t \|(\mathrm{Id} - \Pi) U^\varepsilon(t-s) V_3^\varepsilon(s)\|_{(\mathscr{X}_2^\varepsilon)' \to \mathscr{X}}\, \mathrm{d}s \\ &\lesssim \int_0^t e^{-\sigma(t-s)}\, \|V_3^\varepsilon(s)\|_{(\mathscr{X}_2^\varepsilon)' \to \mathscr{X}}\, \mathrm{d}s.\end{aligned}$$

Corollary 5.9 allows us to conclude that

$$\|(\mathrm{Id} - \Pi)(U^\varepsilon * V_3^\varepsilon)(t)\|_{(\mathscr{X}_2^\varepsilon)' \to (\mathscr{X}_2^\varepsilon)'} \lesssim \int_0^t e^{-\sigma(t-s)} e^{-\sigma s/\varepsilon^2}\, \mathrm{d}s \lesssim e^{-\sigma t}. \tag{5.18}$$

From estimates (5.17) and (5.18), we can conclude that (5.16) holds.

Step 2. From (5.16) and the fact that $U^\varepsilon(t)\Pi = \Pi \in \mathscr{B}((\mathscr{X}_2^\varepsilon)')$, we deduce

$$\|U^\varepsilon(t)\|_{(\mathscr{X}_2^\varepsilon)' \to (\mathscr{X}_2^\varepsilon)'} \lesssim 1.$$

Since $\pi \in \mathscr{B}((\mathscr{X}_2^\varepsilon)')$, it implies that

$$\|U^\varepsilon(t)(\mathrm{Id} - \pi)\|_{(\mathscr{X}_2^\varepsilon)' \to (\mathscr{X}_2^\varepsilon)'} \lesssim 1.$$

Consequently, combining this with Lemma 5.16, we can conclude the proof by interpolation because from (2.16), we have $(\mathscr{X}_1^\varepsilon)' = [\mathscr{X}, (\mathscr{X}_2^\varepsilon)']_{1/2,2}$. □

6. Hydrodynamical limit

We first state a quantitative result which provides estimates on the difference on the solution g^ε to the Landau equation constructed in Theorem 1.1 and the solution g defined in (1.28) whose first macroscopic quantities are solution to the fluid system (1.24). As explained in Subsection 6.2, this theorem combined with a density argument allows to prove Theorem 1.5. It is important to notice that thanks to the estimates obtained on the kinetic equation in Theorem 1.1, under some suitable smallness assumptions on the initial data of both kinetic and fluid equations, only extra-regularity in x on the initial data of the fluid system is needed to obtain a quantitative rate of convergence in ε, as can be seen in (6.1) and (6.2).

Theorem 6.1. *Let $g^\varepsilon_{\rm in} \in \mathscr{X} \cap (\operatorname{Ker}\Lambda_\varepsilon)^\perp$ for $\varepsilon \in (0,1)$ such that $\|g^\varepsilon_{\rm in}\|_{\mathscr{X}} \leqslant \eta_0$ (where η_0 is defined in Theorem 1.1) and $g^\varepsilon \in L^\infty_t(\mathscr{X})$ being the associated solutions of (1.7) with initial data $g^\varepsilon_{\rm in}$ constructed in Theorem 1.1. Consider also $g_0 \in H^{3+\delta}_x L^2_v \cap (\operatorname{Ker}\Lambda_\varepsilon)^\perp$ for some $\delta \in [0,1/2]$ such that $\|g_0\|_{\mathscr{X}} \leqslant \eta_1$ and g defined respectively in (1.27) and (1.28) (where we recall that η_1 has been chosen small enough so that g is defined globally in time).*

There exists $\eta_2 \in (0, \min(\eta_0,\eta_1))$ such that if $\max\left(\|g^\varepsilon_{\rm in}\|_{\mathscr{X}}, \|g_0\|_{\mathscr{X}}\right) \leqslant \eta_2$, then we have

$$\|g^\varepsilon - g\|_{L^\infty_t(\mathscr{X})} \lesssim \varepsilon^\delta C\left(\|g_0\|_{H^{3+\delta}_x L^2_v}, \|g^\varepsilon_{\rm in}\|_{\mathscr{X}}\right) + \|g^\varepsilon_{\rm in} - g_0\|_{\mathscr{X}} \tag{6.1}$$

and

$$\|g^\varepsilon - g\|_{L^1_t(\mathscr{Y}_1)+L^\infty_t(\mathscr{X})} \lesssim \varepsilon^\delta C\left(\|g_0\|_{H^{3+\delta}_x L^2_v}, \|g^\varepsilon_{\rm in}\|_{\mathscr{X}}\right) + \|\pi g^\varepsilon_{\rm in} - g_0\|_{\mathscr{X}} \tag{6.2}$$

where $C\left(\|g_0\|_{H^{3+\delta}_x L^2_v}, \|g^\varepsilon_{\rm in}\|_{\mathscr{X}}\right)$ is a contant only depending on $\|g_0\|_{H^{3+\delta}_x L^2_v}$ and $\|g^\varepsilon_{\rm in}\|_{\mathscr{X}}$.

Remark 6.2. Since $g_0 \in \operatorname{Ker} L$, it decays better than any polynomial in velocity at infinity, it explains the fact that we only use classical Sobolev spaces for g_0 in the RHS of the above inequalities, as already noticed, we have

$$\|g_0\|_{H^3_x L^2_v} \lesssim \|g_0\|_{\mathscr{X}} \lesssim \|g_0\|_{H^3_x L^2_v}$$

and similar inequalities could be obtained for higher order Sobolev spaces.

Remark 6.3. We restrict ourselves to the case $\delta \in [0,1/2]$ in our estimates but one can of course suppose more regularity on the initial data g_0, notice however that we will still have a rate of $\sqrt{\varepsilon}$. It should be noted that we did not look for optimality in terms of rate in our estimates.

6.1. **Reformulation of the hydrodynamical problem.** Before starting the proof of Theorem 6.1, we reformulate the problem. Using the definition of the operator $\Psi^\varepsilon(t)$ in (5.10), we have that the solution g^ε of (1.7) constructed in Theorem 1.1 writes

$$g^\varepsilon(t) = U^\varepsilon(t) g^\varepsilon_{\rm in} + \Psi^\varepsilon(t)(g^\varepsilon, g^\varepsilon).$$

It also follows from [5] that given a well-prepared data $g_0 \in \mathscr{X}$ of the form (1.27), the function g defined in (1.28) satisfies

$$g(t) = U(t) g_0 + \Psi(t)(g,g), \tag{6.3}$$

where, as explained in Subsection 5.4, the operators $U(t)$ and $\Psi(t)$ (defined respectively in Remarks 5.11 and 5.13) are in some sense the limiting operators of $U^\varepsilon(t)$ and $\Psi^\varepsilon(t)$. Formulation (6.3) is thus a way to reformulate the fluid equation in a kinetic fashion.

L^∞_t-estimate. We first reformulate the problem in order to prove the estimate (6.1). To this end, we write the relation satisfied by $h^\varepsilon := g^\varepsilon - g$:

$$\begin{aligned} h^\varepsilon &= U^\varepsilon(t) g^\varepsilon_{\rm in} + \Psi^\varepsilon(t)(g^\varepsilon, g^\varepsilon) - U(t) g_0 - \Psi(t)(g,g) \\ &= (U^\varepsilon(t) - U(t)) g_0 + U^\varepsilon(t)(g^\varepsilon_{\rm in} - g_0) + (\Psi^\varepsilon(t) - \Psi(t))(g,g) \\ &\quad + \Psi^\varepsilon(t)((g^\varepsilon)^\perp, (g^\varepsilon)^\perp) + \Psi^\varepsilon(t)((g^\varepsilon)^\perp, \pi g^\varepsilon) + \Psi^\varepsilon(t)(\pi g^\varepsilon, (g^\varepsilon)^\perp) \\ &\quad + \Psi^\varepsilon(t)(\pi h^\varepsilon, \pi g^\varepsilon) + \Psi^\varepsilon(t)(g, \pi h^\varepsilon). \end{aligned} \tag{6.4}$$

In the next subsection, we are going to study each term in the RHS of the above equality. Some terms are going to vanish in the limit $\varepsilon \to 0$ and other ones will be absorbed in the LHS under suitable smallness assumptions on the initial data. Let us underline that the singularity in ε in the definition of Ψ^ε is going to be handled thanks to Lemma 5.16 which provides a gain of ε when the semigroup $U^\varepsilon(t)$ acts on microscopic quantities. Using that $\pi\Gamma(f_1, f_2) = 0$ for any suitable functions f_1, f_2, we are thus going to be able to remove the singularity in ε in the operator Ψ^ε. In what follows, we shall prove that:

- The three first terms tend to 0 as ε go to 0 (see Lemmas 6.4 and 6.5): For this purpose, we use that the limits of $U^\varepsilon(t)$ and $\Psi^\varepsilon(t)$ as $\varepsilon \to 0$ are $U(t)$ and $\Psi(t)$ (see Lemmas 5.14 and 5.15).

- The fourth, fifth and sixth terms tend to 0 as $\varepsilon \to 0$ (see Lemma 6.6) because those three terms involve the microscopic part of the kinetic solution g^ε, which provides us some extra smallness in ε in $L^2_t(\mathscr{Y}_1)$ thanks to Theorem 1.1.
- The last two terms are bounded by some quantity that involves the norms of the kinetic and fluid initial data multiplied by the $L^\infty_t(\mathscr{X})$-norm of h^ε (see Lemma 6.8). It will thus be absorbed in the LHS of the equality if initial data $g^\varepsilon_{\rm in}$ and g_0 are chosen to be small enough. Notice that one can not hope smallness in ε for those terms because they only involve macroscopic quantities.

$L^1_t + L^\infty_t$-estimate. We now reformulate the problem in order to prove (6.2). To this end, we introduce $R^\varepsilon(t) := U^\varepsilon(t)(g^\varepsilon_{\rm in})^\perp$ and we write that

$$U^\varepsilon(t)g^\varepsilon_{\rm in} - U(t)g_0 = (U^\varepsilon(t) - U(t))g_0 + U^\varepsilon(t)(\pi g^\varepsilon_{\rm in} - g_0) + R^\varepsilon(t).$$

From Corollary 5.17, we have that for $\sigma \in (0, \min(\sigma_1, \sigma_2))$,

$$\|R^\varepsilon(t)\|_{\mathscr{Y}_1} \lesssim \frac{\varepsilon}{\sqrt{t}} e^{-\sigma t} \|g^\varepsilon_{\rm in}\|_{\mathscr{X}} \tag{6.5}$$

and thus

$$\|R^\varepsilon\|_{L^1_t(\mathscr{Y}_1)} \lesssim \varepsilon \|g^\varepsilon_{\rm in}\|_{\mathscr{X}}. \tag{6.6}$$

To conclude, it is thus enough to prove that for $\delta \in [0, 1/2]$,

$$\|g^\varepsilon - g - R^\varepsilon\|_{L^\infty_t(\mathscr{X})} \lesssim \varepsilon^\delta C\left(\|g_0\|_{H^{3+\delta}_x L^2_v}, \|g^\varepsilon_{\rm in}\|_{\mathscr{X}}\right) + \|\pi g^\varepsilon_{\rm in} - g_0\|_{\mathscr{X}}.$$

We then write the relation satisfied by $\widetilde{h}^\varepsilon := g^\varepsilon - g - R^\varepsilon$:

$$\begin{aligned} \widetilde{h}^\varepsilon &= (U^\varepsilon(t) - U(t))g_0 + U^\varepsilon(t)(\pi g^\varepsilon_{\rm in} - g_0) + (\Psi^\varepsilon(t) - \Psi(t))(g, g) \\ &\quad + \Psi^\varepsilon(t)((g^\varepsilon)^\perp, (g^\varepsilon)^\perp) + \Psi^\varepsilon(t)((g^\varepsilon)^\perp, \pi g^\varepsilon) + \Psi^\varepsilon(t)(\pi g^\varepsilon, (g^\varepsilon)^\perp) \\ &\quad + \Psi^\varepsilon(t)(\pi R^\varepsilon, \pi g^\varepsilon) + \Psi^\varepsilon(t)(g, \pi R^\varepsilon) \\ &\quad + \Psi^\varepsilon(t)(\pi \widetilde{h}^\varepsilon, \pi g^\varepsilon) + \Psi^\varepsilon(t)(g, \pi \widetilde{h}^\varepsilon). \end{aligned} \tag{6.7}$$

In what follows, we shall study each term in the RHS of this equality, the ideas between this decomposition being the same as the ones explained after (6.4). Notice furthermore that thanks to estimate (6.5), we are also going to be able to prove that the seventh and eighth terms tend to 0 when $\varepsilon \to 0$ (see Lemma 6.7).

6.2. **Proofs of Theorems 6.1 and 1.5.** From now on, assumptions of Theorem 6.1 are supposed to hold. As explained above, in order to prove Theorem 6.1, we have to estimate the $L^\infty_t(\mathscr{X})$-norm of each term of the decompositions (6.4) and (6.7).

In both decompositions, concerning the first and third terms, Lemmas 5.14 and 5.15 immediately give by interpolation the following lemma:

Lemma 6.4. *We have: For any $t \geqslant 0$ and $\delta \in [0, 1/2]$,*

$$\|(U^\varepsilon(t) - U(t))g_0\|_{\mathscr{X}} + \|(\Psi^\varepsilon(t) - \Psi(t))(g, g)\|_{\mathscr{X}} \lesssim \varepsilon^\delta C\left(\|g_0\|_{H^{3+\delta}_x L^2_v}\right).$$

Concerning the second terms of (6.4) and (6.7), from Theorem 3.1, we have that $U^\varepsilon(t)$ is bounded in $\mathscr{X}$ uniformly in time and ε. As a consequence, we obtain:

Lemma 6.5. *We have:*

$$\|U^\varepsilon(t)(g^\varepsilon_{\rm in} - g_0)\|_{L^\infty_t(\mathscr{X})} \lesssim \|g^\varepsilon_{\rm in} - g_0\|_{\mathscr{X}} \quad \text{and} \quad \|U^\varepsilon(t)(\pi g^\varepsilon_{\rm in} - g_0)\|_{L^\infty_t(\mathscr{X})} \lesssim \|\pi g^\varepsilon_{\rm in} - g_0\|_{\mathscr{X}}.$$

The fourth, fifth and sixth terms of decompositions (6.4) and (6.7) are the most difficult ones to estimate. Indeed, they involve microscopic quantities, we thus have to be sharp in terms of regularity in velocity in order to obtain the following lemma:

Lemma 6.6. *We have: For any $t \geqslant 0$,*

$$\|\Psi^\varepsilon(t)((g^\varepsilon)^\perp, (g^\varepsilon)^\perp)\|_{\mathscr{X}} + \|\Psi^\varepsilon(t)((g^\varepsilon)^\perp, \pi g^\varepsilon)\|_{\mathscr{X}} + \|\Psi^\varepsilon(t)(\pi g^\varepsilon, (g^\varepsilon)^\perp)\|_{\mathscr{X}} \lesssim \sqrt{\varepsilon} \|g^\varepsilon_{\rm in}\|^2_{\mathscr{X}}.$$

Proof. In the whole proof, we fix $\sigma \in (0, \min(\sigma_0, \sigma_1, \sigma_2))$ where σ_j for $j = 0, 1, 2$ are respectively defined in Proposition 3.2, (5.5) and Lemma 5.16. We focus on the first term which is the most intricate. We are going to use (5.9) with $n = 2$ to decompose $\Psi^\varepsilon(t)$ into several parts, it yields

$$\begin{aligned}\Psi^\varepsilon(t)((g^\varepsilon)^\perp, (g^\varepsilon)^\perp) &= \sum_{j=0}^{2} \frac{1}{\varepsilon}\int_0^t V_j^\varepsilon(t-s)\Gamma((g^\varepsilon)^\perp, (g^\varepsilon)^\perp)(s)\,\mathrm{d}s \\ &\quad + \frac{1}{\varepsilon}\int_0^t (V_2^\varepsilon * \mathcal{A}_\varepsilon U^\varepsilon)(t-s)\Gamma((g^\varepsilon)^\perp, (g^\varepsilon)^\perp)(s)\,\mathrm{d}s \\ &=: \sum_{j=0}^{2} \Psi_j^\varepsilon(t)((g^\varepsilon)^\perp, (g^\varepsilon)^\perp) + \Psi_\star^\varepsilon(t)((g^\varepsilon)^\perp, (g^\varepsilon)^\perp).\end{aligned}$$

We first estimate $\Psi_j^\varepsilon(t)((g^\varepsilon)^\perp, (g^\varepsilon)^\perp)$ for $j = 0, 1, 2$. From Propositions 4.2 and Corollary 5.9, we obtain:

$$\|\Psi_j^\varepsilon(t)((g^\varepsilon)^\perp, (g^\varepsilon)^\perp)\|_{\mathscr{X}} \lesssim \frac{1}{\varepsilon}\int_0^t \frac{\varepsilon}{\sqrt{t-s}} e^{-\sigma(t-s)/\varepsilon^2} \|(g^\varepsilon)^\perp(s)\|_{\mathscr{X}} \|(g^\varepsilon)^\perp(s)\|_{\mathscr{Y}_1}\,\mathrm{d}s\,.$$

Then, from (1.22)-(1.23) in Theorem 1.1, we have:

$$\|\Psi_j^\varepsilon(t)((g^\varepsilon)^\perp, (g^\varepsilon)^\perp)\|_{\mathscr{X}} \lesssim \int_0^t \frac{e^{-\sigma(t-s)/\varepsilon^2}}{\sqrt{t-s}} \frac{e^{-3\sigma s/2}}{s^{1/4}} \|(g^\varepsilon)^\perp(s)\|_{\mathscr{Y}_1}^{1/2}\,\mathrm{d}s\,\|g_{\rm in}^\varepsilon\|_{\mathscr{X}}^{3/2}.$$

Using Hölder inequality and (1.22), we obtain

$$\begin{aligned}\|\Psi_j^\varepsilon(t)((g^\varepsilon)^\perp, (g^\varepsilon)^\perp)\|_{\mathscr{X}} &\lesssim \left\| \frac{1}{(t-s)^{1/2} s^{1/4}} \right\|_{L_s^{4/3}([0,t])} \left\| \|(g^\varepsilon)^\perp(s)\|_{\mathscr{Y}_1}^{1/2} \right\|_{L_s^4} \|g_{\rm in}^\varepsilon\|_{\mathscr{X}}^{3/2} \\ &\lesssim \left(\int_0^t \frac{1}{(t-s)^{2/3}} \frac{\mathrm{d}s}{s^{1/3}} \right)^{3/4} \|(g^\varepsilon)^\perp\|_{L_t^2(\mathscr{Y}_1)}^{1/2} \|g_{\rm in}^\varepsilon\|_{\mathscr{X}}^{3/2} \\ &\lesssim \sqrt{\varepsilon}\|g_{\rm in}^\varepsilon\|_{\mathscr{X}}^2.\end{aligned}$$

Let us now deal with $\Psi_\star^\varepsilon(t)((g^\varepsilon)^\perp, (g^\varepsilon)^\perp)$. Performing a change of variable and recalling that $\pi\Gamma((g^\varepsilon)^\perp, (g^\varepsilon)^\perp) = 0$, one can notice that

$$\Psi_\star^\varepsilon(t)((g^\varepsilon)^\perp, (g^\varepsilon)^\perp) = \frac{1}{\varepsilon}\int_0^t \int_0^s V_2^\varepsilon(t-s)\mathcal{A}_\varepsilon U^\varepsilon(s-\tau)(\mathrm{Id}-\pi)\Gamma((g^\varepsilon)^\perp, (g^\varepsilon)^\perp)(\tau)\,\mathrm{d}\tau\,\mathrm{d}s.$$

Remark then that Corollary 5.9 for $j = 2$ implies that for $\sigma' \in (\sigma, \sigma_1)$, we have

$$\|V_2^\varepsilon(t)\|_{(\mathscr{Z}_1^\varepsilon)' \to \mathscr{X}} \lesssim \frac{\sqrt{t}}{\varepsilon} e^{-\sigma' t/\varepsilon^2} \lesssim e^{-\sigma t/\varepsilon^2}.$$

Using now Lemma 5.3, it implies that

$$\begin{aligned}&\|\Psi_\star^\varepsilon(t)((g^\varepsilon)^\perp, (g^\varepsilon)^\perp)\|_{\mathscr{X}} \\ &\quad\lesssim \frac{1}{\varepsilon^3}\int_0^t \int_0^s e^{-\sigma(t-s)/\varepsilon^2} \|U^\varepsilon(s-\tau)(\mathrm{Id}-\pi)\Gamma((g^\varepsilon)^\perp, (g^\varepsilon)^\perp)(\tau)\|_{(\mathscr{Z}_1^\varepsilon)'}\,\mathrm{d}\tau\,\mathrm{d}s.\end{aligned}$$

Lemma 5.18 and the fact that $\mathscr{Y}_1' \hookrightarrow (\mathscr{Z}_1^\varepsilon)'$ (independently of ε) then imply that

$$\begin{aligned}&\|\Psi_\star^\varepsilon(t)((g^\varepsilon)^\perp, (g^\varepsilon)^\perp)\|_{\mathscr{X}} \\ &\quad\lesssim \frac{1}{\varepsilon^3}\int_0^t \int_0^s e^{-\sigma(t-s)/\varepsilon^2} \frac{\sqrt{\varepsilon}}{(s-\tau)^{1/4}} e^{-\sigma(s-\tau)} \|\Gamma((g^\varepsilon)^\perp, (g^\varepsilon)^\perp)(\tau)\|_{\mathscr{Y}_1'}\,\mathrm{d}\tau\,\mathrm{d}s.\end{aligned}$$

From Proposition 4.2 and the fact that $\|(g^\varepsilon)^\perp\|_{\mathscr{X}} \lesssim \|g^\varepsilon\|_{\mathscr{X}}$, we deduce that

$$\begin{aligned}&\|\Psi_\star^\varepsilon(t)((g^\varepsilon)^\perp, (g^\varepsilon)^\perp)\|_{\mathscr{X}} \\ &\quad\lesssim \frac{1}{\varepsilon^{5/2}}\int_0^t \int_0^s e^{-\sigma(t-s)/\varepsilon^2} \frac{e^{-\sigma(s-\tau)}}{(s-\tau)^{1/4}} \|g^\varepsilon(\tau)\|_{\mathscr{X}} \|(g^\varepsilon)^\perp(\tau)\|_{\mathscr{Y}_1}\,\mathrm{d}\tau\,\mathrm{d}s.\end{aligned}$$

Using Cauchy-Schwarz inequality in the variable τ and (1.22) in Theorem 1.1, we obtain

$$\|\Psi^\varepsilon_\star(t)((g^\varepsilon)^\perp,(g^\varepsilon)^\perp)\|_{\mathscr{X}} \lesssim \frac{1}{\varepsilon^{3/2}}\int_0^t e^{-\sigma(t-s)/\varepsilon^2}\,\mathrm{d}s\,\|g^\varepsilon_{\mathrm{in}}\|^2_{\mathscr{X}} \lesssim \sqrt{\varepsilon}\,\|g^\varepsilon_{\mathrm{in}}\|^2_{\mathscr{X}},$$

which concludes the proof of the term $\Psi^\varepsilon(t)((g^\varepsilon)^\perp,(g^\varepsilon)^\perp)$.

The proof for the second and third terms $\Psi^\varepsilon(t)((g^\varepsilon)^\perp,\pi g^\varepsilon)$ and $\Psi^\varepsilon(t)(\pi g^\varepsilon,(g^\varepsilon)^\perp)$ is completely similar once one has noticed that from Proposition 4.2,

$$\begin{aligned}\|\Gamma((g^\varepsilon)^\perp,\pi g^\varepsilon)\|_{\mathscr{Y}_1'} &+ \|\Gamma(\pi g^\varepsilon,(g^\varepsilon)^\perp)\|_{\mathscr{Y}_1'}\\ &\lesssim \|(g^\varepsilon)^\perp\|_{\mathscr{Y}_1}\|\pi g^\varepsilon\|_{\mathscr{X}} + \|(g^\varepsilon)^\perp\|_{\mathscr{X}}\|\pi g^\varepsilon\|_{\mathscr{Y}_1} \lesssim \|(g^\varepsilon)^\perp\|_{\mathscr{Y}_1}\|g^\varepsilon\|_{\mathscr{X}}\end{aligned}$$

where we used the facts that $\mathscr{Y}_1 \hookrightarrow \mathscr{X}$ and $\pi \in \mathscr{B}(\mathscr{X},\mathscr{Y}_1)$. □

For the proof of (6.2), we also need the following lemma:

Lemma 6.7. *We have: For any $t \geqslant 0$,*

$$\|\Psi^\varepsilon(t)(\pi R^\varepsilon,\pi g^\varepsilon)\|_{\mathscr{X}} + \|\Psi^\varepsilon(t)(g,\pi R^\varepsilon)\|_{\mathscr{X}} \lesssim \varepsilon\left(\|g^\varepsilon_{\mathrm{in}}\|_{\mathscr{X}} + \|g_0\|_{\mathscr{X}}\right)\|g^\varepsilon_{\mathrm{in}}\|_{\mathscr{X}}.$$

Proof. From Lemma 5.16, Proposition 4.5, the fact that $\pi \in \mathscr{B}(\mathscr{X},\mathscr{Y}_2)$ and $g = \pi g$, we obtain:

$$\begin{aligned}&\|\Psi^\varepsilon(t)(\pi R^\varepsilon,\pi g^\varepsilon)\|_{\mathscr{X}} + \|\Psi^\varepsilon(t)(g,\pi R^\varepsilon)\|_{\mathscr{X}}\\ &\qquad\lesssim \int_0^t \frac{e^{-\sigma(t-s)}}{\sqrt{t-s}}\|R^\varepsilon(s)\|_{\mathscr{X}}\left(\|g(s)\|_{\mathscr{X}} + \|g^\varepsilon(s)\|_{\mathscr{X}}\right)\mathrm{d}s.\end{aligned}$$

Using now (6.5), (1.22) from Theorem 1.1 and (1.29), we obtain:

$$\begin{aligned}&\|\Psi^\varepsilon(t)(\pi R^\varepsilon,\pi g^\varepsilon)\|_{\mathscr{X}} + \|\Psi^\varepsilon(t)(g,\pi R^\varepsilon)\|_{\mathscr{X}}\\ &\qquad\lesssim \varepsilon\int_0^t \frac{e^{-\sigma(t-s)}}{\sqrt{t-s}}\frac{e^{-\sigma s}}{\sqrt{s}}\,\mathrm{d}s\,\|g^\varepsilon_{\mathrm{in}}\|^2_{\mathscr{X}} + \varepsilon\int_0^t \frac{e^{-\sigma(t-s)}}{\sqrt{t-s}}\frac{e^{-\sigma s}}{\sqrt{s}}\,\mathrm{d}s\,\|g^\varepsilon_{\mathrm{in}}\|_{\mathscr{X}}\|g_0\|_{\mathscr{X}},\end{aligned}$$

which yields the final result. □

Concerning the last two terms of (6.4) and (6.7), we prove the following lemma:

Lemma 6.8. *For any $f \in L^\infty_t(\mathscr{X})$, we have: For any $t \geqslant 0$,*

$$\|\Psi^\varepsilon(t)(\pi f,\pi g^\varepsilon)\|_{\mathscr{X}} + \|\Psi^\varepsilon(t)(g,\pi f)\|_{\mathscr{X}} \lesssim \left(\|g^\varepsilon_{\mathrm{in}}\|_{\mathscr{X}} + \|g_0\|_{\mathscr{X}}\right)\|f\|_{L^\infty_t(\mathscr{X})}.$$

Proof. The proof is similar to the one of Lemma 6.7. We use Lemma 5.16, Proposition 4.5, the fact that $\pi \in \mathscr{B}(\mathscr{X},\mathscr{Y}_2)$ and $g = \pi g$, and we conclude thanks to (1.22) and (1.29). □

End of the proof of Theorem 6.1. Gathering results from Lemmas 6.4, 6.5, 6.6 and 6.8, we can conclude the proof of (6.1) by taking $\|g^\varepsilon_{\mathrm{in}}\|_{\mathscr{X}}$ and $\|g_0\|_{\mathscr{X}}$ small enough. Concerning (6.2), Lemmas 6.4, 6.5, 6.6, 6.7 and 6.8 imply that

$$\|g^\varepsilon - g - R^\varepsilon\|_{L^\infty_t(\mathscr{X})} \lesssim \varepsilon^\delta C\left(\|g_0\|_{H^{3+\delta}_x L^2_v},\|g^\varepsilon_{\mathrm{in}}\|_{\mathscr{X}}\right) + \|\pi g^\varepsilon_{\mathrm{in}} - g_0\|_{\mathscr{X}}.$$

The estimate (6.6) then allows to conclude the proof of (6.2). □

Proof of Theorem 1.5. As mentioned above, in order to obtain results of convergence in Theorem 1.5, we are going to use a density argument that is explained in what follows. We consider a smooth family $(g_{0,\eta})_{\eta\in(0,1)}$ of $(\mathrm{Ker}\,\Lambda_\varepsilon)^\perp$ such that for any $\eta \in (0,1)$, $g_{0,\eta}$ writes

$$g_{0,\eta}(x,v) := \sqrt{M}(v)\left(\rho_{0,\eta}(x) + u_{0,\eta}(x)\cdot v + \frac{|v|^2-3}{2}\theta_{0,\eta}(x)\right)$$
$$\text{with}\quad \nabla_x\cdot u_{0,\eta} = 0 \quad\text{and}\quad \rho_{0,\eta}+\theta_{0,\eta} = 0$$

and

$$\|g_0 - g_{0,\eta}\|_{\mathscr{X}} \leqslant \eta \quad\text{and}\quad \|g_{0,\eta}\|_{\mathscr{X}} \leqslant \eta_1, \quad \forall \eta\in(0,1). \tag{6.8}$$

We have stability for the Navier-Stokes-Fourier system (see for example [24, Appendix B.3]), namely, we know that

$$g_\eta(t) := U(t)g_{0,\eta} + \Psi(t)(g_\eta, g_\eta)$$

satisfies

$$\lim_{\eta\to 0} \|g_\eta - g\|_{L^\infty_t(\mathscr{X})} = 0. \tag{6.9}$$

Then, to study the convergence of g^ε towards g, we write

$$g^\varepsilon - g = g^\varepsilon - g_\eta + g_\eta - g. \tag{6.10}$$

We then apply estimate (6.1) from Theorem 6.1 with $g_{0,\eta}$ and g_η instead of g_0 and g. Notice that for g_0 and η small enough, $g_{0,\eta}$ will also satisfy $\|g_{0,\eta}\|_{\mathscr{X}} \leqslant \eta_2$. Coming back to (6.10), we deduce that

$$\|g^\varepsilon - g\|_{L^\infty_t(\mathscr{X})} \lesssim \|g - g_\eta\|_{L^\infty_t(\mathscr{X})} + \varepsilon^\delta C(\|g^\varepsilon_{\rm in}\|_{\mathscr{X}}, \|g_{0,\eta}\|_{\mathscr{X}}) + \|g^\varepsilon_{\rm in} - g_0\|_{\mathscr{X}} + \|g_0 - g_{0,\eta}\|_{\mathscr{X}}.$$

Using (6.9), we can conclude the proof of (1.31). The proof of (1.33) is similar. □

Appendix A. Proof of Proposition 3.2

We start by recalling that for any suitable function g, we write

$$g = g^\perp + \pi g, \quad \pi g(x,v) = \rho_g(x)\sqrt{M}(v) + u_g(x)\cdot v\sqrt{M}(v) + \theta_g(x)\frac{(|v|^2-3)}{2}\sqrt{M}(v)$$

with ρ_g, u_g, θ_g defined in (1.14), (1.15) and (1.16). In what follows, we also use the following notations: $\rho[g] = \rho_g$, $u[g] = u_g$, $\theta[g] = \theta_g$.

For $g \in L^2_x(\mathbf{T}^3)$, we introduce the following notation: $\langle g\rangle := \int_{\mathbf{T}^3} g \,\mathrm{d}x$. Recall the following classical result (it can be proven straightforwardly thanks to Fourier transform). For any $\phi \in L^2_x(\mathbf{T}^3)$ with null mean (i.e. $\langle \phi\rangle = 0$), there is a unique solution $u \in H^2_x(\mathbf{T}^3)$ to the equation

$$-\Delta_x u = \phi \quad \text{in} \quad \mathbf{T}^3 \quad \text{with} \quad \langle u\rangle = 0.$$

Denote by $(-\Delta_x)^{-1}$ the following bounded operator:

$$\begin{array}{cccc} (-\Delta_x)^{-1} \ : & L^2_x(\mathbf{T}^3)\cap\{\langle\cdot\rangle = 0\} & \longrightarrow & H^2_x(\mathbf{T}^3)\cap\{\langle\cdot\rangle=0\} \\ & \phi & \longmapsto & u. \end{array} \tag{A.1}$$

Notice that in particular, we have that the operator $(-\Delta_x)^{-1}$ is also bounded from $L^2_x(\mathbf{T}^3)\cap\{\langle\cdot\rangle = 0\}$ into $H^1_x(\mathbf{T}^3)\cap\{\langle\cdot\rangle = 0\}$.

Let $f \in \operatorname{Dom}\Lambda_\varepsilon \cap (\operatorname{Ker}\Lambda_\varepsilon)^\perp$. Remark that in particular we have for $k=1,2,3$:

$$\langle \rho[f]\rangle = \langle u_k[f]\rangle = \langle\theta[f]\rangle = 0$$

where we have denoted by $u_k[f]$ the k-th coordinate of $u[f]$. We thus deduce that $(-\Delta_x)^{-1}\rho[f]$, $(-\Delta_x)^{-1}u_k[f]$, and $(-\Delta_x)^{-1}\theta[f]$ are well-defined, and more precisely we have

$$\|(-\Delta_x)^{-1}\rho[f]\|_{H^2_x} \lesssim \|\rho[f]\|_{L^2_x}$$
$$\|(-\Delta_x)^{-1}u_k[f]\|_{H^2_x} \lesssim \|u_k[f]\|_{L^2_x}$$
$$\|(-\Delta_x)^{-1}\theta[f]\|_{H^2_x} \lesssim \|\theta[f]\|_{L^2_x}.$$

In what follows, we shall use without further mention the following explicit computations: for $k \in \{1,2,3\}$,

$$\int_{\mathbf{R}^3} v_k^2 M \,\mathrm{d}v = 1, \quad \int_{\mathbf{R}^3} |v|^2 M\,\mathrm{d}v = 3,$$
$$\int_{\mathbf{R}^3} v_k^2|v|^2 M \,\mathrm{d}v = 5, \quad \int_{\mathbf{R}^3} |v|^4 M\,\mathrm{d}v = 15,$$
$$\int_{\mathbf{R}^3} v_k^2|v|^4 M \,\mathrm{d}v = 35, \quad \int_{\mathbf{R}^3} |v|^6 M\,\mathrm{d}v = 105,$$

as well as that for any odd polynomial function $p = p(v)$, one has $\int_{\mathbf{R}^3} p(v) M\,\mathrm{d}v = 0$.

We split the proof into five steps.

Step 1. Microscopic part. From (2.10) and the skew-adjointness of the transport operator one has

$$\begin{aligned}\langle \Lambda_\varepsilon f, f\rangle_{L^2_{x,v}} &= \frac{1}{\varepsilon^2}\langle Lf, f\rangle_{L^2_{x,v}} - \frac{1}{\varepsilon}\langle v\cdot\nabla_x f, f\rangle_{L^2_{x,v}} \\ &\leqslant -\frac{\sigma_L}{\varepsilon^2}\|f^\perp\|^2_{L^2_x(H^1_{v,*})}.\end{aligned} \tag{A.2}$$

Step 2. Energy estimate. Define for any $k\in\{1,2,3\}$ and any suitable function g

$$\psi_k[g] = \int_{\mathbf{R}^3} v_k\,\frac{(|v|^2-5)}{3}\,g\sqrt{M}\,\mathrm{d}v \tag{A.3}$$

and remark that

$$\psi_k[g] = \psi_k[g^\perp]. \tag{A.4}$$

We first compute

$$\begin{aligned}\theta[\Lambda_\varepsilon f] &= \frac{1}{\varepsilon^2}\theta[Lf^\perp] - \frac{1}{\varepsilon}\theta[v\cdot\nabla_x f] \\ &= -\frac{1}{\varepsilon}\nabla_x\cdot\int_{\mathbf{R}^3}\frac{(|v|^2-3)}{3}\,vf\sqrt{M}\,\mathrm{d}v \\ &= -\frac{1}{\varepsilon}\frac{2}{3}\nabla_x\cdot\int_{\mathbf{R}^3} vf\sqrt{M}\,\mathrm{d}v - \frac{1}{\varepsilon}\nabla_x\cdot\int_{\mathbf{R}^3}\frac{(|v|^2-5)}{3}\,vf\sqrt{M}\,\mathrm{d}v \\ &= -\frac{1}{\varepsilon}\frac{2}{3}\,\nabla_x\cdot u[f] - \frac{1}{\varepsilon}\nabla_x\cdot\psi[f],\end{aligned} \tag{A.5}$$

and using some aforementioned classical results on the moments of M, we also obtain

$$\begin{aligned}\psi_k[\Lambda_\varepsilon f] &= \frac{1}{\varepsilon^2}\psi_k[Lf^\perp] - \frac{1}{\varepsilon}\psi_k[v\cdot\nabla_x f] \\ &= \frac{1}{\varepsilon^2}\psi_k[Lf^\perp] - \frac{1}{\varepsilon}\partial_{x_\ell}\int_{\mathbf{R}^d} v_k v_\ell\,\frac{(|v|^2-5)}{3}\,f\sqrt{M}\,\mathrm{d}v \\ &= \frac{1}{\varepsilon^2}\psi_k[Lf^\perp] - \frac{1}{\varepsilon}\partial_{x_\ell}\int_{\mathbf{R}^d} v_k v_\ell\,\frac{(|v|^2-5)}{3}\left(\rho[f] + u[f]\cdot v + \theta[f]\frac{(|v|^2-3)}{2}\right)M\,\mathrm{d}v \\ &\quad - \frac{1}{\varepsilon}\partial_{x_\ell}\int_{\mathbf{R}^d} v_k v_\ell\,\frac{(|v|^2-5)}{3}\,f^\perp\sqrt{M}\,\mathrm{d}v \\ &= \frac{1}{\varepsilon^2}\psi_k[Lf^\perp] - \frac{1}{\varepsilon}\frac{5}{3}\partial_{x_k}\theta[f] - \frac{1}{\varepsilon}\partial_{x_\ell}\int_{\mathbf{R}^d} v_k v_\ell\,\frac{(|v|^2-5)}{3}\,f^\perp\sqrt{M}\,\mathrm{d}v.\end{aligned} \tag{A.6}$$

We now estimate the term

$$\left\langle\nabla_x(-\Delta_x)^{-1}\theta[\Lambda_\varepsilon f], \psi[f]\right\rangle_{L^2_x} + \left\langle\nabla_x(-\Delta_x)^{-1}\theta[f], \psi[\Lambda_\varepsilon f]\right\rangle_{L^2_x} =: I_1 + I_2.$$

We start by estimating I_1. First, using integrations by parts and (A.5), we have

$$\begin{aligned}\|\nabla_x(-\Delta_x)^{-1}\theta[\Lambda_\varepsilon f]\|^2_{L^2_x} &= \left\langle\theta[\Lambda_\varepsilon f], (-\Delta_x)^{-1}\theta[\Lambda_\varepsilon f]\right\rangle_{L^2_x} \\ &= -\frac{1}{\varepsilon}\left\langle\frac{2}{3}\,\nabla_x\cdot u[f] + \nabla_x\cdot\psi[f], (-\Delta_x)^{-1}\theta[\Lambda_\varepsilon f]\right\rangle_{L^2_x} \\ &= -\frac{1}{\varepsilon}\left\langle\frac{2}{3}u[f] + \psi[f], \nabla_x(-\Delta_x)^{-1}\theta[\Lambda_\varepsilon f]\right\rangle_{L^2_x}.\end{aligned}$$

Using now Cauchy-Schwarz inequality in x and (A.4), we obtain:

$$\|\nabla_x(-\Delta_x)^{-1}\theta[\Lambda_\varepsilon f]\|_{L^2_x} \lesssim \frac{1}{\varepsilon}\left(\|u[f]\|_{L^2_x} + \|\psi[f]\|_{L^2_x}\right).$$

Then, from (A.4) and Cauchy-Schwarz inequality in velocity, we have:

$$\|\psi[f]\|_{L^2_x} = \|\psi[f^\perp]\|_{L^2_x} \lesssim \|f^\perp\|_{L^2_{x,v}}.$$

We deduce that

$$|I_1| \lesssim \frac{1}{\varepsilon}\left(\|u[f]\|_{L^2_x} + \|f^\perp\|_{L^2_{x,v}}\right)\|f^\perp\|_{L^2_{x,v}}.$$

Concerning I_2, from the computation (A.6), we have:

$$\begin{aligned} I_2 &= \frac{1}{\varepsilon^2}\left\langle \partial_{x_k}(-\Delta_x)^{-1}\theta[f], \psi_k[Lf^\perp]\right\rangle_{L^2_x} - \frac{1}{\varepsilon}\frac{5}{3}\left\langle \partial_{x_k}(-\Delta_x)^{-1}\theta[f], \partial_{x_k}\theta[f]\right\rangle_{L^2_x} \\ &\quad - \frac{1}{\varepsilon}\left\langle \partial_{x_k}(-\Delta_x)^{-1}\theta[f], \partial_{x_\ell}\int_{\mathbf{R}^3} v_k v_\ell\,\frac{(|v|^2-5)}{3}\, f^\perp\sqrt{M}\,\mathrm{d}v\right\rangle_{L^2_x} \\ &=: I_{21} + I_{22} + I_{23}. \end{aligned}$$

We treat each term of the above splitting in succession. For I_{21}, we notice first that from the self-adjointness of L in L^2_v:

$$\|\psi[Lf^\perp]\|^2_{L^2_x} \lesssim \int_{\mathbf{T}^3}\left(\int_{\mathbf{R}^3} L\left(v_k\frac{(|v|^2-5)}{3}\sqrt{M}\right) f^\perp\,\mathrm{d}v\right)^2 \mathrm{d}x$$

so that from Cauchy-Schwarz inequality:

$$\|\psi[Lf^\perp]\|_{L^2_x} \lesssim \|f^\perp\|_{L^2_{x,v}}. \tag{A.7}$$

We thus have:

$$I_{21} \lesssim \frac{1}{\varepsilon^2}\|\nabla_x(-\Delta_x)^{-1}\theta[f]\|_{L^2_x}\|f^\perp\|_{L^2_{x,v}} \lesssim \frac{1}{\varepsilon^2}\|\theta[f]\|_{L^2_x}\|f^\perp\|_{L^2_{x,v}}$$

where we used (A.1). The term I_{22} is computed exactly thanks to an integration by parts:

$$I_{22} = \frac{1}{\varepsilon}\frac{5}{3}\left\langle \partial_{x_k}\partial_{x_k}(-\Delta_x)^{-1}(\theta[f]), \theta[f]\right\rangle_{L^2_x} = -\frac{1}{\varepsilon}\frac{5}{3}\|\theta[f]\|^2_{L^2_x}.$$

We bound the last term I_{23} using an integration by parts, (A.1) and Cauchy-Schwarz inequality:

$$\begin{aligned} I_{23} &= \frac{1}{\varepsilon}\left\langle \partial_{x_\ell}\partial_{x_k}(-\Delta_x)^{-1}(\theta[f]), \int_{\mathbf{R}^d} v_k v_\ell\,\frac{(|v|^2-5)}{3}\, f^\perp\sqrt{M}\,\mathrm{d}v\right\rangle_{L^2_x} \\ &\lesssim \frac{1}{\varepsilon}\|\nabla_x^2(-\Delta_x)^{-1}\theta[f]\|_{L^2_x}\|f^\perp\|_{L^2_{x,v}} \\ &\lesssim \frac{1}{\varepsilon}\|\theta[f]\|_{L^2_x}\|f^\perp\|_{L^2_{x,v}}. \end{aligned}$$

Finally, we have obtained for some $\kappa, C > 0$:

$$\begin{aligned} &\left\langle \nabla_x(-\Delta_x)^{-1}\theta[\Lambda_\varepsilon f], \psi[f]\right\rangle_{L^2_x} + \left\langle \nabla_x(-\Delta_x)^{-1}\theta[f], \psi[\Lambda_\varepsilon f]\right\rangle_{L^2_x} \\ &\qquad \leqslant -\frac{\kappa}{\varepsilon}\|\theta[f]\|^2_{L^2_x} + \frac{C}{\varepsilon}\|u[f]\|_{L^2_x}\|f^\perp\|_{L^2_{x,v}} + \frac{C}{\varepsilon^3}\|f^\perp\|^2_{L^2_{x,v}}. \end{aligned} \tag{A.8}$$

Step 3. Momentum estimate. Define, for any $k,\ell \in \{1,2,3\}$ and any suitable function g

$$\Theta_{k\ell}[g] = \begin{cases} \dfrac{1}{7}\displaystyle\int_{\mathbf{R}^3} v_k v_\ell |v|^2 g\sqrt{M}\,\mathrm{d}v & \text{if} \quad k \neq \ell, \\ \displaystyle\int_{\mathbf{R}^3}\left(\frac{1}{2} + v_k^2 - \frac{1}{2}|v|^2\right) g\sqrt{M}\,\mathrm{d}v & \text{if} \quad k = \ell, \end{cases} \tag{A.9}$$

and remark that, using some classical aforementioned results on the moments of M, we have

$$\Theta_{k\ell}[g] = \Theta_{k\ell}[g^\perp] \quad \text{if} \quad k \neq \ell \tag{A.10}$$

and

$$\begin{aligned} \Theta_{kk}[g] &= \Theta_{kk}[g^\perp] + \int_{\mathbf{R}^3}\left(\frac{1}{2} + v_k^2 - \frac{1}{2}|v|^2\right)\left(\rho[g] + u[g]\cdot v + \theta[g]\frac{(|v|^2-3)}{2}\right) M\,\mathrm{d}v \\ &= \Theta_{kk}[g^\perp] - \frac{1}{2}\theta[g]. \end{aligned} \tag{A.11}$$

We now compute for any $k \in \{1,2,3\}$, writing $f = \pi f + f^{\perp}$,

$$
\begin{aligned}
u_k[\Lambda_\varepsilon f] &= \frac{1}{\varepsilon^2} u_k[L f^{\perp}] - \frac{1}{\varepsilon} u_k[v \cdot \nabla_x f] \\
&= -\frac{1}{\varepsilon} \partial_{x_\ell} \int_{\mathbf{R}^3} v_k v_\ell \left(\rho[f] + u[f] \cdot v + \theta[f] \frac{(|v|^2 - 3)}{2} \right) M \,\mathrm{d}v \\
&\quad - \frac{1}{\varepsilon} \partial_{x_\ell} \int_{\mathbf{R}^3} v_k v_\ell f^{\perp} \sqrt{M} \,\mathrm{d}v \\
&= -\frac{1}{\varepsilon} \partial_{x_k} \rho[f] - \frac{1}{\varepsilon} \partial_{x_k} \theta[f] - \frac{1}{\varepsilon} \partial_{x_\ell} \int_{\mathbf{R}^3} v_k v_\ell f^{\perp} \sqrt{M} \,\mathrm{d}v.
\end{aligned} \tag{A.12}
$$

We also obtain

$$
\begin{aligned}
\Theta_{k\ell}[\Lambda_\varepsilon f] &= \frac{1}{\varepsilon^2} \Theta_{k\ell}[L f^{\perp}] - \frac{1}{\varepsilon} \Theta_{k\ell}[v \cdot \nabla_x \pi f] - \frac{1}{\varepsilon} \Theta_{k\ell}[v \cdot \nabla_x f^{\perp}] \\
&= \frac{1}{\varepsilon^2} \Theta_{k\ell}[L f^{\perp}] - \frac{1}{\varepsilon} \Theta_{k\ell} \left[v \cdot \nabla_x \left(u[f] \cdot v \sqrt{M} \right) \right] - \frac{1}{\varepsilon} \Theta_{k\ell}[v \cdot \nabla_x f^{\perp}].
\end{aligned} \tag{A.13}
$$

If $k \neq \ell$, then

$$
\begin{aligned}
&\Theta_{k\ell} \left[v \cdot \nabla_x \left(u[f] \cdot v \sqrt{M} \right) \right] \\
&= \partial_{x_p} u_q[f] \frac{1}{7} \int_{\mathbf{R}^3} v_k v_\ell v_p v_q |v|^2 M \,\mathrm{d}v \\
&= \partial_{x_k} u_\ell[f] \frac{1}{7} \int_{\mathbf{R}^3} v_k^2 v_\ell^2 |v|^2 M \,\mathrm{d}v + \partial_{x_\ell} u_k[f] \frac{1}{7} \int_{\mathbf{R}^3} v_k^2 v_\ell^2 |v|^2 M \,\mathrm{d}v \\
&= \partial_{x_k} u_\ell[f] + \partial_{x_\ell} u_k[f]
\end{aligned} \tag{A.14}
$$

because

$$
\int_{\mathbf{R}^3} v_k^2 v_\ell^2 |v|^2 M \,\mathrm{d}v = \int_{\mathbf{R}^3} v_k^2 v_\ell^2 (v_k^2 + v_\ell^2 + v_m^2) M \,\mathrm{d}v = 7.
$$

If $k = \ell$, then

(A.15)

$$
\begin{aligned}
&\Theta_{kk} \left[v \cdot \nabla_x \left(u[f] \cdot v \sqrt{M} \right) \right] \\
&\quad = \partial_{x_p} u_q[f] \int_{\mathbf{R}^3} \left(\frac{1}{2} + v_k^2 - \frac{1}{2}|v|^2 \right) v_p v_q M \,\mathrm{d}v \\
&\quad = \partial_{x_k} u_k[f] \int_{\mathbf{R}^3} \left(\frac{1}{2} + v_k^2 - \frac{1}{2}|v|^2 \right) v_k^2 M \,\mathrm{d}v + \sum_{p \neq k} \partial_{x_p} u_p[f] \int_{\mathbf{R}^3} \left(\frac{1}{2} + v_k^2 - \frac{1}{2}|v|^2 \right) v_p^2 M \,\mathrm{d}v \\
&\quad = \partial_{x_k} u_k[f] - \sum_{p \neq k} \partial_{x_p} u_p[f],
\end{aligned}
$$

because

$$
\int_{\mathbf{R}^3} \left(\frac{1}{2} + v_k^2 - \frac{1}{2}|v|^2 \right) v_k^2 M \,\mathrm{d}v = 1 \quad \text{and} \quad \int_{\mathbf{R}^3} \left(\frac{1}{2} + v_k^2 - \frac{1}{2}|v|^2 \right) v_p^2 M \,\mathrm{d}v = -1.
$$

We now estimate the term

$$
\left\langle \partial_{x_\ell} (-\Delta_x)^{-1} u_k[\Lambda_\varepsilon f], \Theta_{k\ell}[f] \right\rangle_{L^2_x} + \left\langle \partial_{x_\ell} (-\Delta_x)^{-1} u_k[f], \Theta_{k\ell}[\Lambda_\varepsilon f] \right\rangle_{L^2_x} =: J_1 + J_2.
$$

For J_1, we first notice that using (A.10) and (A.11) and Cauchy-Schwarz inequality, one can prove that

$$
\|\Theta_{k\ell}[f]\|_{L^2_x} \lesssim \|\theta[f]\|_{L^2_x} + \|f^{\perp}\|_{L^2_{x,v}}.
$$

Performing integrations by parts and using (A.12), we also have that

$$
\begin{aligned}
&\|\nabla_x (-\Delta_x)^{-1} u_k[\Lambda_\varepsilon f]\|_{L^2_x}^2 = \left\langle u_k[\Lambda_\varepsilon f], (-\Delta_x)^{-1} u_k[\Lambda_\varepsilon f] \right\rangle_{L^2_x} \\
&\quad = -\frac{1}{\varepsilon} \left\langle \partial_{x_k} \rho[f] + \partial_{x_k} \theta[f] + \partial_{x_\ell} \int_{\mathbf{R}^3} v_k v_\ell f^{\perp} \sqrt{M} \,\mathrm{d}v, (-\Delta_x)^{-1} u_k[\Lambda_\varepsilon f] \right\rangle_{L^2_x} \\
&\quad = \frac{1}{\varepsilon} \left\langle \rho[f] + \theta[f], \partial_{x_k} (-\Delta_x)^{-1} u_k[\Lambda_\varepsilon f] \right\rangle_{L^2_x}
\end{aligned}
$$

$$+\frac{1}{\varepsilon}\left\langle \int_{\mathbf{R}^3} v_k v_\ell f^\perp \sqrt{M}\,\mathrm{d}v, \partial_{x_\ell}(-\Delta_x)^{-1}u_k[\Lambda_\varepsilon f]\right\rangle_{L^2_x}.$$

From Cauchy-Schwarz inequality (in x and also in v for the third term), we obtain:

$$\|\nabla_x(-\Delta_x)^{-1}u_k[\Lambda_\varepsilon f]\|_{L^2_x} \lesssim \frac{1}{\varepsilon}\left(\|\rho[f]\|_{L^2_x} + \|\theta[f]\|_{L^2_x} + \|f^\perp\|_{L^2_{x,v}}\right).$$

Gathering the two previous estimates, by Cauchy-Schwarz inequality, we deduce

$$\begin{aligned}|J_1| &\lesssim \|(-\Delta_x)^{-1}u_k[\Lambda_\varepsilon f]\|_{H^1_x}\|\Theta_{k\ell}[f]\|_{L^2_x}\\ &\lesssim \frac{1}{\varepsilon}\left(\|\rho[f]\|_{L^2_x} + \|\theta[f]\|_{L^2_x} + \|f^\perp\|_{L^2_{x,v}}\right)\left(\|\theta[f]\|_{L^2_x} + \|f^\perp\|_{L^2_{x,v}}\right).\end{aligned}$$

For the term J_2, the computation made in (A.13) yields

$$\begin{aligned}J_2 &= \frac{1}{\varepsilon^2}\left\langle \partial_{x_\ell}(-\Delta_x)^{-1}u_k[f], \Theta_{k\ell}[Lf^\perp]\right\rangle_{L^2_x}\\ &\quad - \frac{1}{\varepsilon}\left\langle \partial_{x_\ell}(-\Delta_x)^{-1}u_k[f], \Theta_{k\ell}\left[v\cdot\nabla_x(u[f]\cdot v\sqrt{M})\right]\right\rangle_{L^2_x}\\ &\quad - \frac{1}{\varepsilon}\left\langle \partial_{x_\ell}(-\Delta_x)^{-1}u_k[f], \Theta_{k\ell}[v\cdot\nabla_x f^\perp]\right\rangle_{L^2_x}\\ &=: J_{21}+J_{22}+J_{23}.\end{aligned}$$

To bound J_{21}, we first notice that as in (A.7), one can prove that

$$\|\Theta_{k\ell}[Lf^\perp]\|_{L^2_x} \lesssim \|f^\perp\|_{L^2_{x,v}}.$$

Then, (A.1) and Cauchy-Schwarz inequality give

$$J_{21} \lesssim \frac{1}{\varepsilon^2}\|\partial_{x_\ell}(-\Delta_x)^{-1}u_k[f]\|_{L^2_x}\|f^\perp\|_{L^2_{x,v}} \lesssim \frac{1}{\varepsilon^2}\|u[f]\|_{L^2_x}\|f^\perp\|_{L^2_{x,v}}.$$

The term J_{22} is computed explicitly thanks to (A.14) and (A.15) and integrations by parts:

$$\begin{aligned}J_{22} &= -\frac{1}{\varepsilon}\sum_k\sum_{\ell\neq k}\left\langle \partial_{x_\ell}(-\Delta_x)^{-1}u_k[f], \partial_{x_k}u_\ell[f] + \partial_{x_\ell}u_k[f]\right\rangle_{L^2_x}\\ &\quad - \frac{1}{\varepsilon}\sum_k\left\langle \partial_{x_k}(-\Delta_x)^{-1}u_k[f], \partial_{x_k}u_k[f] - \sum_{p\neq k}\partial_{x_p}u_p[f]\right\rangle_{L^2_x}\\ &= -\frac{1}{\varepsilon}\sum_k\sum_{\ell\neq k}\left\langle -\partial_{x_k}\partial_{x_\ell}(-\Delta_x)^{-1}u_k[f], u_\ell[f]\right\rangle_{L^2_x}\\ &\quad - \frac{1}{\varepsilon}\sum_k\sum_{\ell\neq k}\left\langle -\partial_{x_\ell}\partial_{x_\ell}(-\Delta_x)^{-1}u_k[f], u_k[f]\right\rangle_{L^2_x}\\ &\quad - \frac{1}{\varepsilon}\sum_k\left\langle -\partial_{x_k}\partial_{x_k}(-\Delta_x)^{-1}u_k[f], u_k[f]\right\rangle_{L^2_x}\\ &\quad + \frac{1}{\varepsilon}\sum_k\sum_{p\neq k}\left\langle -\partial_{x_p}\partial_{x_k}(-\Delta_x)^{-1}u_k[f], u_p[f]\right\rangle_{L^2_x}\end{aligned}$$

so that

$$J_{22} = -\frac{1}{\varepsilon}\sum_k\left\langle -\Delta_x(-\Delta_x)^{-1}u_k[f], u_k[f]\right\rangle_{L^2_x} = -\frac{1}{\varepsilon}\sum_k\|u_k[f]\|^2_{L^2_x} = -\frac{1}{\varepsilon}\|u[f]\|^2_{L^2_x}.$$

The term J_{23} is treated thanks to an integration by parts, (A.1) and Cauchy-Schwarz inequality:

$$\begin{aligned}J_{23} &\lesssim \frac{1}{\varepsilon}\|\nabla_x^2(-\Delta_x)^{-1}u[f]\|_{L^2_x}\|f^\perp\|_{L^2_{x,v}}\\ &\lesssim \frac{1}{\varepsilon}\|u[f]\|_{L^2_x}\|f^\perp\|_{L^2_{x,v}}.\end{aligned}$$

Finally, we have obtained for some constants $\kappa, C > 0$

$$
\begin{aligned}
&\left\langle \partial_{x_\ell}(-\Delta_x)^{-1} u_k[\Lambda_\varepsilon f], \Theta_{k\ell}[f] \right\rangle_{L^2_x} + \left\langle \partial_{x_\ell}(-\Delta_x)^{-1} u_k[f], \Theta_{k\ell}[\Lambda_\varepsilon f] \right\rangle_{L^2_x} \\
&\qquad \leqslant -\frac{\kappa}{\varepsilon}\|u[f]\|^2_{L^2_x} + \frac{C}{\varepsilon}\|\rho[f]\|_{L^2_x}\|\theta[f]\|_{L^2_x} + \frac{C}{\varepsilon}\|\rho[f]\|_{L^2_x}\|f^\perp\|_{L^2_{x,v}} \\
&\qquad\quad + \frac{C}{\varepsilon}\|\theta[f]\|^2_{L^2_x} + \frac{C}{\varepsilon^3}\|f^\perp\|^2_{L^2_{x,v}}.
\end{aligned} \tag{A.16}
$$

Step 4. Mass estimate. We first compute

$$
\begin{aligned}
\rho[\Lambda_\varepsilon f] &= \frac{1}{\varepsilon^2}\rho[Lf^\perp] - \frac{1}{\varepsilon}\rho[v\cdot\nabla_x f] \\
&= -\frac{1}{\varepsilon}\nabla_x \cdot \int_{\mathbf{R}^d} v f\sqrt{M}\,\mathrm{d}v \\
&= -\frac{1}{\varepsilon}\nabla_x\cdot u[f].
\end{aligned} \tag{A.17}
$$

We now estimate the term

$$
\left\langle \partial_{x_k}(-\Delta_x)^{-1}\rho[\Lambda_\varepsilon f], u_k[f] \right\rangle_{L^2_x} + \left\langle \partial_{x_k}(-\Delta_x)^{-1}\rho[f], u_k[\Lambda_\varepsilon f] \right\rangle_{L^2_x} =: R_1 + R_2.
$$

For R_1, from the computation (A.17) and (A.1), we have

$$
R_1 \lesssim \|(-\Delta_x)^{-1}\rho[\Lambda_\varepsilon f]\|_{H^1_x}\|u[f]\|_{L^2_x} \lesssim \frac{1}{\varepsilon}\|u[f]\|^2_{L^2_x}.
$$

From (A.12) we rewrite the term R_2 as

$$
\begin{aligned}
R_2 &= -\frac{1}{\varepsilon}\left\langle \partial_{x_k}(-\Delta_x)^{-1}\rho[f], \partial_{x_k}\rho[f]\right\rangle_{L^2_x} - \frac{1}{\varepsilon}\frac{2}{3}\left\langle \partial_{x_k}(-\Delta_x)^{-1}\rho[f], \partial_{x_k}\theta[f]\right\rangle_{L^2_x} \\
&\quad - \frac{1}{\varepsilon}\left\langle \partial_{x_k}(-\Delta_x)^{-1}\rho[f], \partial_{x_\ell}\int_{\mathbf{R}^d} v_k v_\ell f^\perp\sqrt{M}\,\mathrm{d}v\right\rangle_{L^2_x} \\
&=: R_{21} + R_{22} + R_{23}.
\end{aligned}
$$

The first term R_{21} is computed exactly thanks to an integration by parts:

$$
R_{21} = -\frac{1}{\varepsilon}\left\langle -\partial_{x_k}\partial_{x_k}(-\Delta_x)^{-1}\rho[f], \rho[f]\right\rangle_{L^2_x} = -\frac{1}{\varepsilon}\|\rho[f]\|^2_{L^2_x}.
$$

The second one is estimated thanks to and integration by parts, Cauchy-Schwarz inequality and (A.1):

$$
\begin{aligned}
R_{22} &\lesssim \frac{1}{\varepsilon}\|\nabla^2_x(-\Delta_x)^{-1}\rho_f\|_{L^2_x}\|\theta[f]\|_{L^2_x} \\
&\lesssim \frac{1}{\varepsilon}\|\rho[f]\|_{L^2_x}\|\theta[f]\|_{L^2_x}.
\end{aligned}
$$

Similarly, we obtain

$$
\begin{aligned}
R_{23} &\lesssim \frac{1}{\varepsilon}\|\nabla^2_x(-\Delta_x)^{-1}\rho[f]\|_{L^2_x}\left\|\int_{\mathbf{R}^d} v_k v_\ell f^\perp\sqrt{M}\,\mathrm{d}v\right\|_{L^2_x} \\
&\lesssim \frac{1}{\varepsilon}\|\rho[f]\|_{L^2_x}\|f^\perp\|_{L^2_{x,v}}.
\end{aligned}
$$

Gathering the previous estimates, we obtain for some constants $\kappa, C > 0$

$$
\begin{aligned}
&\left\langle \partial_{x_k}(-\Delta_x)^{-1}\rho[\Lambda_\varepsilon f], u_k[f]\right\rangle_{L^2_x} + \left\langle \partial_{x_k}(-\Delta_x)^{-1}\rho[f], u_k[\Lambda_\varepsilon f]\right\rangle_{L^2_x} \\
&\qquad \leqslant -\frac{\kappa}{\varepsilon}\|\rho[f]\|^2_{L^2_x} + \frac{C}{\varepsilon}\|u[f]\|^2_{L^2_x} + \frac{C}{\varepsilon}\|\theta[f]\|^2_{L^2_x} + \frac{C}{\varepsilon}\|f^\perp\|^2_{L^2_{x,v}}.
\end{aligned} \tag{A.18}
$$

Step 5. Conclusion. We introduce the following inner product on $L^2_{x,v}$:
(A.19)
$$\begin{aligned}\langle\!\langle f,g\rangle\!\rangle_{L^2_{x,v}} &:= \langle f,g\rangle_{L^2_{x,v}} + \eta_1\varepsilon\left\langle \partial_{x_k}(-\Delta_x)^{-1}\theta[f],\psi_k[g]\right\rangle_{L^2_x} + \eta_1\varepsilon\left\langle \partial_{x_k}(-\Delta_x)^{-1}\theta[g],\psi_k[f]\right\rangle_{L^2_x}\\ &\quad + \eta_2\varepsilon\left\langle \partial_{x_\ell}(-\Delta_x)^{-1}u_k[f],\Theta_{k\ell}[g]\right\rangle_{L^2_x} + \eta_2\varepsilon\left\langle \partial_{x_\ell}(-\Delta_x)^{-1}u_k[g],\Theta_{k\ell}[f]\right\rangle_{L^2_x}\\ &\quad + \eta_3\varepsilon\left\langle \partial_{x_k}(-\Delta_x)^{-1}\rho[f],u_k[g]\right\rangle_{L^2_x} + \eta_3\varepsilon\left\langle \partial_{x_k}(-\Delta_x)^{-1}\rho[g],u_k[f]\right\rangle_{L^2_x}\end{aligned}$$
with constants $0<\eta_3\ll\eta_2\ll\eta_1\ll 1$ to be chosen below, and denote by $|\!|\!| g|\!|\!|^2_{L^2_{x,v}} = \langle\!\langle g,g\rangle\!\rangle_{L^2_{x,v}}$ the associated norm. We observe that
$$\|g\|_{L^2_{x,v}} \lesssim |\!|\!| g|\!|\!|_{L^2_{x,v}} \lesssim \|g\|_{L^2_{x,v}}$$
where the multiplicative constants are uniform in $\varepsilon\in(0,1]$. The norms $\|\cdot\|_{L^2_{x,v}}$ and $|\!|\!|\cdot|\!|\!|_{L^2_{x,v}}$ are thus equivalent independently of $\varepsilon\in(0,1]$. Gathering estimates (A.2)–(A.8)–(A.16)–(A.18) and using Young's inequality, we obtain:
$$\begin{aligned}\langle\!\langle \Lambda_\varepsilon f,f\rangle\!\rangle_{L^2_{x,v}} \leqslant &-\frac{1}{\varepsilon^2}\left(\frac{\sigma_L}{2}-C\eta_1-C\eta_2-C\eta_3\right)\|f^\perp\|^2_{L^2_x(H^1_{v,*})}\\ &-\left(\frac{\kappa\eta_1}{2}-C\eta_2-C\eta_3\right)\|\theta[f]\|^2_{L^2_x}\\ &-\left(\kappa\eta_2-C\eta_1^2-C\eta_3\right)\|u[f]\|^2_{L^2_x}\\ &-\left(\kappa\eta_3-C\eta_2^2-C\frac{\eta_2^2}{\eta_1}\right)\|\rho[f]\|^2_{L^2_x}.\end{aligned}$$
By choosing $\eta_1:=\eta$, $\eta_2:=\eta^{\frac32}$, $\eta_3:=\eta^{\frac74}$ with $\eta>0$ small enough and recalling that $\|f\|^2_{L^2_{x,v}} = \|f^\perp\|^2_{L^2_{x,v}}+\|\rho[f]\|^2_{L^2_x}+\|u[f]\|^2_{L^2_x}+\|\theta[f]\|^2_{L^2_x}$, we thus obtain
$$\langle\!\langle \Lambda_\varepsilon f,f\rangle\!\rangle_{L^2_{x,v}} \leqslant -\sigma|\!|\!| f|\!|\!|^2_{L^2_{x,v}} - \frac{\kappa}{\varepsilon^2}\|f^\perp\|^2_{L^2_x(H^1_{v,*})}$$
for some constants $\sigma,\kappa>0$, which completes the proof. □

Appendix B. Proofs of Lemmas 5.5 and 5.6

Proof of Lemma 5.5. Let us first recall that $\mathcal{B}_\varepsilon = \varepsilon^{-2}\mathcal{B}-\varepsilon^{-1}v\cdot\nabla_x$ with $\mathcal{B}$ and $v\cdot\nabla_x$ that are respectively self-adjoint and skew-adjoint operators in $L^2_{x,v}$. Proving that $S_{\mathcal{B}_\varepsilon}$ regularizes from $\mathscr{Y}_1'$ to $\mathscr{X}$ and from $(\mathscr{Z}_1^\varepsilon)'$ to $\mathscr{X}$ is similar to prove that $S_{\mathcal{B}_\varepsilon}$ regularizes from $\mathscr{X}$ to $\mathscr{Y}_1$ and from $\mathscr{X}$ to $\mathscr{Z}_1^\varepsilon$ with same rates. We will only focus on the proof of (5.7) and explain the adaptation to make to prove (5.8) in the first step of the proof.

Moreover, to prove (5.7), it is sufficient to prove that for any $\alpha\in\mathbf{R}$ and $\langle v\rangle^\alpha f_{\rm in}\in L^2_{x,v}$, one has: For any $t\in(0,\varepsilon^2]$,
$$\|\langle v\rangle^\alpha S_{\mathcal{B}_\varepsilon}(t)f_{\rm in}\|_{L^2_x(H^1_{v,*})} \lesssim \frac{\varepsilon}{\sqrt t}\|\langle v\rangle^\alpha f_{\rm in}\|_{L^2_{x,v}} \tag{B.1}$$
and
$$\|\langle v\rangle^\alpha S_{\mathcal{B}_\varepsilon}(t)f_{\rm in}\|_{L^2_{x,v}} + \|\langle v\rangle^\alpha S_{\mathcal{B}_\varepsilon}(t)f_{\rm in}\|_{L^2_x(H^1_{v,*})} + \varepsilon\|\widetilde\nabla_x(\langle v\rangle^\alpha S_{\mathcal{B}_\varepsilon}(t)f_{\rm in})\|_{L^2_{x,v}} \lesssim \frac{\varepsilon^3}{t^{3/2}}\|\langle v\rangle^\alpha f_{\rm in}\|_{L^2_{x,v}}. \tag{B.2}$$
Indeed, since ∇_x commutes with $\mathcal{B}_\varepsilon$, from estimate (B.1) (resp. (B.2)), we already obtain the first (resp. second) estimate of (5.7) for $t\in(0,\varepsilon^2]$. Then, fix $\sigma\in(0,\sigma_1)$. We obtain the estimates given in (5.7) for all $t>0$ by using the exponential decay of $S_{\mathcal{B}_\varepsilon}$ in $\mathscr{X}$ given in Lemma 5.4. More precisely, using that for $t\geqslant\varepsilon^2$, $S_{\mathcal{B}_\varepsilon}(t)=S_{\mathcal{B}_\varepsilon}(\varepsilon^2)S_{\mathcal{B}_\varepsilon}(t-\varepsilon^2)$, we obtain that for $\sigma'\in(\sigma,\sigma_1)$ and for any $t>0$,
$$\|S_{\mathcal{B}_\varepsilon}(t)\|_{\mathscr{X}\to\mathscr{Y}_1} \lesssim \frac{\varepsilon}{\min(\varepsilon,\sqrt t)}e^{-\sigma' t/\varepsilon^2} \quad\text{and}\quad \|S_{\mathcal{B}_\varepsilon}(t)\|_{\mathscr{X}\to\mathscr{Z}_1^\varepsilon} \lesssim \frac{\varepsilon^3}{\min(\varepsilon^3,t^{3/2})}e^{-\sigma' t/\varepsilon^2}.$$

It implies the wanted conclusion using that $\sigma' > \sigma$.

For the remainder of the proof, we let $\alpha \in \mathbf{R}$ and $f_{\rm in}$ be such that $\langle v\rangle^\alpha f_{\rm in} \in L^2_{x,v}$, and consider the solution $f(t) = S_{\mathcal{B}_\varepsilon}(t) f_{\rm in}$ to the equation $\partial_t f = \mathcal{B}_\varepsilon f$ with initial data $f_{\rm in}$, and we shall prove (B.1) and (B.2).

Step 1. Define the functional

$$
\begin{aligned}
\text{(B.3)}\quad \mathcal{E}_\varepsilon(t) := \|\langle v\rangle^\alpha f\|^2_{L^2_{x,v}} &+ \alpha_1 \frac{t}{\varepsilon^2}\Big(K \|\langle v\rangle^\alpha \langle v\rangle^{\frac{\gamma}{2}+1} f\|^2_{L^2_{x,v}} + \|\langle v\rangle^\alpha \widetilde\nabla_v f\|^2_{L^2_{x,v}}\Big) \\
&+ \varepsilon\alpha_2 \left(\frac{t}{\varepsilon^2}\right)^2 \Big\langle \langle v\rangle^\alpha \widetilde\nabla_v f, \langle v\rangle^\alpha \widetilde\nabla_x f\Big\rangle_{L^2_{x,v}} \\
&+ \varepsilon^2 \alpha_3 \left(\frac{t}{\varepsilon^2}\right)^3 \Big(\|\langle v\rangle^\alpha \widetilde\nabla_x f\|^2_{L^2_{x,v}} + K \|\langle v\rangle^\alpha \langle v\rangle^{\frac{\gamma}{2}} \nabla_x f\|^2_{L^2_{x,v}}\Big),
\end{aligned}
$$

where $\alpha_1, \alpha_2, \alpha_3, K > 0$ are positive constants such that $0 < \alpha_3 \ll \alpha_2 \ll \alpha_1 \ll 1$ and $\alpha_2 \leqslant \sqrt{\alpha_1\alpha_3}$. Notice that in order to prove (5.8), one just has to change the sign in front of the term which mixes derivatives in x and v in the definition of the functional $\mathcal{E}_\varepsilon$. It will allow to conclude that (5.7) holds for the adjoint of $\mathcal{B}_\varepsilon$ instead of $\mathcal{B}_\varepsilon$ and thus imply (5.8). The constants α_i will be chosen small enough in Step 6, and K will be chosen large enough in Step 3 and Step 5. We remark that

$$
\|\langle v\rangle^\alpha f\|^2_{L^2_x(H^1_{v,*})} \lesssim K \|\langle v\rangle^\alpha \langle v\rangle^{\frac{\gamma}{2}+1} f\|^2_{L^2_{x,v}} + \|\langle v\rangle^\alpha \widetilde\nabla_v f\|^2_{L^2_{x,v}} \lesssim \|\langle v\rangle^\alpha f\|^2_{L^2_x(H^1_{v,*})}
$$

and

$$
\|\langle v\rangle^\alpha \widetilde\nabla_x f\|^2_{L^2_{x,v}} \lesssim \|\langle v\rangle^\alpha \widetilde\nabla_x f\|^2_{L^2_{x,v}} + K \|\langle v\rangle^\alpha \langle v\rangle^{\frac{\gamma}{2}} \nabla_x f\|^2_{L^2_{x,v}} \lesssim \|\langle v\rangle^\alpha \widetilde\nabla_x f\|^2_{L^2_{x,v}}.
$$

Therefore, we can already observe that for any $t \in (0, \varepsilon^2]$, one has the following lower bounds

$$
\frac{t}{\varepsilon^2} \|\langle v\rangle^\alpha f\|^2_{L^2_x(H^1_{v,*})} \lesssim \mathcal{E}_\varepsilon(t)
$$

and

$$
\left(\frac{t}{\varepsilon^2}\right)^3 \Big(\|\langle v\rangle^\alpha f\|^2_{L^2_{x,v}} + \|\langle v\rangle^\alpha f\|^2_{L^2_x(H^1_{v,*})} + \varepsilon^2 \|\langle v\rangle^\alpha \widetilde\nabla_x f\|^2_{L^2_{x,v}}\Big) \lesssim \mathcal{E}_\varepsilon(t).
$$

Therefore, in order to prove (B.1) and (B.2), it is sufficient to prove that $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_\varepsilon(t) \leqslant 0$ for all $t \in (0, \varepsilon^2]$. We then compute

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_\varepsilon(t) = \frac{\mathrm{d}}{\mathrm{d}t}\|\langle v\rangle^\alpha f\|^2_{L^2_{x,v}} &+ \frac{\alpha_1}{\varepsilon^2}\Big(K \|\langle v\rangle^\alpha \langle v\rangle^{\frac{\gamma}{2}+1} f\|^2_{L^2_{x,v}} + \|\langle v\rangle^\alpha \widetilde\nabla_v f\|^2_{L^2_{x,v}}\Big) \\
&+ \frac{\alpha_1}{\varepsilon^2} t \frac{\mathrm{d}}{\mathrm{d}t}\Big(K \|\langle v\rangle^\alpha \langle v\rangle^{\frac{\gamma}{2}+1} f\|^2_{L^2_{x,v}} + \|\langle v\rangle^\alpha \widetilde\nabla_v f\|^2_{L^2_{x,v}}\Big) \\
&+ 2\frac{\alpha_2}{\varepsilon^3} t \Big\langle \langle v\rangle^\alpha \widetilde\nabla_v f, \langle v\rangle^\alpha \widetilde\nabla_x f\Big\rangle_{L^2_{x,v}} + \frac{\alpha_2}{\varepsilon^3} t^2 \frac{\mathrm{d}}{\mathrm{d}t} \Big\langle \langle v\rangle^\alpha \widetilde\nabla_v f, \langle v\rangle^\alpha \widetilde\nabla_x f\Big\rangle_{L^2_{x,v}} \\
&+ 3\frac{\alpha_3}{\varepsilon^4} t^2 \Big(\|\langle v\rangle^\alpha \widetilde\nabla_x f\|^2_{L^2_{x,v}} + K \|\langle v\rangle^\alpha \langle v\rangle^{\frac{\gamma}{2}} \nabla_x f\|^2_{L^2_{x,v}}\Big) \\
&+ \frac{\alpha_3}{\varepsilon^4} t^3 \frac{\mathrm{d}}{\mathrm{d}t}\Big(\|\langle v\rangle^\alpha \widetilde\nabla_x f\|^2_{L^2_{x,v}} + K \|\langle v\rangle^\alpha \langle v\rangle^{\frac{\gamma}{2}} \nabla_x f\|^2_{L^2_{x,v}}\Big)
\end{aligned}
$$

and we shall estimate each term separately in the sequel.

Step 2. From Lemma 5.4, we already have

$$
\text{(B.4)}\qquad \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\langle v\rangle^\alpha f\|^2_{L^2_{x,v}} \leqslant -\frac{\kappa}{\varepsilon^2}\|\langle v\rangle^\alpha f\|^2_{L^2_x(H^1_{v,*})}
$$

for some constant $\kappa > 0$.

Step 3. We prove in this step that, choosing $K > 0$ large enough, we have

$$
\begin{aligned}
\text{(B.5)}\qquad &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big\{ K \|\langle v\rangle^\alpha \langle v\rangle^{\frac{\gamma}{2}+1} f\|^2_{L^2_{x,v}} + \|\langle v\rangle^\alpha \widetilde\nabla_v f\|^2_{L^2_{x,v}}\Big\} \\
&\qquad \leqslant -\frac{\kappa}{2\varepsilon^2}\|\langle v\rangle^\alpha f\|^2_{L^2_x(H^2_{v,*})} + \frac{C}{\varepsilon}\|\langle v\rangle^\alpha f\|_{L^2_x(H^1_{v,*})} \|\langle v\rangle^\alpha \widetilde\nabla_x f\|_X,
\end{aligned}
$$

for some constants $\kappa, C > 0$. From Lemma 5.4, we already have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\langle v\rangle^{\alpha}\langle v\rangle^{\frac{\gamma}{2}+1}f\|^2_{L^2_{x,v}} \leqslant -\frac{\kappa}{\varepsilon^2}\|\langle v\rangle^{\alpha}\langle v\rangle^{\frac{\gamma}{2}+1}f\|^2_{L^2_x(H^1_{v,*})}$$

for some $\kappa > 0$. Moreover, we have

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\langle v\rangle^{\alpha}\widetilde\nabla_v f\|^2_{L^2_{x,v}} &= \left\langle \langle v\rangle^{2\alpha}\widetilde\nabla_{v_i} f, \widetilde\nabla_{v_i}(\mathcal{B}_\varepsilon f)\right\rangle_{L^2_{x,v}} \\ &= \left\langle \langle v\rangle^{2\alpha}\widetilde\nabla_{v_i} f, \mathcal{B}_\varepsilon(\widetilde\nabla_{v_i} f)\right\rangle_{L^2_{x,v}} + \left\langle \langle v\rangle^{2\alpha}\widetilde\nabla_{v_i} f, [\widetilde\nabla_{v_i}, \mathcal{B}_\varepsilon] f\right\rangle_{L^2_{x,v}}.\end{aligned}$$

Thanks to Lemma 5.4, there holds

$$\left\langle \langle v\rangle^{2\alpha}\widetilde\nabla_{v_i} f, \mathcal{B}_\varepsilon(\widetilde\nabla_{v_i} f)\right\rangle_{L^2_{x,v}} \leqslant -\frac{\kappa}{\varepsilon^2}\|\langle v\rangle^{\alpha}\widetilde\nabla_v f\|^2_{L^2_x(H^1_{v,*})},$$

for some constant $\kappa > 0$. From Lemma 5.2-(i), we have

$$\begin{aligned}&\left\langle \langle v\rangle^{2\alpha}\widetilde\nabla_{v_i} f, [\widetilde\nabla_{v_i}, \mathcal{B}_\varepsilon] f\right\rangle_{L^2_{x,v}} \\ &\quad= -\frac{1}{\varepsilon^2}\left\langle \widetilde\nabla_{v_j}(\langle v\rangle^{2\alpha}\widetilde\nabla_{v_i} f), [\widetilde\nabla_{v_i}, \widetilde\nabla_{v_j}] f\right\rangle_{L^2_{x,v}} - \frac{1}{\varepsilon^2}\left\langle \widetilde\nabla^*_{v_j}(\langle v\rangle^{2\alpha}\widetilde\nabla_{v_i} f), [\widetilde\nabla_{v_i}, \widetilde\nabla^*_{v_j}] f\right\rangle_{L^2_{x,v}} \\ &\qquad - \frac{1}{\varepsilon^2}\left\langle \langle v\rangle^{2\alpha}\widetilde\nabla_{v_i} f, (\widetilde\nabla_{v_i} m^2) f\right\rangle_{L^2_{x,v}} - \frac{1}{\varepsilon^2}\left\langle \langle v\rangle^{2\alpha}\widetilde\nabla_{v_i} f, \left[[\widetilde\nabla_{v_i}, \widetilde\nabla^*_{v_j}], \widetilde\nabla_{v_j}\right] f\right\rangle_{L^2_{x,v}} \\ &\qquad - \frac{1}{\varepsilon}\left\langle \langle v\rangle^{2\alpha}\widetilde\nabla_{v_i} f, \widetilde\nabla_{x_i} f\right\rangle_{L^2_{x,v}} \\ &\quad=: -\frac{1}{\varepsilon^2}(T_1 + T_2 + T_3 + T_4) - \frac{1}{\varepsilon}\left\langle \langle v\rangle^{2\alpha}\widetilde\nabla_{v_i} f, \widetilde\nabla_{x_i} f\right\rangle_{L^2_{x,v}}.\end{aligned}$$

Writing $\widetilde\nabla_{v_j}(\langle v\rangle^{2\alpha}\widetilde\nabla_{v_i} f) = \langle v\rangle^{\alpha}\widetilde\nabla_{v_j}(\langle v\rangle^{\alpha}\widetilde\nabla_{v_i} f) + (\widetilde\nabla_{v_j}\langle v\rangle^{\alpha})\langle v\rangle^{\alpha}\widetilde\nabla_{v_i} f$, using that $|\widetilde\nabla_{v_j}\langle v\rangle^{\alpha}| \lesssim \langle v\rangle^{\frac{\gamma}{2}-1}\langle v\rangle^{\alpha}$ and thanks to Lemma 2.2-(iii), we obtain

$$T_1 \leqslant \frac{\kappa}{8}\|\langle v\rangle^{\alpha}\widetilde\nabla_v f\|^2_{L^2_x(H^1_{v,*})} + C\|\langle v\rangle^{\alpha}\langle v\rangle^{\gamma+1}\nabla_v f\|^2_{L^2_{x,v}}.$$

In a similar way, observing now that

$$\widetilde\nabla^*_{v_j}(\langle v\rangle^{2\alpha}\widetilde\nabla_{v_i} f) = -\langle v\rangle^{\alpha}\widetilde\nabla_{v_j}(\langle v\rangle^{\alpha}\widetilde\nabla_{v_i} f) - \left[(\partial_{v_\ell} B_{j\ell})\langle v\rangle^{\alpha} + (\widetilde\nabla_{v_j}\langle v\rangle^{\alpha})\right]\langle v\rangle^{\alpha}\widetilde\nabla_{v_i} f$$

and using Lemma 2.2-(iv), we get

$$T_2 \leqslant \frac{\kappa}{8}\|\langle v\rangle^{\alpha}\widetilde\nabla_v f\|^2_{L^2_x(H^1_{v,*})} + C\|\langle v\rangle^{\alpha}\langle v\rangle^{\gamma} f\|^2_{L^2_{x,v}} + C\|\langle v\rangle^{\alpha}\langle v\rangle^{\gamma+1}\nabla_v f\|^2_{L^2_{x,v}}.$$

Using that $|\widetilde\nabla_{v_i} m^2| \lesssim \langle v\rangle^{\frac{3\gamma}{2}+2}$ from Lemma 5.1, we also get

$$T_3 \leqslant \frac{\kappa}{8}\|\langle v\rangle^{\alpha}\widetilde\nabla_v f\|^2_{L^2_x(H^1_{v,*})} + C\|\langle v\rangle^{\alpha}\langle v\rangle^{\gamma+1} f\|^2_{L^2_{x,v}}.$$

For the term T_4, we use Lemma 2.2-(vii) to obtain

$$T_4 \leqslant \frac{\kappa}{8}\|\langle v\rangle^{\alpha}\widetilde\nabla_v f\|^2_{L^2_x(H^1_{v,*})} + C\|\langle v\rangle^{\alpha}\langle v\rangle^{\gamma-1} f\|^2_{L^2_{x,v}} + C\|\langle v\rangle^{\alpha}\langle v\rangle^{\gamma}\nabla_v f\|^2_{L^2_{x,v}}.$$

Hence, we obtain

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\langle v\rangle^{\alpha}\widetilde\nabla_v f\|^2_{L^2_{x,v}} &\leqslant -\frac{\kappa}{2\varepsilon^2}\|\langle v\rangle^{\alpha}\widetilde\nabla_v f\|^2_{L^2_x(H^1_{v,*})} + \frac{C}{\varepsilon^2}\|\langle v\rangle^{\alpha}\langle v\rangle^{\gamma+1} f\|^2_{L^2_{x,v}} \\ &\quad + \frac{C}{\varepsilon^2}\|\langle v\rangle^{\alpha}\langle v\rangle^{\gamma+1}\nabla_v f\|^2_{L^2_{x,v}} + \frac{C}{\varepsilon}\|\langle v\rangle^{\alpha}\widetilde\nabla_v f\|_{L^2_{x,v}}\|\langle v\rangle^{\alpha}\widetilde\nabla_x f\|_{L^2_{x,v}}.\end{aligned}$$

We conclude to (B.5) by gathering previous estimates, observing that $\|\langle v\rangle^{\alpha}\langle v\rangle^{\gamma+1} f\|_{L^2_{x,v}} + \|\langle v\rangle^{\alpha}\langle v\rangle^{\gamma+1}\nabla_v f\|_{L^2_{x,v}} \lesssim \|\langle v\rangle^{\alpha}\langle v\rangle^{\frac{\gamma}{2}+1} f\|_{L^2_x(H^1_{v,*})}$ and taking $K > 0$ large enough.

Step 4. In this step, we show that

(B.6)
$$\frac{\mathrm{d}}{\mathrm{d}t}\left\langle \langle v\rangle^{\alpha}\widetilde\nabla_v f, \langle v\rangle^{\alpha}\widetilde\nabla_x f\right\rangle_{L^2_{x,v}} \leqslant -\frac{1}{\varepsilon}\|\langle v\rangle^{\alpha}\widetilde\nabla_x f\|^2_{L^2_{x,v}} + \frac{C}{\varepsilon^2}\|\langle v\rangle^{\alpha} f\|_{L^2_x(H^2_{v,*})}\|\langle v\rangle^{\alpha}\widetilde\nabla_x f\|_{L^2_x(H^1_{v,*})},$$

for some constant $C>0$. We compute, using (5.4),

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left\langle \langle v\rangle^\alpha \widetilde\nabla_v f, \langle v\rangle^\alpha \widetilde\nabla_x f\right\rangle_{L^2_{x,v}} &= \left\langle \langle v\rangle^{2\alpha}\widetilde\nabla_{v_i} f, \widetilde\nabla_{x_i}\left\{\frac{1}{\varepsilon^2}\mathcal{B}f - \frac{1}{\varepsilon} v\cdot\nabla_x f\right\}\right\rangle_{L^2_{x,v}} \\ &\quad + \left\langle \langle v\rangle^{2\alpha}\widetilde\nabla_{x_i} f, \widetilde\nabla_{v_i}\left\{\frac{1}{\varepsilon^2}\mathcal{B}f - \frac{1}{\varepsilon} v\cdot\nabla_x f\right\}\right\rangle_{L^2_{x,v}} \\ &= -\frac{1}{\varepsilon^2}\left\langle \widetilde\nabla_{x_i}(\langle v\rangle^{2\alpha}\widetilde\nabla_{v_i} f), \mathcal{B}f\right\rangle_{L^2_{x,v}} + \frac{1}{\varepsilon^2}\left\langle \widetilde\nabla^*_{v_i}(\langle v\rangle^{2\alpha}\widetilde\nabla_{x_i} f), \mathcal{B}f\right\rangle_{L^2_{x,v}} - \frac{1}{\varepsilon}\|\langle v\rangle^\alpha\widetilde\nabla_x f\|^2_{L^2_{x,v}},\end{aligned}$$

where we have used Lemma 2.2-(i). We hence conclude to (B.6) by using the fact that $\|\langle v\rangle^\alpha \mathcal{B}f\|_{L^2_{x,v}} \lesssim \|\langle v\rangle^\alpha f\|_{L^2_x(H^2_{v,*})}$ (see (3.19) for a similar estimate) together with estimates (3.15) and (3.18).

Step 5. We prove in this step that, choosing $K>0$ large enough, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big\{\|\langle v\rangle^\alpha\widetilde\nabla_x f\|^2_{L^2_{x,v}} + K\|\langle v\rangle^\alpha\langle v\rangle^{\frac{\gamma}{2}}\nabla_x f\|^2_{L^2_{x,v}}\Big\} \leqslant -\frac{\kappa}{2\varepsilon^2}\|\langle v\rangle^\alpha\widetilde\nabla_x f\|^2_{L^2_x(H^1_{v,*})}, \tag{B.7}$$

for some constants $\kappa, C>0$. We first remark that, since ∇_x commutes with $\mathcal{B}_\varepsilon$, we already have from Lemma 5.4 that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\langle v\rangle^\alpha\langle v\rangle^{\frac{\gamma}{2}}\nabla_x f\|^2_{L^2_{x,v}} \leqslant -\frac{\kappa}{\varepsilon^2}\|\langle v\rangle^\alpha\langle v\rangle^{\frac{\gamma}{2}}\nabla_x f\|^2_{L^2_x(H^1_{v,*})}$$

for some constant $\kappa>0$.

We then write

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\langle v\rangle^\alpha\widetilde\nabla_x f\|^2_{L^2_{x,v}} &= \left\langle \langle v\rangle^{2\alpha}\widetilde\nabla_{x_i} f, \widetilde\nabla_{x_i}(\mathcal{B}_\varepsilon f)\right\rangle_{L^2_{x,v}} \\ &= \left\langle \langle v\rangle^{2\alpha}\widetilde\nabla_{x_i} f, \mathcal{B}_\varepsilon(\widetilde\nabla_{x_i} f)\right\rangle_{L^2_{x,v}} + \left\langle \langle v\rangle^{2\alpha}\widetilde\nabla_{x_i} f, [\widetilde\nabla_{x_i}, \mathcal{B}_\varepsilon] f\right\rangle_{L^2_{x,v}}.\end{aligned}$$

Thanks to Lemma 5.4, there holds

$$\left\langle \langle v\rangle^{2\alpha}\widetilde\nabla_{x_i} f, \mathcal{B}_\varepsilon(\widetilde\nabla_{x_i} f)\right\rangle_{L^2_{x,v}} \leqslant -\frac{\kappa}{\varepsilon^2}\|\langle v\rangle^\alpha\widetilde\nabla_x f\|^2_{L^2_x(H^1_{v,*})},$$

for some constant $\kappa>0$. From Lemma 5.2-(ii), we have

$$\begin{aligned}&\left\langle \langle v\rangle^{2\alpha}\widetilde\nabla_{x_i} f, [\widetilde\nabla_{x_i}, \mathcal{B}_\varepsilon] f\right\rangle_{L^2_{x,v}} \\ &= -\frac{1}{\varepsilon^2}\left\langle \widetilde\nabla_{v_j}(\langle v\rangle^{2\alpha}\widetilde\nabla_{x_i} f), [\widetilde\nabla_{x_i}, \widetilde\nabla_{v_j}] f\right\rangle_{L^2_{x,v}} - \frac{1}{\varepsilon^2}\left\langle \widetilde\nabla^*_{v_j}(\langle v\rangle^{2\alpha}\widetilde\nabla_{x_i} f), [\widetilde\nabla_{x_i}, \widetilde\nabla^*_{v_j}] f\right\rangle_{L^2_{x,v}} \\ &\quad - \frac{1}{\varepsilon^2}\left\langle \langle v\rangle^{2\alpha}\widetilde\nabla_{x_i} f, \left[[\widetilde\nabla_{x_i}, \widetilde\nabla^*_{v_j}], \widetilde\nabla_{v_j}\right] f\right\rangle_{L^2_{x,v}} \\ &=: -\frac{1}{\varepsilon^2}(R_1+R_2+R_3).\end{aligned}$$

Writing $\widetilde\nabla_{v_j}(\langle v\rangle^{2\alpha}\widetilde\nabla_{x_i} f) = \langle v\rangle^\alpha\widetilde\nabla_{v_j}(\langle v\rangle^\alpha\widetilde\nabla_{x_i} f) + (\widetilde\nabla_{v_j}\langle v\rangle^\alpha)\langle v\rangle^\alpha\widetilde\nabla_{x_i} f$, using the fact that $|\widetilde\nabla_{v_j}\langle v\rangle^\alpha| \lesssim \langle v\rangle^{\frac{\gamma}{2}-1}\langle v\rangle^\alpha$ and thanks to Lemma 2.2-(v), we obtain

$$R_1 \leqslant \frac{\kappa}{6}\|\langle v\rangle^\alpha\widetilde\nabla_x f\|^2_{L^2_x(H^1_{v,*})} + C\|\langle v\rangle^\alpha\langle v\rangle^{\gamma+1}\nabla_x f\|^2_{L^2_{x,v}}.$$

In a similar way, observing now that

$$\widetilde\nabla^*_{v_j}(\langle v\rangle^{2\alpha}\widetilde\nabla_{x_i} f) = -\langle v\rangle^\alpha\widetilde\nabla_{v_j}(\langle v\rangle^\alpha\widetilde\nabla_{x_i} f) - \left[(\partial_{v_\ell}B_{j\ell})\langle v\rangle^\alpha + (\widetilde\nabla_{v_j}\langle v\rangle^\alpha)\right]\langle v\rangle^\alpha\widetilde\nabla_{x_i} f,$$

we get

$$R_2 \leqslant \frac{\kappa}{6}\|\langle v\rangle^\alpha\widetilde\nabla_x f\|^2_{L^2_x(H^1_{v,*})} + C\|\langle v\rangle^\alpha\langle v\rangle^{\gamma+1}\nabla_x f\|^2_{L^2_{x,v}}.$$

For the term R_3, we use Lemma 2.2-(viii) to obtain

$$R_3 \leqslant \frac{\kappa}{6}\|\langle v\rangle^\alpha\widetilde\nabla_x f\|^2_{L^2_x(H^1_{v,*})} + C\|\langle v\rangle^\alpha\langle v\rangle^{\gamma}\nabla_x f\|^2_{L^2_{x,v}}.$$

Hence, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\langle v\rangle^\alpha\widetilde\nabla_x f\|^2_{L^2_{x,v}} \leqslant -\frac{\kappa}{2\varepsilon^2}\|\langle v\rangle^\alpha\widetilde\nabla_x f\|^2_{L^2_x(H^1_{v,*})} + \frac{C}{\varepsilon^2}\|\langle v\rangle^\alpha\langle v\rangle^{\gamma+1}\nabla_x f\|^2_{L^2_{x,v}}.$$

We conclude to (B.7) by gathering previous estimates as well as noticing the fact that $\|\langle v\rangle^\alpha \langle v\rangle^{\gamma+1}\nabla_x f\|_{L^2_{x,v}} \lesssim \|\langle v\rangle^\alpha \langle v\rangle^{\frac{\gamma}{2}}\nabla_x f\|_{L^2_x(H^1_{v,*})}$ and taking $K>0$ large enough.

Step 6. Proof of (B.1) *and* (B.2). Gathering (B.4)–(B.5)–(B.6)–(B.7), we obtain

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_\varepsilon(t) \leqslant & -\frac{2\kappa}{\varepsilon^2}\|\langle v\rangle^\alpha f\|^2_{L^2_x(H^1_{v,*})} + \frac{\alpha_1 C}{\varepsilon^2}\|\langle v\rangle^\alpha f\|^2_{L^2_x(H^1_{v,*})} \\
& + \frac{\alpha_1}{\varepsilon^2} t\left(-\frac{\kappa}{\varepsilon^2}\|\langle v\rangle^\alpha f\|^2_{L^2_x(H^2_{v,*})} + \frac{C}{\varepsilon}\|\langle v\rangle^\alpha f\|_{L^2_x(H^1_{v,*})}\|\langle v\rangle^\alpha \widetilde\nabla_x f\|_X\right) \\
& + 2\frac{\alpha_2}{\varepsilon^3} t\left\langle \langle v\rangle^\alpha \widetilde\nabla_v f, \langle v\rangle^\alpha \widetilde\nabla_x f\right\rangle_{L^2_{x,v}} \\
& + \frac{\alpha_2}{\varepsilon^3} t^2\left(-\frac{1}{\varepsilon}\|\langle v\rangle^\alpha \widetilde\nabla_x f\|^2_{L^2_{x,v}} + \frac{C}{\varepsilon^2}\|\langle v\rangle^\alpha f\|_{L^2_x(H^2_{v,*})}\|\langle v\rangle^\alpha \widetilde\nabla_x f\|_{L^2_x(H^1_{v,*})}\right) \\
& + \frac{\alpha_3 C}{\varepsilon^4} t^2\|\langle v\rangle^\alpha \widetilde\nabla_x f\|^2_{L^2_{x,v}} - \frac{\alpha_3}{\varepsilon^4} t^3\frac{\kappa}{\varepsilon^2}\|\langle v\rangle^\alpha \widetilde\nabla_x f\|^2_{L^2_x(H^1_{v,*})}.
\end{aligned}
$$

Using Young's inequality, we have

$$
\begin{aligned}
C\alpha_1\frac{t}{\varepsilon^3}\|\langle v\rangle^\alpha f\|_{L^2_x(H^1_{v,*})}\|\langle v\rangle^\alpha \widetilde\nabla_x f\|_{L^2_{x,v}} &\leqslant \frac{\alpha_2}{4}\frac{t^2}{\varepsilon^4}\|\langle v\rangle^\alpha \widetilde\nabla_x f\|^2_{L^2_{x,v}} + C\frac{\alpha_1^2}{\alpha_2}\frac{1}{\varepsilon^2}\|\langle v\rangle^\alpha f\|^2_{L^2_x(H^1_{v,*})} \\
2\frac{\alpha_2}{\varepsilon^3} t\left\langle \langle v\rangle^\alpha \widetilde\nabla_v f, \langle v\rangle^\alpha \widetilde\nabla_x f\right\rangle_{L^2_{x,v}} &\leqslant \frac{\alpha_2}{4}\frac{t^2}{\varepsilon^4}\|\langle v\rangle^\alpha \widetilde\nabla_x f\|^2_{L^2_{x,v}} + C\frac{\alpha_2}{\varepsilon^2}\|\langle v\rangle^\alpha f\|^2_{L^2_x(H^1_{v,*})} \\
C\alpha_2\frac{t^2}{\varepsilon^5}\|\langle v\rangle^\alpha f\|_{L^2_x(H^2_{v,*})}\|\langle v\rangle^\alpha \widetilde\nabla_x f\|_{L^2_x(H^1_{v,*})} &\leqslant \frac{\alpha_3\kappa}{2}\frac{t^3}{\varepsilon^6}\|\langle v\rangle^\alpha \widetilde\nabla_x f\|^2_{L^2_x(H^1_{v,*})} \\
&\quad + C\frac{\alpha_2^2}{\alpha_3}\frac{t}{\varepsilon^4}\|\langle v\rangle^\alpha f\|^2_{L^2_x(H^2_{v,*})}.
\end{aligned}
$$

We thus deduce, for any $t\in(0,\varepsilon^2]$, that

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_\varepsilon(t) \leqslant & -\frac{1}{\varepsilon^2}\left(2\kappa - C\alpha_1 - C\alpha_2 - C\frac{\alpha_1^2}{\alpha_2}\right)\|\langle v\rangle^\alpha f\|^2_{L^2_x(H^1_{v,*})} \\
& - \frac{t}{\varepsilon^4}\left(\alpha_1\kappa - C\frac{\alpha_2^2}{\alpha_3}\right)\|\langle v\rangle^\alpha f\|^2_{L^2_x(H^2_{v,*})} - \frac{t^2}{\varepsilon^4}\left(\frac{\alpha_2}{2} - C\alpha_3\right)\|\langle v\rangle^\alpha \widetilde\nabla_x f\|^2_{L^2_{x,v}} \\
& - \frac{\alpha_3\kappa}{2}\frac{t^3}{\varepsilon^6}\|\langle v\rangle^\alpha \widetilde\nabla_x f\|^2_{L^2_x(H^1_{v,*})}.
\end{aligned}
$$

We now choose $\alpha_1=\eta$, $\alpha_2=\eta^{3/2}$, and $\alpha_3=\eta^{5/3}$ with $\eta\in(0,1)$ small enough such that each quantity appearing inside the parentheses in above inequality is positive. Therefore one obtains that $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_\varepsilon(t)\leqslant 0$, for any $t\in(0,\varepsilon^2]$, which concludes the proof as explained in Step 1. ☐

Proof of Lemma 5.6. The proof follows the same lines as the proof of Lemma 5.5. More precisely, as explained in the Step 1 of the proof of Lemma 5.5, in order to obtain the desired result it is sufficient to prove that $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}^1_\varepsilon(t)\leqslant 0$ for all $t\in(0,\varepsilon^2]$ for some well-chosen functional $\mathcal{E}^1_\varepsilon$.

Let $\alpha\in\mathbf{R}$ and f_{in} be such that $\langle v\rangle^\alpha f_{\mathrm{in}}\in L^2_{x,v}$. We then consider the solution $f(t)=S_{\mathcal{B}_\varepsilon}(t)f_{\mathrm{in}}$ to the equation $\partial_t f=\mathcal{B}_\varepsilon f$ with initial data f_{in}. Recalling that $\mathcal{E}_\varepsilon$ is defined in (B.3), we then define the functional

$$
\begin{aligned}
\mathcal{E}^1_\varepsilon(t) := {} & \mathcal{E}_\varepsilon(t) \\
& + \beta_1\left(\frac{t}{\varepsilon^2}\right)^2\left(K_1\|\langle v\rangle^\alpha\langle v\rangle^{\gamma+2}f\|^2_{L^2_{x,v}} + K_2\|\langle v\rangle^\alpha\langle v\rangle^{\frac{\gamma}{2}+1}\widetilde\nabla_v f\|^2_{L^2_{x,v}} + \|\langle v\rangle^\alpha\widetilde\nabla_v\widetilde\nabla_v f\|^2_{L^2_{x,v}}\right) \\
& + \varepsilon^2\beta_2\left(\frac{t}{\varepsilon^2}\right)^4\Big(K_1\|\langle v\rangle^\alpha\langle v\rangle^{\gamma+1}\nabla_x f\|^2_{L^2_{x,v}} + K_2\|\langle v\rangle^\alpha\langle v\rangle^{\frac{\gamma}{2}+1}\widetilde\nabla_x f\|^2_{L^2_{x,v}} \\
& \qquad + \|\langle v\rangle^\alpha\widetilde\nabla_v\widetilde\nabla_x f\|^2_{L^2_{x,v}}\Big)
\end{aligned}
$$

$$
\begin{aligned}
&+ \varepsilon^3\beta_3\left(\frac{t}{\varepsilon^2}\right)^5 \left\langle \langle v\rangle^\alpha \widetilde\nabla_v\widetilde\nabla_x f, \langle v\rangle^\alpha \widetilde\nabla_x\widetilde\nabla_x f\right\rangle_{L^2_{x,v}} \\
&+ \varepsilon^4\beta_4\left(\frac{t}{\varepsilon^2}\right)^6 \Big(K_1\|\langle v\rangle^\alpha\langle v\rangle^\gamma \nabla_x\nabla_x f\|^2_{L^2_{x,v}} + K_2\|\langle v\rangle^\alpha\langle v\rangle^{\frac{\gamma}{2}} \nabla_x\widetilde\nabla_x f\|^2_{L^2_{x,v}} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad + \|\langle v\rangle^\alpha\widetilde\nabla_x\widetilde\nabla_x f\|^2_{L^2_{x,v}}\Big)
\end{aligned}
$$

where $\beta_1,\beta_2,\beta_3,\beta_4,K_1,K_2>0$ are positive constants to be chosen later such that $0<\beta_4\ll\beta_3\ll\beta_2\ll\beta_1\ll 1$, $\beta_4\leqslant\sqrt{\beta_3\beta_2}$ and $K_1\gg K_2\gg 1$.

Step 1. From the proof of Lemma 5.5 (Step 6), we already have that for any $t\in(0,\varepsilon^2]$, there holds

$$
\begin{aligned}
\frac{\mathrm d}{\mathrm dt}\mathcal E_\varepsilon(t) &\leqslant -\frac{\kappa}{2\varepsilon^2}\|\langle v\rangle^\alpha f\|^2_{L^2_x(H^1_{v,*})} - \frac{\eta\kappa t}{2\varepsilon^4}\|\langle v\rangle^\alpha f\|^2_{L^2_x(H^2_{v,*})} \\
&\quad - \frac{\eta^{3/2}t^2}{4\varepsilon^4}\|\langle v\rangle^\alpha\widetilde\nabla_x f\|^2_{L^2_{x,v}} - \frac{\eta^{5/3}\kappa}{2}\frac{t^3}{\varepsilon^6}\|\langle v\rangle^\alpha\widetilde\nabla_x f\|^2_{L^2_x(H^1_{v,*})}.
\end{aligned}
\tag{B.8}
$$

for some constant $\kappa>0$ and where $\eta\in(0,1)$ is small enough.

Step 2. From Lemma 5.4, we already have

$$
\frac{\mathrm d}{\mathrm dt}\|\langle v\rangle^\alpha\langle v\rangle^{\gamma+2} f\|^2_{L^2_{x,v}} \leqslant -\frac{\kappa}{\varepsilon^2}\|\langle v\rangle^\alpha\langle v\rangle^{\gamma+2} f\|^2_{L^2_x(H^1_{v,*})}
\tag{B.9}
$$

for some $\kappa>0$. Moreover, from the proof of Lemma 5.5 (Step 3), we already know that

$$
\begin{aligned}
&\frac{\mathrm d}{\mathrm dt}\|\langle v\rangle^\alpha\langle v\rangle^{\frac{\gamma}{2}+1}\widetilde\nabla_v f\|^2_{L^2_{x,v}} \\
&\leqslant -\frac{\kappa}{2\varepsilon^2}\|\langle v\rangle^\alpha\langle v\rangle^{\frac{\gamma}{2}+1}\widetilde\nabla_v f\|^2_{L^2_x(H^1_{v,*})} + \frac{C}{\varepsilon^2}\|\langle v\rangle^\alpha\langle v\rangle^{\frac{3\gamma}{2}+2} f\|^2_{L^2_{x,v}} \\
&\quad + \frac{C}{\varepsilon^2}\|\langle v\rangle^\alpha\langle v\rangle^{\frac{3\gamma}{2}+2}\nabla_v f\|^2_{L^2_{x,v}} + \frac{C}{\varepsilon}\|\langle v\rangle^\alpha\langle v\rangle^{\frac{\gamma}{2}+1}\widetilde\nabla_v f\|_{L^2_{x,v}}\|\langle v\rangle^\alpha\langle v\rangle^{\frac{\gamma}{2}+1}\widetilde\nabla_x f\|_{L^2_{x,v}}
\end{aligned}
\tag{B.10}
$$

for some constants $\kappa, C>0$. We now compute

$$
\begin{aligned}
&\frac12\frac{\mathrm d}{\mathrm dt}\|\langle v\rangle^\alpha\widetilde\nabla_v\widetilde\nabla_v f\|^2_{L^2_{x,v}} \\
&= \left\langle \langle v\rangle^\alpha\widetilde\nabla_{v_i}\widetilde\nabla_{v_j} f, \langle v\rangle^\alpha\widetilde\nabla_{v_i}\widetilde\nabla_{v_j}\mathcal B_\varepsilon f\right\rangle_{L^2_{x,v}} \\
&= \left\langle \langle v\rangle^\alpha\widetilde\nabla_{v_i}\widetilde\nabla_{v_j} f, \langle v\rangle^\alpha\widetilde\nabla_{v_i}\mathcal B_\varepsilon(\widetilde\nabla_{v_j} f)\right\rangle_{L^2_{x,v}} + \left\langle \langle v\rangle^\alpha\widetilde\nabla_{v_i}\widetilde\nabla_{v_j} f, \langle v\rangle^\alpha\widetilde\nabla_{v_i}[\widetilde\nabla_{v_j},\mathcal B_\varepsilon] f\right\rangle_{L^2_{x,v}} \\
&= \left\langle \langle v\rangle^\alpha\widetilde\nabla_{v_i}\widetilde\nabla_{v_j} f, \langle v\rangle^\alpha\widetilde\nabla_{v_i}\mathcal B_\varepsilon(\widetilde\nabla_{v_j} f)\right\rangle_{L^2_{x,v}} + \left\langle \langle v\rangle^\alpha\widetilde\nabla_{v_i}\widetilde\nabla_{v_j} f, \langle v\rangle^\alpha[\widetilde\nabla_{v_j},\mathcal B_\varepsilon]\widetilde\nabla_{v_i} f\right\rangle_{L^2_{x,v}} \\
&\quad + \left\langle \langle v\rangle^\alpha\widetilde\nabla_{v_i}\widetilde\nabla_{v_j} f, \langle v\rangle^\alpha[\widetilde\nabla_{v_i},[\widetilde\nabla_{v_j},\mathcal B_\varepsilon]] f\right\rangle_{L^2_{x,v}} \\
&=: I_1+I_2+I_3.
\end{aligned}
$$

From the proof of Lemma 5.5 (Step 3), we already know that

$$
\begin{aligned}
I_1+I_2 &\leqslant -\frac{\kappa}{2\varepsilon^2}\|\langle v\rangle^\alpha\widetilde\nabla_v\widetilde\nabla_v f\|^2_{L^2_x(H^1_{v,*})} + \frac{C}{\varepsilon^2}\|\langle v\rangle^\alpha\langle v\rangle^{\gamma+1}\widetilde\nabla_v f\|^2_{L^2_{x,v}} \\
&\quad + \frac{C}{\varepsilon^2}\|\langle v\rangle^\alpha\langle v\rangle^{\gamma+1}\nabla_v\widetilde\nabla_v f\|^2_{L^2_{x,v}} + \frac{C}{\varepsilon}\|\langle v\rangle^\alpha\widetilde\nabla_v\widetilde\nabla_v f\|_{L^2_{x,v}}\|\langle v\rangle^\alpha\widetilde\nabla_x\widetilde\nabla_v f\|_{L^2_{x,v}}.
\end{aligned}
$$

For the term I_3, we use Lemma 2.2 to get

$$
\begin{aligned}
\left[\widetilde\nabla_{v_i},[\widetilde\nabla_{v_j},\mathcal B_\varepsilon]\right] f &= -\frac{1}{\varepsilon^2}\left[\widetilde\nabla_{v_i},\widetilde\nabla^*_{v_k}[\widetilde\nabla_{v_j},\widetilde\nabla_{v_k}]\right] f - \frac{1}{\varepsilon^2}\left[\widetilde\nabla_{v_i},\widetilde\nabla_{v_k}[\widetilde\nabla_{v_j},\widetilde\nabla^*_{v_k}]\right] f \\
&\quad - \frac{1}{\varepsilon^2}\left[\widetilde\nabla_{v_i},[[\widetilde\nabla_{v_j},\widetilde\nabla^*_{v_k}],\widetilde\nabla_{v_k}]\right] f - \frac{1}{\varepsilon^2}(\widetilde\nabla_{v_i}\widetilde\nabla_{v_j} m^2) f \\
&\quad - \frac{1}{\varepsilon}[\widetilde\nabla_{v_i},\widetilde\nabla_{x_j}] f.
\end{aligned}
$$

Expanding the first two terms, we observe that

$$\begin{aligned}
&\Big[\widetilde\nabla_{v_i}, \widetilde\nabla^*_{v_k}[\widetilde\nabla_{v_j}, \widetilde\nabla_{v_k}]\Big] + \Big[\widetilde\nabla_{v_i}, \widetilde\nabla_{v_k}[\widetilde\nabla_{v_j}, \widetilde\nabla^*_{v_k}]\Big] \\
&\quad = \widetilde\nabla_{v_i}\widetilde\nabla^*_{v_k}[\widetilde\nabla_{v_j}, \widetilde\nabla_{v_k}] - \widetilde\nabla^*_{v_k}[\widetilde\nabla_{v_j}, \widetilde\nabla_{v_k}]\widetilde\nabla_{v_i} + \widetilde\nabla_{v_i}\widetilde\nabla_{v_k}[\widetilde\nabla_{v_j}, \widetilde\nabla^*_{v_k}] - \widetilde\nabla_{v_k}[\widetilde\nabla_{v_j}, \widetilde\nabla^*_{v_k}]\widetilde\nabla_{v_i} \\
&\quad = \widetilde\nabla^*_{v_k}\Big[\widetilde\nabla_{v_i}, [\widetilde\nabla_{v_j}, \widetilde\nabla_{v_k}]\Big] + \widetilde\nabla_{v_k}\Big[\widetilde\nabla_{v_i}, [\widetilde\nabla_{v_j}, \widetilde\nabla^*_{v_k}]\Big] + [\widetilde\nabla_{v_i}, \widetilde\nabla^*_{v_k}][\widetilde\nabla_{v_j}, \widetilde\nabla_{v_k}] \\
&\qquad\qquad + [\widetilde\nabla_{v_i}, \widetilde\nabla_{v_k}][\widetilde\nabla_{v_j}, \widetilde\nabla^*_{v_k}],
\end{aligned}$$

whence

$$\begin{aligned}
I_3 &= -\frac{1}{\varepsilon^2}\Big\langle \widetilde\nabla_{v_k}\Big(\langle v\rangle^{2\alpha}\widetilde\nabla_{v_i}\widetilde\nabla_{v_j} f\Big), \Big[\widetilde\nabla_{v_i}, [\widetilde\nabla_{v_j}, \widetilde\nabla_{v_k}]\Big] f\Big\rangle_{L^2_{x,v}} \\
&\quad - \frac{1}{\varepsilon^2}\Big\langle \widetilde\nabla^*_{v_k}\Big(\langle v\rangle^{2\alpha}\widetilde\nabla_{v_i}\widetilde\nabla_{v_j} f\Big), \Big[\widetilde\nabla_{v_i}, [\widetilde\nabla_{v_j}, \widetilde\nabla^*_{v_k}]\Big] f\Big\rangle_{L^2_{x,v}} \\
&\quad - \frac{1}{\varepsilon^2}\Big\langle \langle v\rangle^{\alpha}\widetilde\nabla_{v_i}\widetilde\nabla_{v_j} f, \langle v\rangle^{\alpha}[\widetilde\nabla_{v_i}, \widetilde\nabla^*_{v_k}][\widetilde\nabla_{v_j}, \widetilde\nabla_{v_k}] f\Big\rangle_{L^2_{x,v}} \\
&\quad - \frac{1}{\varepsilon^2}\Big\langle \langle v\rangle^{\alpha}\widetilde\nabla_{v_i}\widetilde\nabla_{v_j} f, \langle v\rangle^{\alpha}[\widetilde\nabla_{v_i}, \widetilde\nabla_{v_k}][\widetilde\nabla_{v_j}, \widetilde\nabla^*_{v_k}] f\Big\rangle_{L^2_{x,v}} \\
&\quad - \frac{1}{\varepsilon^2}\Big\langle \langle v\rangle^{\alpha}\widetilde\nabla_{v_i}\widetilde\nabla_{v_j} f, \langle v\rangle^{\alpha}\Big[\widetilde\nabla_{v_i}, [[\widetilde\nabla_{v_j}, \widetilde\nabla^*_{v_k}], \widetilde\nabla_{v_k}]\Big] f\Big\rangle_{L^2_{x,v}} \\
&\quad - \frac{1}{\varepsilon^2}\Big\langle \langle v\rangle^{\alpha}\widetilde\nabla_{v_i}\widetilde\nabla_{v_j} f, \langle v\rangle^{\alpha}(\widetilde\nabla_{v_i}\widetilde\nabla_{v_j} m^2) f\Big\rangle_{L^2_{x,v}} \\
&\quad - \frac{1}{\varepsilon}\Big\langle \langle v\rangle^{\alpha}\widetilde\nabla_{v_i}\widetilde\nabla_{v_j} f, \langle v\rangle^{\alpha}[\widetilde\nabla_{v_i}, \widetilde\nabla_{x_j}] f\Big\rangle_{L^2_{x,v}} \\
&=: -\frac{1}{\varepsilon^2}(I_{31} + I_{32} + I_{33} + I_{34} + I_{35} + I_{36}) - \frac{1}{\varepsilon}\Big\langle \langle v\rangle^{\alpha}\widetilde\nabla_{v_i}\widetilde\nabla_{v_j} f, \langle v\rangle^{\alpha}[\widetilde\nabla_{v_i}, \widetilde\nabla_{x_j}] f\Big\rangle_{L^2_{x,v}}.
\end{aligned}$$

Now remark that

$$\|\langle v\rangle^{-\alpha}\widetilde\nabla^*_{v_k}(\langle v\rangle^{2\alpha}\widetilde\nabla_{v_i}\widetilde\nabla_{v_j} f)\|_{L^2_{x,v}} + \|\langle v\rangle^{-\alpha}\widetilde\nabla_{v_k}(\langle v\rangle^{2\alpha}\widetilde\nabla_{v_i}\widetilde\nabla_{v_j} f)\|_{L^2_{x,v}} \lesssim \|\langle v\rangle^{\alpha}\widetilde\nabla_{v}\widetilde\nabla_{v} f\|_{L^2_x(H^1_{v,*})}$$

and using Lemma 2.2, we obtain

$$I_{31} + I_{32} \lesssim \|\langle v\rangle^{\alpha}\widetilde\nabla_{v}\widetilde\nabla_{v} f\|_{L^2_x(H^1_{v,*})}\Big(\|\langle v\rangle^{\alpha}\langle v\rangle^{\frac{3\gamma}{2}+1}\nabla_v f\|_{L^2_{x,v}} + \|\langle v\rangle^{\alpha}\langle v\rangle^{\frac{3\gamma}{2}} f\|_{L^2_{x,v}}\Big).$$

Thanks to Lemma 2.2, we observe that

$$|[\widetilde\nabla_{v_i}, \widetilde\nabla^*_{v_k}][\widetilde\nabla_{v_j}, \widetilde\nabla_{v_k}] f| + |[\widetilde\nabla_{v_i}, \widetilde\nabla_{v_k}][\widetilde\nabla_{v_j}, \widetilde\nabla^*_{v_k}] f| \lesssim \langle v\rangle^{2\gamma+2}|\nabla_v\nabla_v f| + \langle v\rangle^{2\gamma+1}|\nabla_v f|$$

and thus we obtain

$$\begin{aligned}
I_{33} + I_{34} &\lesssim \|\langle v\rangle^{\alpha}\langle v\rangle^{\frac{\gamma}{2}+1}\widetilde\nabla_{v}\widetilde\nabla_{v} f\|_{L^2_{x,v}}\Big(\|\langle v\rangle^{\alpha}\langle v\rangle^{\frac{3\gamma}{2}+1}\nabla_v\nabla_v f\|_{L^2_{x,v}} + \|\langle v\rangle^{\alpha}\langle v\rangle^{\frac{3\gamma}{2}}\nabla_v f\|_{L^2_{x,v}}\Big) \\
&\lesssim \|\langle v\rangle^{\alpha}\widetilde\nabla_{v}\widetilde\nabla_{v} f\|_{L^2_x(H^1_{v,*})}\Big(\|\langle v\rangle^{\alpha}\langle v\rangle^{\frac{3\gamma}{2}+1}\nabla_v\nabla_v f\|_{L^2_{x,v}} + \|\langle v\rangle^{\alpha}\langle v\rangle^{\frac{3\gamma}{2}}\nabla_v f\|_{L^2_{x,v}}\Big).
\end{aligned}$$

For the term I_{35}, we write

$$\begin{aligned}
I_{35} &= \Big\langle \widetilde\nabla^*_{v_i}(\langle v\rangle^{2\alpha}\widetilde\nabla_{v_i}\widetilde\nabla_{v_j} f), [[\widetilde\nabla_{v_j}, \widetilde\nabla^*_{v_k}], \widetilde\nabla_{v_k}] f\Big\rangle_{L^2_{x,v}} \\
&\qquad - \Big\langle \langle v\rangle^{\alpha}\widetilde\nabla_{v_i}\widetilde\nabla_{v_j} f, \langle v\rangle^{\alpha}[[\widetilde\nabla_{v_j}, \widetilde\nabla^*_{v_k}], \widetilde\nabla_{v_k}]\widetilde\nabla_{v_i} f\Big\rangle_{L^2_{x,v}}
\end{aligned}$$

and using Lemma 2.2, we thus get

$$\begin{aligned}
I_{35} &\lesssim \|\langle v\rangle^{\alpha}\widetilde\nabla_{v}\widetilde\nabla_{v} f\|_{L^2_x(H^1_{v,*})}\Big(\|\langle v\rangle^{\alpha}\langle v\rangle^{\frac{3\gamma}{2}+1}\nabla_v f\|_{L^2_{x,v}} + \|\langle v\rangle^{\alpha}\langle v\rangle^{\frac{3\gamma}{2}} f\|_{L^2_{x,v}} \\
&\qquad + \|\langle v\rangle^{\alpha}\langle v\rangle^{\gamma}\nabla_v\widetilde\nabla_v f\|_{L^2_{x,v}}\Big).
\end{aligned}$$

Using that $|\widetilde\nabla_{v_i}\widetilde\nabla_{v_j} m^2| \lesssim \langle v\rangle^{2\gamma+2}$, we also obtain

$$I_{36} \lesssim \|\langle v\rangle^{\alpha}\widetilde\nabla_{v}\widetilde\nabla_{v} f\|_{L^2_x(H^1_{v,*})}\|\langle v\rangle^{\alpha}\langle v\rangle^{\frac{3\gamma}{2}+1} f\|_{L^2_{x,v}}.$$

Gathering previous estimates and using Young's inequality, we thus get

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\|\langle v\rangle^{\alpha}\widetilde{\nabla}_v\widetilde{\nabla}_v f\|^2_{L^2_{x,v}} \leqslant & -\frac{\kappa}{4\varepsilon^2}\|\langle v\rangle^{\alpha}\widetilde{\nabla}_v\widetilde{\nabla}_v f\|^2_{L^2_x(H^1_{v,*})} + \frac{C}{\varepsilon^2}\|\langle v\rangle^{\alpha}\langle v\rangle^{\gamma+1}\widetilde{\nabla}_v f\|^2_{L^2_{x,v}} \\
& + \frac{C}{\varepsilon^2}\|\langle v\rangle^{\alpha}\langle v\rangle^{\gamma+1}\nabla_v\widetilde{\nabla}_v f\|^2_{L^2_{x,v}} + \frac{C}{\varepsilon^2}\|\langle v\rangle^{\alpha}\langle v\rangle^{\frac{3\gamma}{2}+1}\nabla_v\nabla_v f\|^2_{L^2_{x,v}} \\
& + \frac{C}{\varepsilon^2}\|\langle v\rangle^{\alpha}\langle v\rangle^{\frac{3\gamma}{2}+1}\nabla_v f\|^2_{L^2_{x,v}} + \frac{C}{\varepsilon^2}\|\langle v\rangle^{\alpha}\langle v\rangle^{\frac{3\gamma}{2}+2} f\|^2_{L^2_{x,v}} \\
& + \frac{C}{\varepsilon}\|\langle v\rangle^{\alpha}\widetilde{\nabla}_v\widetilde{\nabla}_v f\|_{L^2_{x,v}}\|\langle v\rangle^{\alpha}\widetilde{\nabla}_x f\|_{L^2_x(H^1_{v,*})}.
\end{aligned}
\tag{B.11}
$$

Observing that

$$
\begin{aligned}
\|\langle v\rangle^{\alpha}\langle v\rangle^{\frac{3\gamma}{2}+2} f\|_{L^2_{x,v}} + \|\langle v\rangle^{\alpha}\langle v\rangle^{\frac{3\gamma}{2}+2}\nabla_v f\|_{L^2_{x,v}} + \|\langle v\rangle^{\alpha}\langle v\rangle^{\gamma+1}\widetilde{\nabla}_v f\|_{L^2_{x,v}} \\
\lesssim \|\langle v\rangle^{\alpha}\langle v\rangle^{\gamma+2} f\|_{L^2_x(H^1_{v,*})}
\end{aligned}
$$

and

$$
\|\langle v\rangle^{\alpha}\langle v\rangle^{\gamma+1}\nabla_v\widetilde{\nabla}_v f\|_{L^2_{x,v}} + \|\langle v\rangle^{\alpha}\langle v\rangle^{\frac{3\gamma}{2}+1}\nabla_v\nabla_v f\|_{L^2_{x,v}} \lesssim \|\langle v\rangle^{\alpha}\langle v\rangle^{\frac{\gamma}{2}+1}\widetilde{\nabla}_v f\|_{L^2_x(H^1_{v,*})},
$$

we then gather (B.9)–(B.10)–(B.11) choosing $K_1 \gg K_2 \gg 1$ large enough, which yields

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\Big(K_1\|\langle v\rangle^{\alpha}\langle v\rangle^{\gamma+2} f\|^2_{L^2_{x,v}} + K_2\|\langle v\rangle^{\alpha}\langle v\rangle^{\frac{\gamma}{2}+1}\widetilde{\nabla}_v f\|^2_{L^2_{x,v}} + \|\langle v\rangle^{\alpha}\widetilde{\nabla}_v\widetilde{\nabla}_v f\|^2_{L^2_{x,v}}\Big) \\
&\quad\leqslant -\frac{\kappa}{\varepsilon^2}\left(\|\langle v\rangle^{\alpha}\langle v\rangle^{\gamma+2} f\|^2_{L^2_x(H^1_{v,*})} + \|\langle v\rangle^{\alpha}\langle v\rangle^{\frac{\gamma}{2}+1}\widetilde{\nabla}_v f\|^2_{L^2_x(H^1_{v,*})} + \|\langle v\rangle^{\alpha}\widetilde{\nabla}_v\widetilde{\nabla}_v f\|^2_{L^2_x(H^1_{v,*})}\right) \\
&\qquad + \frac{C}{\varepsilon}\|\langle v\rangle^{\alpha} f\|_{L^2_x(H^2_{v,*})}\|\langle v\rangle^{\alpha}\widetilde{\nabla}_x f\|_{L^2_x(H^1_{v,*})}.
\end{aligned}
\tag{B.12}
$$

Step 3. From the proof of Lemma 5.5 (Step 5) we already have, with K_1 large enough with respect to K_2,

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left\{K_1\|\langle v\rangle^{\alpha}\langle v\rangle^{\gamma+1}\nabla_x f\|^2_{L^2_{x,v}} + K_2\|\langle v\rangle^{\alpha}\langle v\rangle^{\frac{\gamma}{2}+1}\widetilde{\nabla}_x f\|^2_{L^2_{x,v}}\right\} \\
&\qquad \leqslant -\frac{\kappa}{\varepsilon^2}\|\langle v\rangle^{\alpha}\langle v\rangle^{\frac{\gamma}{2}+1}\widetilde{\nabla}_x f\|^2_{L^2_x(H^1_{v,*})}.
\end{aligned}
\tag{B.13}
$$

We now compute

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\langle v\rangle^{\alpha}\widetilde{\nabla}_v\widetilde{\nabla}_x f\|^2_{L^2_{x,v}} \\
&\quad = \left\langle \langle v\rangle^{\alpha}\widetilde{\nabla}_{v_i}\widetilde{\nabla}_{x_j} f, \langle v\rangle^{\alpha}\widetilde{\nabla}_{v_i}\widetilde{\nabla}_{x_j}\mathcal{B}_\varepsilon f\right\rangle_{L^2_{x,v}} \\
&\quad = \left\langle \langle v\rangle^{\alpha}\widetilde{\nabla}_{v_i}\widetilde{\nabla}_{x_j} f, \langle v\rangle^{\alpha}\widetilde{\nabla}_{v_i}\mathcal{B}_\varepsilon(\widetilde{\nabla}_{x_j} f)\right\rangle_{L^2_{x,v}} + \left\langle \langle v\rangle^{\alpha}\widetilde{\nabla}_{v_i}\widetilde{\nabla}_{x_j} f, \langle v\rangle^{\alpha}\widetilde{\nabla}_{v_i}[\widetilde{\nabla}_{x_j}, \mathcal{B}_\varepsilon] f\right\rangle_{L^2_{x,v}} \\
&\quad = \left\langle \langle v\rangle^{\alpha}\widetilde{\nabla}_{v_i}\widetilde{\nabla}_{x_j} f, \langle v\rangle^{\alpha}\widetilde{\nabla}_{v_i}\mathcal{B}_\varepsilon(\widetilde{\nabla}_{x_j} f)\right\rangle_{L^2_{x,v}} + \left\langle \langle v\rangle^{\alpha}\widetilde{\nabla}_{v_i}\widetilde{\nabla}_{x_j} f, \langle v\rangle^{\alpha}[\widetilde{\nabla}_{x_j}, \mathcal{B}_\varepsilon]\widetilde{\nabla}_{v_i} f\right\rangle_{L^2_{x,v}} \\
&\qquad + \left\langle \langle v\rangle^{\alpha}\widetilde{\nabla}_{v_i}\widetilde{\nabla}_{x_j} f, \langle v\rangle^{\alpha}[\widetilde{\nabla}_{v_i}, [\widetilde{\nabla}_{x_j}, \mathcal{B}_\varepsilon]] f\right\rangle_{L^2_{x,v}} \\
&\quad =: J_1 + J_2 + J_3.
\end{aligned}
$$

From the proof of Lemma 5.5 (Step 3), we already know that

$$
\begin{aligned}
J_1 + J_2 \leqslant & -\frac{\kappa}{2\varepsilon^2}\|\langle v\rangle^{\alpha}\widetilde{\nabla}_v\widetilde{\nabla}_x f\|^2_{L^2_x(H^1_{v,*})} + \frac{C}{\varepsilon^2}\|\langle v\rangle^{\alpha}\langle v\rangle^{\gamma+1}\widetilde{\nabla}_x f\|^2_{L^2_{x,v}} \\
& + \frac{C}{\varepsilon^2}\|\langle v\rangle^{\alpha}\langle v\rangle^{\gamma+1}\nabla_v\widetilde{\nabla}_x f\|^2_{L^2_{x,v}} + \frac{C}{\varepsilon}\|\langle v\rangle^{\alpha}\widetilde{\nabla}_v\widetilde{\nabla}_x f\|_{L^2_{x,v}}\|\langle v\rangle^{\alpha}\widetilde{\nabla}_x\widetilde{\nabla}_x f\|_{L^2_{x,v}}.
\end{aligned}
$$

The term J_3 can be estimated as the term I_3 in Step 2 above. Therefore, we obtain, gathering these estimates and using Young's inequality,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{dt}}\|\langle v\rangle^\alpha\widetilde\nabla_v\widetilde\nabla_x f\|^2_{L^2_{x,v}} &\leqslant -\frac{\kappa}{4\varepsilon^2}\|\langle v\rangle^\alpha\widetilde\nabla_v\widetilde\nabla_x f\|^2_{L^2_x(H^1_{v,*})}+\frac{C}{\varepsilon^2}\|\langle v\rangle^\alpha\langle v\rangle^{\gamma+1}\widetilde\nabla_x f\|^2_{L^2_{x,v}}\\ &\quad+\frac{C}{\varepsilon^2}\|\langle v\rangle^\alpha\langle v\rangle^{\gamma+1}\nabla_v\widetilde\nabla_x f\|^2_{L^2_{x,v}}+\frac{C}{\varepsilon^2}\|\langle v\rangle^\alpha\langle v\rangle^{\frac{3\gamma}{2}+1}\nabla_v\nabla_x f\|^2_{L^2_{x,v}}\\ &\quad+\frac{C}{\varepsilon^2}\|\langle v\rangle^\alpha\langle v\rangle^{\frac{3\gamma}{2}+1}\nabla_x f\|^2_{L^2_{x,v}}+\frac{C}{\varepsilon}\|\langle v\rangle^\alpha\widetilde\nabla_v\widetilde\nabla_x f\|_{L^2_{x,v}}\|\langle v\rangle^\alpha\widetilde\nabla_x\widetilde\nabla_x f\|_{L^2_{x,v}}.\end{aligned}\tag{B.14}$$

Observing that

$$\begin{aligned}&\|\langle v\rangle^\alpha\langle v\rangle^{\gamma+1}\widetilde\nabla_x f\|_{L^2_{x,v}}+\|\langle v\rangle^\alpha\langle v\rangle^{\gamma+1}\nabla_v\widetilde\nabla_x f\|_{L^2_{x,v}}\\ &+\|\langle v\rangle^\alpha\langle v\rangle^{\frac{3\gamma}{2}+1}\nabla_v\nabla_x f\|_{L^2_{x,v}}+\|\langle v\rangle^\alpha\langle v\rangle^{\frac{3\gamma}{2}+1}\nabla_x f\|_{L^2_{x,v}}\lesssim\|\langle v\rangle^\alpha\langle v\rangle^{\frac{\gamma}{2}+1}\widetilde\nabla_x f\|_{L^2_x(H^1_{v,*})}\end{aligned}$$

we then gather (B.13)–(B.14) and choose $K_1\gg K_2\gg 1$ large enough, which yields

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{dt}}\Big(K_1\|\langle v\rangle^\alpha\langle v\rangle^{\gamma+1}\nabla_x f\|^2_{L^2_{x,v}}+K_2\|\langle v\rangle^\alpha\langle v\rangle^{\frac{\gamma}{2}+1}\widetilde\nabla_x f\|^2_{L^2_{x,v}}+\|\langle v\rangle^\alpha\widetilde\nabla_v\widetilde\nabla_x f\|^2_{L^2_{x,v}}\Big)\\ &\quad\leqslant-\frac{\kappa}{\varepsilon^2}\Big(\|\langle v\rangle^\alpha\langle v\rangle^{\frac{\gamma}{2}+1}\widetilde\nabla_x f\|^2_{L^2_x(H^1_{v,*})}+\|\langle v\rangle^\alpha\widetilde\nabla_v\widetilde\nabla_x f\|^2_{L^2_x(H^1_{v,*})}\Big)\\ &\qquad+\frac{C}{\varepsilon}\|\langle v\rangle^\alpha\widetilde\nabla_x f\|_{L^2_x(H^1_{v,*})}\|\langle v\rangle^\alpha\widetilde\nabla_x\widetilde\nabla_x f\|_{L^2_{x,v}}.\end{aligned}\tag{B.15}$$

Step 4. Arguing as in the proof of Lemma 5.5 (Step 4), we obtain

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{dt}}\Big\langle\langle v\rangle^\alpha\widetilde\nabla_v\widetilde\nabla_x f,\langle v\rangle^\alpha\widetilde\nabla_x\widetilde\nabla_x f\Big\rangle_{L^2_{x,v}}\\ &\qquad\leqslant-\frac{1}{\varepsilon}\|\langle v\rangle^\alpha\widetilde\nabla_x\widetilde\nabla_x f\|^2_{L^2_{x,v}}+\frac{C}{\varepsilon^2}\|\langle v\rangle^\alpha\widetilde\nabla_x f\|_{L^2_x(H^2_{v,*})}\|\langle v\rangle^\alpha\widetilde\nabla_x\widetilde\nabla_x f\|_{L^2_x(H^1_{v,*})}.\end{aligned}\tag{B.16}$$

Step 5. Since ∇_x commutes with $\widetilde\nabla_x$ and $\mathcal{B}_\varepsilon$, we already know from the proof of Lemma 5.5 (Step 5) that, with K_1 large enough with respect to K_2,

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{dt}}\Big\{K_1\|\langle v\rangle^\alpha\langle v\rangle^{\gamma}\nabla_x\nabla_x f\|^2_{L^2_{x,v}}+K_2\|\langle v\rangle^\alpha\langle v\rangle^{\frac{\gamma}{2}}\nabla_x\widetilde\nabla_x f\|^2_{L^2_{x,v}}\Big\}\\ &\qquad\leqslant-\frac{\kappa}{\varepsilon^2}\|\langle v\rangle^\alpha\langle v\rangle^{\frac{\gamma}{2}}\nabla_x\widetilde\nabla_x f\|^2_{L^2_x(H^1_{v,*})}\end{aligned}\tag{B.17}$$

for some constant $\kappa>0$. We now compute

$$\begin{aligned}&\frac12\frac{\mathrm{d}}{\mathrm{dt}}\|\langle v\rangle^\alpha\widetilde\nabla_x\widetilde\nabla_x f\|^2_{L^2_{x,v}}\\ &=\Big\langle\langle v\rangle^\alpha\widetilde\nabla_{x_i}\widetilde\nabla_{x_j}f,\langle v\rangle^\alpha\widetilde\nabla_{x_i}\widetilde\nabla_{x_j}\mathcal{B}_\varepsilon f\Big\rangle_{L^2_{x,v}}\\ &=\Big\langle\langle v\rangle^\alpha\widetilde\nabla_{x_i}\widetilde\nabla_{x_j}f,\langle v\rangle^\alpha\widetilde\nabla_{x_i}\mathcal{B}_\varepsilon(\widetilde\nabla_{x_j}f)\Big\rangle_{L^2_{x,v}}+\Big\langle\langle v\rangle^\alpha\widetilde\nabla_{x_i}\widetilde\nabla_{x_j}f,\langle v\rangle^\alpha\widetilde\nabla_{x_i}[\widetilde\nabla_{x_j},\mathcal{B}_\varepsilon]f\Big\rangle_{L^2_{x,v}}\\ &=\Big\langle\langle v\rangle^\alpha\widetilde\nabla_{x_i}\widetilde\nabla_{x_j}f,\langle v\rangle^\alpha\widetilde\nabla_{x_i}\mathcal{B}_\varepsilon(\widetilde\nabla_{x_j}f)\Big\rangle_{L^2_{x,v}}+\Big\langle\langle v\rangle^\alpha\widetilde\nabla_{x_i}\widetilde\nabla_{x_j}f,\langle v\rangle^\alpha[\widetilde\nabla_{x_j},\mathcal{B}_\varepsilon]\widetilde\nabla_{x_i}f\Big\rangle_{L^2_{x,v}}\\ &\quad+\Big\langle\langle v\rangle^\alpha\widetilde\nabla_{x_i}\widetilde\nabla_{x_j}f,\langle v\rangle^\alpha[\widetilde\nabla_{x_i},[\widetilde\nabla_{x_j},\mathcal{B}_\varepsilon]]f\Big\rangle_{L^2_{x,v}}\\ &=:R_1+R_2+R_3.\end{aligned}$$

From the proof of Lemma 5.5 (Step 5), we already know that

$$R_1+R_2\leqslant-\frac{\kappa}{2\varepsilon^2}\|\langle v\rangle^\alpha\widetilde\nabla_x\widetilde\nabla_x f\|^2_{L^2_x(H^1_{v,*})}+\frac{C}{\varepsilon^2}\|\langle v\rangle^\alpha\langle v\rangle^{\gamma+1}\nabla_x\widetilde\nabla_x f\|^2_{L^2_{x,v}}.$$

The term R_3 can be estimated in a similar way as the term I_3 in Step 2 above. Therefore we obtain, using Young's inequality,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{dt}}\|\langle v\rangle^\alpha\widetilde\nabla_x\widetilde\nabla_x f\|^2_{L^2_{x,v}}&\leqslant-\frac{\kappa}{2\varepsilon^2}\|\langle v\rangle^\alpha\widetilde\nabla_x\widetilde\nabla_x f\|^2_{L^2_x(H^1_{v,*})}+\frac{C}{\varepsilon^2}\|\langle v\rangle^\alpha\langle v\rangle^{\gamma+1}\nabla_x\widetilde\nabla_x f\|^2_{L^2_{x,v}}\\ &\quad+\frac{C}{\varepsilon^2}\|\langle v\rangle^\alpha\langle v\rangle^{\frac{3\gamma}{2}+1}\nabla_x\nabla_x f\|^2_{L^2_{x,v}}.\end{aligned}\tag{B.18}$$

Observing that

$$\|\langle v\rangle^{\alpha}\langle v\rangle^{\gamma+1}\nabla_x\widetilde{\nabla}_x f\|_{L^2_{x,v}} + \|\langle v\rangle^{\alpha}\langle v\rangle^{\frac{3\gamma}{2}+1}\nabla_x\nabla_x f\|_{L^2_{x,v}} \lesssim \|\langle v\rangle^{\alpha}\langle v\rangle^{\frac{\gamma}{2}}\nabla_x\widetilde{\nabla}_x f\|_{L^2_x(H^1_{v,*})}$$

we then gather (B.17)–(B.18) and choose $K_1 \gg K_2 \gg 1$ large enough, which yields

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Big(K_1\|\langle v\rangle^{\alpha}\langle v\rangle^{\gamma}\nabla_x\nabla_x f\|^2_{L^2_{x,v}} + K_2\|\langle v\rangle^{\alpha}\langle v\rangle^{\frac{\gamma}{2}}\nabla_x\widetilde{\nabla}_x f\|^2_{L^2_{x,v}} + \|\langle v\rangle^{\alpha}\widetilde{\nabla}_x\widetilde{\nabla}_x f\|^2_{L^2_{x,v}}\Big)\\ \leqslant -\frac{\kappa}{\varepsilon^2}\|\langle v\rangle^{\alpha}\widetilde{\nabla}_x\widetilde{\nabla}_x f\|^2_{L^2_x(H^1_{v,*})}.\end{aligned} \tag{B.19}$$

Step 6. We can then conclude the proof as in Step 6 of the proof of Lemma 5.5. □

References

[1] Alexandre, R., and Villani, C. On the Landau approximation in plasma physics. *Ann. Inst. H. Poincaré Anal. Non Linéaire 21*, 1 (2004), 61–95.

[2] Alonso, R., Lods, B., and Tristani, I. Fluid dynamic limit of Boltzmann equation for granular hard-spheres in a nearly elastic regime. *arXiv preprint arXiv:2008.05173*, 2021.

[3] Bardos, C., Golse, F., and Levermore, C. D. Fluid dynamic limits of kinetic equations. II. Convergence proofs for the Boltzmann equation. *Comm. Pure Appl. Math. 46*, 5 (1993), 667–753.

[4] Bardos, C., Golse, F., and Levermore, D. Fluid dynamic limits of kinetic equations. I. Formal derivations. *J. Statist. Phys. 63*, 1-2 (1991), 323–344.

[5] Bardos, C., and Ukai, S. The classical incompressible Navier-Stokes limit of the Boltzmann equation. *Math. Models Methods Appl. Sci. 1*, 2 (1991), 235–257.

[6] Bergh, J., and Löfström, J. *Interpolation spaces. An introduction.* Springer-Verlag, Berlin-New York, 1976. Grundlehren der Mathematischen Wissenschaften, No. 223.

[7] Bernou, A., Carrapatoso, K., Mischler, S., and Tristani, I. Hypocoercivity for kinetic linear equations in bounded domains with general Maxwell boundary condition. *arXiv preprint arXiv:2102.07709*, to appear in *Annales de l'Institut Henri Poincaré Analyse Non Linéaire*, 2022.

[8] Briant, M. From the Boltzmann equation to the incompressible Navier-Stokes equations on the torus: a quantitative error estimate. *J. Differential Equations 259*, 11 (2015), 6072–6141.

[9] Briant, M., Merino-Aceituno, S., and Mouhot, C. From Boltzmann to incompressible Navier-Stokes in Sobolev spaces with polynomial weight. *Anal. Appl. (Singap.) 17*, 1 (2019), 85–116.

[10] Caflisch, R. E. The fluid dynamic limit of the nonlinear Boltzmann equation. *Comm. Pure Appl. Math. 33*, 5 (1980), 651–666.

[11] Carrapatoso, K. Exponential convergence to equilibrium for the homogeneous Landau equation with hard potentials. *Bull. Sci. Math. 139*, 7 (2015), 777–805.

[12] Carrapatoso, K., and Mischler, S. Landau equation for very soft and Coulomb potentials near Maxwellians. *Ann. PDE 3*, 1 (2017), Paper No. 1, 65.

[13] Carrapatoso, K., Tristani, I., and Wu, K.-C. Cauchy problem and exponential stability for the inhomogeneous Landau equation. *Arch. Ration. Mech. Anal. 221*, 1 (2016), 363–418.

[14] Cercignani, C. *The Boltzmann equation and its applications*, vol. 67 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1988.

[15] Cercignani, C., Illner, R., and Pulvirenti, M. *The mathematical theory of dilute gases*, vol. 106 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1994.

[16] Chapman, S., and Cowling, T. G. *The mathematical theory of non-uniform gases: An account of the kinetic theory of viscosity, thermal conduction, and diffusion in gases*. Cambridge University Press, New York, 1960.

[17] De Masi, A., Esposito, R., and Lebowitz, J. L. Incompressible Navier-Stokes and Euler limits of the Boltzmann equation. *Comm. Pure Appl. Math. 42*, 8 (1989), 1189–1214.

[18] Degond, P., and Lemou, M. Dispersion relations for the linearized Fokker-Planck equation. *Arch. Rational Mech. Anal. 138*, 2 (1997), 137–167.

[19] Desvillettes, L., and Villani, C. On the trend to global equilibrium for spatially inhomogeneous kinetic systems: the Boltzmann equation. *Invent. Math. 159*, 2 (2005), 245–316.

[20] DiPerna, R. J., and Lions, P.-L. On the Cauchy problem for Boltzmann equations: global existence and weak stability. *Ann. of Math. (2) 130*, 2 (1989), 321–366.

[21] Dolbeault, J., Mouhot, C., and Schmeiser, C. Hypocoercivity for kinetic equations with linear relaxation terms. *Comptes Rendus Mathematique 347*, 9-10 (2009), 511 – 516.

[22] Ellis, R. S., and Pinsky, M. A. The first and second fluid approximations to the linearized Boltzmann equation. *J. Math. Pures Appl. (9) 54* (1975), 125–156.

[23] Fujita, H., and Kato, T. On the Navier-Stokes initial value problem. I. *Arch. Rational Mech. Anal. 16* (1964), 269–315.

[24] Gallagher, I., and Tristani, I. On the convergence of smooth solutions from Boltzmann to Navier-Stokes. *Ann. H. Lebesgue 3* (2020), 561–614.

[25] Gervais, P. On the convergence from boltzmann to Navier-Stokes-Fourier for general initial data. *arXiv preprint arXiv:2201.02825*, 2021.

[26] Gervais, P. A spectral study of the linearized boltzmann operator in L^2-spaces with polynomial and Gaussian weights. *Kinetic & Related Models 14*, 4 (2021), 725–747.

[27] Golse, F., and Saint-Raymond, L. The Navier-Stokes limit of the Boltzmann equation for bounded collision kernels. *Invent. Math. 155*, 1 (2004), 81–161.

[28] Golse, F., and Saint-Raymond, L. The incompressible Navier-Stokes limit of the Boltzmann equation for hard cutoff potentials. *J. Math. Pures Appl. (9) 91*, 5 (2009), 508–552.

[29] Grad, H. Asymptotic theory of the Boltzmann equation. II. In *Rarefied Gas Dynamics (Proc. 3rd Internat. Sympos., Palais de l'UNESCO, Paris, 1962), Vol. I* (1963), Academic Press, New York, pp. 26–59.

[30] Grad, H. Asymptotic equivalence of the Navier-Stokes and nonlinear Boltzmann equations. In *Proc. Sympos. Appl. Math., Vol. XVII* (1965), Amer. Math. Soc., Providence, R.I., pp. 154–183.

[31] Gualdani, M. P., Mischler, S., and Mouhot, C. Factorization of non-symmetric operators and exponential H-theorem. *Mém. Soc. Math. Fr. (N.S.)*, 153 (2017), 137.

[32] Guo, Y. The Landau equation in a periodic box. *Comm. Math. Phys. 231*, 3 (2002), 391–434.

[33] Guo, Y. Boltzmann diffusive limit beyond the Navier-Stokes approximation. *Comm. Pure Appl. Math. 59*, 5 (2006), 626–687.

[34] Guo, Y. Decay and continuity of the Boltzmann equation in bounded domains. *Arch. Ration. Mech. Anal. 197*, 3 (2010), 713–809.

[35] Hérau, F. Hypocoercivity and exponential time decay for the linear inhomogeneous relaxation Boltzmann equation. *Asymptot. Anal. 46*, 3-4 (2006), 349–359.

[36] Hérau, F. Introduction to hypocoercive methods and applications for simple linear inhomogeneous kinetic models. In *Lectures on the analysis of nonlinear partial differential equations. Part 5*, vol. 5 of *Morningside Lect. Math.* Int. Press, Somerville, MA, 2018, pp. 119–147.

[37] Hérau, F., and Nier, F. Isotropic hypoellipticity and trend to equilibrium for the Fokker-Planck equation with a high-degree potential. *Arch. Ration. Mech. Anal. 171*, 2 (2004), 151–218.

[38] Hérau, F., Tonon, D., and Tristani, I. Short time diffusion properties of inhomogeneous kinetic equations with fractional collision kernel. *arXiv preprint arXiv:1709.09943*, 2018.

[39] Hérau, F., Tonon, D., and Tristani, I. Regularization estimates and Cauchy theory for inhomogeneous Boltzmann equation for hard potentials without cut-off. *Comm. Math. Phys. 377*, 1 (2020), 697–771.

[40] Hilbert, D. *Sur les problèmes futurs des mathématiques.* Les Grands Classiques Gauthier-Villars. [Gauthier-Villars Great Classics]. Éditions Jacques Gabay, Sceaux, 1990. Les 23 problèmes. [The 23 problems], Translated from the 1900 German original by M. L. Laugel and revised by the author, Reprint of the 1902 French translation.

[41] Jiang, N., and Xiong, L. Diffusive limit of the Boltzmann equation with fluid initial layer in the periodic domain. *SIAM J. Math. Anal. 47*, 3 (2015), 1747–1777.

[42] Jiang, N., Xu, C.-J., and Zhao, H. Incompressible Navier-Stokes-Fourier limit from the Boltzmann equation: classical solutions. *Indiana Univ. Math. J. 67*, 5 (2018), 1817–1855.

[43] Lachowicz, M. On the initial layer and the existence theorem for the nonlinear Boltzmann equation. *Math. Methods Appl. Sci. 9*, 3 (1987), 342–366.

[44] Lemarié-Rieusset, P. G. *Recent developments in the Navier-Stokes problem*, vol. 431 of *Chapman & Hall/CRC Research Notes in Mathematics.* Chapman & Hall/CRC, Boca Raton, FL, 2002.

[45] Lemarié-Rieusset, P. G. *The Navier-Stokes problem in the 21st century.* CRC Press, Boca Raton, FL, 2016.

[46] Levermore, C. D., and Masmoudi, N. From the Boltzmann equation to an incompressible Navier-Stokes-Fourier system. *Arch. Ration. Mech. Anal. 196*, 3 (2010), 753–809.

[47] Lions, P.-L., and Masmoudi, N. From the Boltzmann equations to the equations of incompressible fluid mechanics. I, II. *Arch. Ration. Mech. Anal. 158*, 3 (2001), 173–193, 195–211.

[48] Mischler, S., and Mouhot, C. Exponential stability of slowly decaying solutions to the kinetic-Fokker-Planck equation. *Arch. Ration. Mech. Anal. 221*, 2 (2016), 677–723.

[49] Mouhot, C. Explicit coercivity estimates for the linearized Boltzmann and Landau operators. *Comm. Partial Differential Equations 31*, 7-9 (2006), 1321–1348.

[50] Mouhot, C., and Neumann, L. Quantitative perturbative study of convergence to equilibrium for collisional kinetic models in the torus. *Nonlinearity 19*, 4 (2006), 969–998.

[51] Mouhot, C., and Strain, R. M. Spectral gap and coercivity estimates for linearized Boltzmann collision operators without angular cutoff. *J. Math. Pures Appl. (9) 87*, 5 (2007), 515–535.

[52] Nicolaenko, B. Dispersion Laws for Plane Wave Propagation. *Courant Institute of Mathematical Sciences, New York University* The Boltzmann equation (1971), 125–172.

[53] Nishida, T. Fluid dynamical limit of the nonlinear Boltzmann equation to the level of the compressible Euler equation. *Comm. Math. Phys. 61*, 2 (1978), 119–148.

[54] Rachid, M. Hypoelliptic and spectral estimates for the linearized Landau operator. *arXiv preprint arXiv:2004:09300*, 2020.

[55] Rachid, M. Incompressible navier-stokes-fourier limit from the landau equation. *Kinetic & Related Models 14*, 4 (2021), 599–638.
[56] Saint-Raymond, L. *Hydrodynamic limits of the Boltzmann equation*, vol. 1971 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2009.
[57] Ukai, S. Solutions of the Boltzmann equation. In *Patterns and waves*, vol. 18 of *Stud. Math. Appl.* North-Holland, Amsterdam, 1986, pp. 37–96.
[58] Ukai, S., and Yang, T. Mathematical theory of the Boltzmann equation. In *Lecture Notes Ser No 8, Liu Bie Ju Center for Math. Sci.* City University of Hong-Kong, 2006.
[59] Villani, C. Hypocoercivity. *Mem. Amer. Math. Soc. 202* (2009), iv+141.
[60] Yang, T., and Yu, H. Spectrum analysis of some kinetic equations. *Arch. Ration. Mech. Anal. 222*, 2 (2016), 731–768.

(K. Carrapatoso) Centre de Mathématiques Laurent Schwartz, École Polytechnique, Institut Polytechnique de Paris, 91128 Palaiseau cedex, France
E-mail address: kleber.carrapatoso@polytechnique.edu

(M. Rachid) Laboratoire de Mathématiques Jean Leray, Université de Nantes, 2 rue de la Houssinière BP 92208 F-44322 Nantes Cedex 3, France
E-mail address: Mohamad.Rachid@univ-nantes.fr

(I. Tristani) Département de Mathématiques et Applications, École Normale Supérieure, CNRS, PSL University, 75005 Paris, France
E-mail address: Isabelle.Tristani@ens.fr