ON THE SELF-SIMILAR STABILITY OF THE PARABOLIC-PARABOLIC KELLER-SEGEL EQUATION

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Abstract. We consider the parabolic-parabolic Keller-Segel equation in the plane and prove the nonlinear exponential stability of the self-similar profile in a quasi parabolic-elliptic regime. We first perform a perturbation argument in order to obtain exponential stability for the semigroup associated to part of the first component of the linearized operator, by exploiting the exponential stability of the linearized operator for the parabolic-elliptic Keller-Segel equation. We finally employ a purely semigroup analysis to prove linear, and then nonlinear, exponential stability of the system in appropriated functional spaces with polynomial weights.

Contents

1. Introduction

In this paper we are concerned with the parabolic-parabolic Keller-Segel system in self-similar variables in the plane

$$\begin{cases} \partial_t f = \Delta f + \operatorname{div}(\mu x f - f \nabla u) \\ \partial_t u = \dfrac{1}{\varepsilon}(\Delta u + f) + \mu x \cdot \nabla u, \end{cases} \tag{1.1}$$

with fixed drift parameter $\mu > 0$ and with small time scale parameter $\varepsilon > 0$, which aims to give the time evolution of the collective motion of cells (described by the *cells density* $f = f(t, x)$) that are attracted by a chemical substance (described by the *chemo-attractant concentration* $u = u(t, x)$) they are able to emit ([24, 17]). Here $t \geq 0$ is the time variable and $x \in \mathbb{R}^2$ stands for the space variable. We refer to the work [6] as well as to the reviews [15, 26] and the references quoted therein for biological motivation and mathematical introduction.

We establish in a convenient weighted Sobolev space the exponential stability of the *normalized self-similar profile* in the quasi parabolic-elliptic regime, that is for small values of the time scale $\varepsilon > 0$, without assuming any radial symmetry property on the initial datum. This extends similar results obtained in [8] in a radially symmetric framework. As in that last reference, the proof of the stability is based on a perturbation argument which takes advantage of the exponential stability of the self-similar profile for the parabolic-elliptic Keller-Segel equation established in [7, 11]. The proof however differs from [8] because

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it uses among other things (1) a different, and somehow more standard, perturbation argument performed at the level of the main part of the first component of the linearized operator instead of at the level of the whole linearized system and (2) a purely semigroup analysis of the linear and nonlinear stability of the system.

Our result implies that in a quasi-parabolic-elliptic regime and for some class of initial data without assuming any radial symmetry property, the associated solution to the parabolic-parabolic KS system in standard variables (corresponding to $\mu = 0$) has a self-similar long-time behavior, which in particular means that no concentration occurs in large time and thus the diffusion mechanism is really the dominant phenomenon all along the time evolution.

It is worth mentioning that as far as the existence problem is concerned, an alternative possible approach has been developed in [6] where weak solutions have been proved to exist for a very general class of initial data, see also [21, 20]. The associated uniqueness result has been solved in [8], see also [2, 9], but the accurate analysis of the long-time behavior of these solutions is still lacking. On the other hand, mild solutions have been proved to exist under a smallness condition in the initial datum for instance in [3, 12, 10] with associated self-similar behavior result in the longtime asymptotic in [23, 22, 10] or under a large time scale parameter for instance in [10, 5].

The two, and only two, general properties satisfied (at least formally) by the solutions of the parabolic-parabolic Keller-Segel equation are the positivity preservation of the cells density, i.e.

$$f(t,\cdot) \geq 0 \quad \text{if} \quad f(0,\cdot) \geq 0,$$

and a similar positivity property for the chemo-attractant concentration u, as well as the mass conservation of the cells density, namely

$$\langle\!\langle f(t,\cdot)\rangle\!\rangle = \langle\!\langle f(0,\cdot)\rangle\!\rangle, \quad \forall\, t \geq 0, \quad \langle\!\langle h\rangle\!\rangle := \int_{\mathbb{R}^2} h\,dx. \tag{1.2}$$

That mass conservation (1.2) is known to be violated in some *supercritical mass* situation. However, we will only be concerned in this paper with a *subcritical mass framework* that we describe now.

We denote by $Q = Q^\mu_\varepsilon$ and $P = P^\mu_\varepsilon$ the normalized stationary solutions to the Keller-Segel system (1.1), that is

$$\begin{cases} 0 = \Delta Q + \operatorname{div}(\mu x Q - Q\nabla P), & Q(0) = 8, \\ 0 = \Delta P + Q + \varepsilon\mu x \cdot \nabla P, \end{cases} \tag{1.3}$$

which existence, uniqueness, radially symmetric property and smoothness have been established in [23, 4, 10]. It is worth emphasizing that we adopt here the normalizing convention of [14] motivated by the fact that in the vanishing drift limit

$$Q^\mu_\varepsilon \to Q^0, \quad \nabla P^\mu_\varepsilon \to \nabla P^0, \quad \text{as} \quad \mu \to 0,$$

where (Q^0, P^0) is defined by

$$Q^0(x) := \frac{8}{(1+|x|^2)^2}, \quad \Delta P^0 = Q^0,$$

and thus Q^0 is the well-known 8π critical mass solution to the parabolic-elliptic Keller-Segel equation in standard variables (corresponding thus to $\mu = \varepsilon = 0$). Because for any $\varepsilon > 0$ there exists a one-to-one mapping

$$\mathscr{M}_\varepsilon : (0,\infty) \to (0, 8\pi), \quad \mu \mapsto \mathscr{M}_\varepsilon(\mu) := \langle\!\langle Q^\mu_\varepsilon\rangle\!\rangle,$$

another possible (and more standard) normalization convention should be to fix the drift term $\mu := 1$ and to normalize the stationary solution by its subcritical mass in the interval $(0, 8\pi)$.

We next introduce the perturbation (g, v) of the stationary state (Q, P) defined by

$$f = Q + g, \quad \langle\!\langle g\rangle\!\rangle = 0, \quad u = P + v,$$

in such a way that the mass compatibility condition $\langle\!\langle f \rangle\!\rangle = \langle\!\langle Q \rangle\!\rangle$ is satisfied. If (f, u) is a solution to (1.1) then (g, v) satisfies the system

$$\begin{cases} \partial_t g = \Delta g + \operatorname{div}(\mu x g - g\nabla P - Q\nabla v) - \operatorname{div}(g\nabla v) \\ \partial_t v = \dfrac{1}{\varepsilon}(\Delta v + g) + \mu x \cdot \nabla v, \end{cases} \tag{1.4}$$

and reciprocally. Instead of working with solutions (g, v) to (1.4) we shall rather work with the unknown (g, w) defined by

$$w := v - \kappa * g,$$

where κ is the Laplace kernel in the plane

$$\kappa(z) := -\frac{1}{2\pi}\log|z|, \quad \nabla\kappa(z) = -\frac{1}{2\pi}\frac{z}{|z|^2}, \tag{1.5}$$

so that $\kappa * \Omega$ is a solution to the Laplace equation $-\Delta(\kappa * \Omega) = \Omega$ in $\mathbb{R}^2$. We will therefore consider the modified system

$$\begin{cases} \partial_t g = \Delta g + \operatorname{div}(\mu x g - g\nabla P - Q\nabla\kappa * g - Q\nabla w) - \operatorname{div}(g\nabla\kappa * g) - \operatorname{div}(g\nabla w) \\ \partial_t w = \dfrac{1}{\varepsilon}\Delta w + \mu x \cdot \nabla w + g \\ \qquad + \nabla\kappa * [g\nabla P + Q\nabla\kappa * g + Q\nabla w] + \nabla\kappa * [g\nabla w + g\nabla\kappa * g], \end{cases} \tag{1.6}$$

satisfied by (g, w), that we complement with an initial condition (g_0, w_0).

We introduce the Banach spaces $\mathcal{X} := L^2_k \times (L^p \cap \dot H^1)$ and $\mathcal{Y} = H^1_k \times (L^p \cap \dot H^2)$ endowed with the norms

$$\|(g, w)\|_{\mathcal{X}} := \|g\|_{L^2_k} + \|w\|_{L^p} + \|w\|_{\dot H^1},$$
$$\|(g, v)\|_{\mathcal{Y}} := \|g\|_{H^1_k} + \|w\|_{L^p} + \|w\|_{\dot H^2},$$

where the weighted Lebesgue space $L^p_k(\mathbb{R}^2)$, for $1 \le p \le \infty$ and $k \ge 0$, is defined by

$$L^p_k(\mathbb{R}^2) := \{f \in L^1_{loc}(\mathbb{R}^2);\ \|f\|_{L^p_k} := \|\langle x\rangle^k f\|_{L^p} < \infty\}, \quad \langle x\rangle := (1 + |x|^2)^{1/2},$$

and the norm of the higher-order Sobolev spaces $W^{\ell,p}_k(\mathbb{R}^2)$ is defined by

$$\|f\|^p_{W^{\ell,p}_k} := \sum_{|\alpha|\le\ell} \|\langle x\rangle^k \partial^\alpha f\|^p_{L^p}.$$

We define the homogeneous seminorm $f \mapsto \|f\|_{\dot H^\ell} := \|D^\ell f\|_{L^2}$ and we write $f \in \dot H^\ell$ if $\|f\|_{\dot H^\ell} < \infty$. We also denote by H^{-1}_k the duality space of H^1_k for the scalar product $\langle\cdot,\cdot\rangle_{L^2_k}$, namely

$$\|\phi\|_{H^{-1}_k} = \sup_{\|f\|_{H^1_k}\le 1} \langle \phi, f\rangle_{L^2_k} = \sup_{\|g\|_{H^1}\le 1} \Big\langle \langle x\rangle^k \phi, g\Big\rangle_{L^2} = \|\langle x\rangle^k \phi\|_{H^{-1}},$$

so that we may identify

$$H^{-1}_k = \Big\{F_0 + \operatorname{div} F_1;\ F_i \in L^2_k\Big\}.$$

For $k > 1$, so that $L^2_k \subset L^1$, we finally denote

$$L^2_{k,0} := \Big\{f \in L^2_k;\ \langle\!\langle f \rangle\!\rangle = 0\Big\}.$$

We may now state our main result.

Theorem 1.1. *Let us fix $\mu \in (0, \infty)$, $k > 3$ and $p > 2$. There are $\varepsilon_0, \eta_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$ and any initial data $(g_0, w_0) \in L^2_{k,0} \times (L^p \cap \dot H^1)$ with $\|(g_0, w_0)\|_{\mathcal{X}} \le \eta_0$, there exists a unique global solution $(g, w) \in L^\infty_t(\mathcal{X}) \cap L^2_t(\mathcal{Y})$ to (1.6) which verifies*

$$\|(g, w)\|_{L^\infty_t(\mathcal{X})} + \|(g, w)\|_{L^2_t(\mathcal{Y})} \lesssim \|(g_0, w_0)\|_{\mathcal{X}}. \tag{1.7}$$

Moreover, for any $\lambda \in (0, \frac{2\mu}{p})$, we have the decay estimate

$$\|(g(t), w(t))\|_{\mathcal{X}} \lesssim e^{-\lambda t}\|(g_0, w_0)\|_{\mathcal{X}}, \quad \forall\, t \ge 0. \tag{1.8}$$

This result improves [8, Theorem 1.4] where similar estimates are established with the restriction that the initial datum is radially symmetric (and satisfies additional regularity and confinement conditions) and also improves [23, 22, 10] which deal with small initial data and arbitrary time scale parameter $\varepsilon > 0$. It is worth emphasizing that because of the one-to-one mapping $\mu \mapsto \mathscr{M}_\varepsilon(\mu)$, the choice of a given drift parameter $\mu \in (0,\infty)$ and its associated steady state Q^μ_ε here is equivalent to the choice of a given subcritical mass in $(0, 8\pi)$ for the initial datum in [8, 23, 22, 10]. It is also worth underlining that Theorem 1.1 implies that the corresponding solution (F, U) to the parabolic-parabolic Keller-Segel equation in standard variables (i.e. $\mu = 0$) satisfies

$$F(t,x) \sim \frac{1}{R(t)^2} Q\left(\frac{x}{R(t)}\right), \quad U(t,x) \sim P\left(\frac{x}{R(t)}\right), \quad \text{as} \quad t \to \infty,$$

with $R(t) := (1 + \mu t)^{1/2}$, and we refer to [8, Sec 1.] for further discussions.

As said above, we shall always work with the unknown (g, w) and it is worth stressing why the corresponding evolution system is given by (1.6). We may indeed observe that if (g, v) satisfies (1.4) then the function $w := v - \kappa * g$ straightforwardly satisfies

$$\begin{aligned}\partial_t w &= \frac{1}{\varepsilon}\Delta w + \mu x\cdot\nabla w + \mu x \cdot \nabla\kappa * g - \nabla\kappa * [\nabla g + \mu x g - g\nabla P - Q\nabla\kappa * g - Q\nabla w] \\ &+ \nabla\kappa * [g\nabla w + g\nabla\kappa * g]\,.\end{aligned}$$

Using that

$$x\cdot\nabla\kappa * g - \nabla\kappa * (xg) = -\frac{1}{2\pi}\int \frac{(x-y)}{|x-y|^2}\{x\,g(y) - y\,g(y)\}\,dy = -\frac{1}{2\pi}\int_{\mathbb{R}^2} g(y)\,dy = 0,$$

because of the mass vanishing condition on g, the equation on w simplifies and thus (g, w) satisfies (1.6). Equivalently, defining the operator

$$\mathcal{L}(g,w) = (\mathcal{L}_1(g,w), \mathcal{L}_2(g,w))$$

by

$$\mathcal{L}_i(g,w) = \mathcal{L}_{i,1} g + \mathcal{L}_{i,2} w, \quad i = 1, 2,$$

with

$$\text{(1.9)} \quad \begin{cases} \mathcal{L}_{1,1} g = \Delta g + \operatorname{div}(\mu x g - g\nabla P - Q\nabla\kappa * g), & \mathcal{L}_{1,2} w = -\operatorname{div}(Q\nabla w), \\ \mathcal{L}_{2,1} g = g + \nabla\kappa * [g\nabla P + Q\nabla\kappa * g]\,, & \mathcal{L}_{2,2} w = \frac{1}{\varepsilon}\Delta w + \mu x\cdot\nabla w + \nabla\kappa * [Q\nabla w]\,, \end{cases}$$

the system (1.6) on (g, w) rewrites as

$$\text{(1.10)} \quad \begin{cases} \partial_t(g,w) = \mathcal{L}(g,w) + (-\operatorname{div}(g\nabla\kappa * g) - \operatorname{div}(g\nabla w), \nabla\kappa * [g\nabla w + g\nabla\kappa * g]) \\ (g,w)_{|t=0} = (g_0, w_0). \end{cases}$$

In the initial Sections 2 and 3, we present some estimates on the family of steady states (Q, P) and some functional inequalities that will be useful throughout the paper. In Section 4, we establish the dissipativity of the operator $\mathcal{L}_{1,1}$ and next the exponential decay of the associated semigroup for small enough values of $\varepsilon > 0$, thanks to a perturbation argument and by taking advantage of the dissipativity of the limit operator for $\varepsilon = 0$ corresponding to the usual linearized parabolic-elliptic Keller-Segel operator. In Section 5, we prove in a more direct way the dissipativity of the operator $\mathcal{L}_{2,2}$. In Section 6, we deduce then the decay of the semigroup $S_\mathcal{L}$ associated to $\mathcal{L}$ by writing in a proper accurate enough semigroup way the two decay estimates of $S_{\mathcal{L}_{i,i}}$ and by showing that both out of the diagonal contributions $\mathcal{L}_{i,j}$, $i \neq j$, are small enough. The above two arguments significantly differ from those used in the proof of [8, Theorem 1.4]. In Section 7, we finally present the proof of Theorem 1.1 which is based on a classical nonlinear stability trick.

In the sequel, for two functions S and T defined on $\mathbb{R}_+$, we define the convolution $S * T$ by

$$(S * T)(t) = \int_0^t S(t-s)T(s)\,\mathrm{d}s, \quad \text{for all } t \geq 0,$$

so that in particular the Duhamel formula associated to an evolution equation

$$\partial_t g = \Lambda g + G, \quad g(0) = g_0,$$

writes

$$g = S_\Lambda g_0 + S_\Lambda * G.$$

Moreover, for $\lambda \in \mathbb{R}$, we denote $\mathbf{e}_\lambda : t \mapsto e^{\lambda t}$. We also write $A \simeq B$ if $A = cB$ for a numerical constant c and $A \lesssim B$ when $A \leq cB$ for a numerical constant $c > 0$ and $A, B \geq 0$.

2. Estimates over Q and P

We present some estimates on the steady states $Q = Q^\mu_\varepsilon$ and $P = P^\mu_\varepsilon$, that we recall satisfy (1.3), which will be useful in the next sections.

Proposition 2.1. *There exist $\varepsilon_0 > 0$ and $\alpha_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ and $\alpha \in (\alpha_0, 1)$ we have:*

(1) *(Bounds over P) For all $x \in \mathbb{R}^2$ there holds*

$$P^0(x) - \frac{\mu\alpha|x|^2}{2} < P(x) - \frac{\mu\alpha|x|^2}{2} < P^0(x) < P(x) < 0, \tag{2.1}$$

$$x \cdot \nabla P(x) - \mu\alpha|x|^2 < x \cdot \nabla P^0(x) < x \cdot \nabla P(x) < 0. \tag{2.2}$$

(2) *(Bounds over Q) For all $x \in \mathbb{R}^2$, there holds*

$$Q^0(x)e^{-\mu\frac{|x|^2}{2}} < Q(x) < Q^0(x)e^{-\mu(1-\alpha)\frac{|x|^2}{2}}. \tag{2.3}$$

Proof of Proposition 2.1. In order to prove (2.1) and (2.2), we follow the same ideas as in the proof of [14, Proposition 4.1], but including the necessary modifications to handle the terms depending on ε that appear for this new problem.

First notice that P^0 and P are radial functions solving the equations

$$\Delta P^0(x) = -Q^0(x) = -8e^{P^0(x)}, \quad P^0(0) = 0,$$

$$\Delta P(x) + \mu\varepsilon x \cdot \nabla P(x) = -Q(x) = -8e^{P(x)-\mu\frac{|x|^2}{2}}, \quad P(0) = 0.$$

In polar variables, these equations read as

$$(P^0)''(r) + \frac{1}{r}(P^0)'(r) = -8e^{P^0(r)}, \quad P^0(0) = (P^0)'(0) = 0,$$

$$(P)''(r) + \left(\frac{1}{r} + \mu\varepsilon r\right)(P)'(r) = -8e^{P(r)-\mu\frac{|r|^2}{2}}, \quad P(0) = P'(0) = 0. \tag{2.4}$$

Solving these two equations, we get

$$P^0(r) = -8\int_0^r \frac{1}{\rho}\int_0^\rho \tau e^{P^0(\tau)} d\tau d\rho, \tag{2.5}$$

$$P(r) = -8\int_0^r \frac{e^{-\mu\varepsilon\frac{\rho^2}{2}}}{\rho}\int_0^\rho \tau e^{P(\tau)-\mu(1-\varepsilon)\frac{\tau^2}{2}} d\tau d\rho. \tag{2.6}$$

Plugging an expansion in powers of r up to order 4 for $P(r)$ in (2.4), the coefficients of such expansion can be computed, which gives

$$P(r) - \mu\alpha\frac{r^2}{2} = -(2 + \frac{\mu\alpha}{2})r^2 + (1 + \frac{\mu}{4}(1+\varepsilon))r^4 + o(r^4),$$

$$P^0(r) = -2r^2 + r^4 + o(r^4),$$

$$P(r) = -2r^2 + (1 + \frac{\mu}{4}(1+\varepsilon))r^4 + o(r^4),$$

for any given $\alpha \in (0, 1)$. This implies that there exists $r_0 = r_0(\alpha) > 0$ such that the following relation holds true

$$P^0(r) - \frac{\mu\alpha|r|^2}{2} < P(r) - \frac{\mu\alpha|r|^2}{2} < P^0(r) < P(r) < 0, \quad \forall r \in (0, r_0). \tag{2.7}$$

Set $\alpha = 1-\varepsilon$ and assume now, by contradiction, that there exists $r_1 > 0$ such that $P^0(r_1) = P(r_1)$ and

$$P(r) - \frac{\mu(1-\varepsilon)|r|^2}{2} < P^0(r) < P(r), \quad \forall\, r \in (0, r_1).$$

Using (2.5) and (2.6), we get

$$\begin{aligned} 0 &= P^0(r_1) - P(r_1) \\ &= -8 \int_0^{r_1} \frac{1}{\rho} \int_0^\rho \tau \left(e^{P^0(\tau)} - e^{P(\tau)-\mu(1-\varepsilon)\frac{\tau^2}{2} - \mu\varepsilon\frac{\rho^2}{2}} \right) d\tau d\rho \\ &\le -8 \int_0^{r_1} \frac{1}{\rho} \int_0^\rho \tau \left(e^{P^0(\tau)} - e^{P(\tau)-\mu(1-\varepsilon)\frac{\tau^2}{2}} \right) d\tau d\rho < 0, \end{aligned}$$

the last strict inequality being due to the second inequality in (2.7). This is a contradiction and therefore $P_0(r) < P(r)$ for all $r > 0$.

On the other hand, suppose by contradiction again that there exist $\alpha \in (0,1)$ and $r_\alpha > 0$ such that

$$\Psi^\mu_\varepsilon(\alpha, r_\alpha) := P(r_\alpha) - \frac{\mu\alpha|r_\alpha|^2}{2} = P^0(r_\alpha)$$

and

$$P(r) - \frac{\mu\alpha|r|^2}{2} < P^0(r) < P(r), \quad \forall\, r \in (0, r_\alpha).$$

Using (2.6), we have

$$\begin{aligned} \Psi^\mu_\varepsilon(\alpha, r_\alpha) &= -8 \int_0^{r_\alpha} \frac{1}{\rho} \int_0^\rho \tau e^{P(\tau)-\mu(1-\varepsilon)\frac{\tau^2}{2} - \mu\varepsilon\frac{\rho^2}{2}} d\tau d\rho - \frac{\mu\alpha|r_\alpha|^2}{2} \\ &< -8 \int_0^{r_\alpha} \frac{1}{\rho} \int_0^\rho \tau e^{P^0(\tau)-\mu(1-\varepsilon)\frac{\tau^2}{2} - \mu\varepsilon\frac{\rho^2}{2}} d\tau d\rho - \frac{\mu\alpha|r_\alpha|^2}{2} =: \tilde\Psi(\alpha, r_\alpha, \mu). \end{aligned}$$

Notice that $\tilde\Psi(\alpha, r_\alpha, 0) = P^0(r_\alpha)$. If we prove that there exist values of α such that $\partial_\mu \tilde\Psi(\alpha, r_\alpha, 0) < 0$ then, in a neighborhood of $\mu = 0$, we would have

$$\Psi^\mu_\varepsilon(\alpha, r_\alpha) < \tilde\Psi(\alpha, r_\alpha, \mu) < \tilde\Psi(\alpha, r_\alpha, 0) = P^0(r_\alpha),$$

which would be a contradiction. Since there exist $\varepsilon_0 > 0$ and $\alpha_0 > 0$ such that for all $(\varepsilon, \alpha) \in [0, \varepsilon_0] \times [\alpha_0, 1]$, the function

$$\begin{aligned} \partial_r \partial_\mu \tilde\Psi(\alpha, r, 0) &= \frac{4(1-\varepsilon)}{r} \int_0^r \tau^3 e^{P^0(\tau)} d\tau + 4\varepsilon r \int_0^r \tau e^{P^0(\tau)} d\tau - \alpha r \\ &= \frac{2(1-\varepsilon)}{r} \left(\ln(1+r^2) + \frac{1}{1+r^2} - 1 \right) + 4\varepsilon r \left(1 - \frac{1}{1+r^2} \right) - \alpha r \end{aligned}$$

is less than 0 for all $r > 0$, we deduce that

$$\partial_\mu \tilde\Psi(\alpha, r_\alpha, 0) = \int_0^{r_\alpha} \partial_r \partial_\mu \tilde\Psi(\alpha, r, 0) dr < 0,$$

which leads to the desired contradiction and finishes the proof of (2.1). We may establish (2.2) in a very similar way, but using the expressions for $x \cdot \nabla P^0$ and $x \cdot \nabla P$. We finally prove (2.3) by taking the exponential of the estimate (2.1). ☐

Lemma 2.2. *There exist some constants $C_i > 0$, $i = 0, \dots, 3$, $\varepsilon_0 > 0$ and $\vartheta \in (0,1)$, such that for any $\mu \in (0, \infty)$ and any $\varepsilon \in (0, \varepsilon_0]$, there hold*

$$0 \le Q(x) \le C_0\, e^{-\mu\vartheta|x|^2/2} \langle x \rangle^{-4}, \tag{2.8}$$

$$\sup_{x \in \mathbb{R}^2} \Big(\frac{1}{|x|} + \langle x \rangle\Big)\, |\nabla P(x)| \le C_1, \tag{2.9}$$

and

$$|\Delta P| \le C_2 \mu\varepsilon + C_3 \langle x \rangle^{-1}. \tag{2.10}$$

Proof of Lemma 2.2. Consider the values of $\varepsilon_0 > 0$ and $\alpha_0 > 0$ given in Proposition 2.1, so that from its proof, $\vartheta := 1 - \alpha_0 \in (0,1)$ is independent from μ and ε. The estimate (2.8) is then nothing but (2.3). Computing the explicit expression for ∇P gives

$$\nabla P = -e^{-\mu\varepsilon\frac{|x|^2}{2}} \frac{x}{|x|^2} \int_0^{|x|} Q(r) e^{\mu\varepsilon\frac{r^2}{2}} r dr,$$

and hence

$$|\nabla P| \leq \frac{1}{|x|} \int_0^{|x|} Q(r) e^{\mu\varepsilon\frac{r^2}{2}} dr \leq \frac{1}{|x|} \int_0^{|x|} \langle r\rangle^{-4} e^{-\mu(\vartheta-\varepsilon)\frac{r^2}{2}} r dr.$$

Taking ε small enough, we deduce

$$|\nabla P| \leq \frac{1}{|x|} \int_0^{|x|} \langle r\rangle^{-4} r dr \leq |x| \langle x\rangle^{-2},$$

which directly implies (2.9). Finally, writing

$$\Delta P = -\mu\varepsilon x \cdot \nabla P - Q,$$

we conclude to (2.10) thanks to (2.2) and (2.8). □

Lemma 2.3. *There exist some constants $\vartheta \in (0,1)$, $C_i > 0$, $i = 1, \dots, 4$, such that for any $\mu \in (0,\infty)$, any $\varepsilon \in (0, \varepsilon_0]$ and any $x \in \mathbb{R}^2$, there holds*

$$|\nabla P^\mu_\varepsilon - \nabla P^\mu_0| \leq \mu\varepsilon C_1 |x|, \tag{2.11}$$

$$|\Delta P^\mu_\varepsilon - \Delta P^\mu_0| \leq \mu\varepsilon C_2, \tag{2.12}$$

$$|Q^\mu_\varepsilon - Q^\mu_0| \leq \mu\varepsilon C_3 e^{-\frac{\vartheta\mu|x|^2}{2}}, \tag{2.13}$$

$$|\nabla Q^\mu_\varepsilon - \nabla Q^\mu_0| \leq \mu\varepsilon C_4 e^{-\frac{\vartheta\mu|x|^2}{2}}. \tag{2.14}$$

Proof of Lemma 2.3. We recall that in radial variables, we have the expressions

$$\begin{aligned} P^\mu_0(r) &= -8 \int_0^r \frac{1}{\rho} \int_0^\rho e^{P^\mu_0 - \mu\frac{\tau^2}{2}} \tau d\tau d\rho, \\ P^\mu_\varepsilon(r) &= -8 \int_0^r \frac{e^{-\mu\varepsilon\frac{\rho^2}{2}}}{\rho} \int_0^\rho e^{P^\mu_\varepsilon - \mu(1-\varepsilon)\frac{\tau^2}{2}} \tau d\tau d\rho \end{aligned}$$

which imply

$$\begin{aligned} P^\mu_\varepsilon - P^\mu_0 &= \left(P^\mu_\varepsilon - \int_0^r (P^\mu_\varepsilon)' e^{\mu\varepsilon\frac{\tau^2}{2}} d\tau \right) + \left(\int_0^r (P^\mu_\varepsilon)' e^{\mu\varepsilon\frac{\tau^2}{2}} d\tau - P^\mu_0 \right) \\ &= -\int_0^r \frac{e^{-\mu\varepsilon\frac{\rho^2}{2}} - 1}{\rho} \int_0^\rho Q^\mu_\varepsilon(\tau) e^{\mu\varepsilon\frac{\tau^2}{2}} \tau d\tau d\rho \\ &\quad - 8 \int_0^r \frac{1}{\rho} \int_0^\rho (e^{P^\mu_\varepsilon - \mu(1-\varepsilon)\frac{\tau^2}{2}} - e^{P^\mu_0 - \mu\frac{\tau^2}{2}}) \tau d\tau. \end{aligned}$$

Directly from Proposition 2.1, we know that $\int_0^\rho Q^\mu_\varepsilon(\tau) e^{\mu\varepsilon\frac{\tau^2}{2}} \tau d\tau \leq \int_0^\rho Q^0(\tau) \tau d\tau \leq 8\pi$, and the mean value theorem gives us that

$$e^{P^\mu_\varepsilon - \mu(1-\varepsilon)\frac{\tau^2}{2}} - e^{P^\mu_0 - \mu\frac{\tau^2}{2}} = (P^\mu_\varepsilon - P^\mu_0 + \mu\varepsilon\frac{\tau^2}{2}) e^{h(\tau)},$$

with $h(\tau)$ satisfying

$$h(\tau) \leq \max\{P^\mu_\varepsilon - \mu(1-\varepsilon)\frac{\tau^2}{2}, P^\mu_0 - \mu\frac{\tau^2}{2}\} \leq P^0 - \frac{\vartheta\mu\tau^2}{2}.$$

Thanks to Proposition 2.1, we deduce

$$8 e^{h(\tau)} \leq Q^0(\tau) e^{-\frac{\vartheta\mu\tau^2}{2}}.$$

Putting everything together, we get

$$|P^\mu_\varepsilon - P^\mu_0| \le 8\pi \int_0^r \frac{1-e^{-\frac{\mu\varepsilon\rho^2}{2}}}{\rho} d\rho + \int_0^r \frac{1}{\rho}\int_0^\rho |P^\mu_\varepsilon - P^\mu_0 + \mu\varepsilon \frac{\tau^2}{2}|Q^0(\tau)\tau d\tau d\rho$$
$$\le \mu\varepsilon K r^2 + \int_0^r \int_0^\rho |P^\mu_\varepsilon - P^\mu_0|Q^0(\tau) d\tau d\rho,$$

where K is a constant independent of μ and ε. Integrating by parts the integral term, we get

$$|P^\mu_\varepsilon - P^\mu_0| \le \mu\varepsilon K r^2 + r\int_0^r |P^\mu_\varepsilon - P^\mu_0|Q^0(\tau)d\tau,$$

which thanks to Grönwall's Lemma gives

$$|P^\mu_\varepsilon - P^\mu_0| \le \mu\varepsilon C_0 |x|^2.$$

This estimate together with a similar manipulation on the gradients of P^μ_ε and P^μ_0 gives (2.11). On the other hand, we have

$$\begin{aligned}|Q^\mu_\varepsilon - Q^\mu_0| &= 8|e^{P^\mu_\varepsilon - \mu(1-\varepsilon)\frac{r^2}{2}} - e^{P^\mu_0 - \mu\frac{r^2}{2}}| \\ &= |P^\mu_\varepsilon - P^\mu_0 + \mu\varepsilon\frac{r^2}{2}| 8 e^{h(r)} \\ &\le \mu\varepsilon C_2 |r|^2 Q^0(r) e^{-\frac{\vartheta\mu r^2}{2}},\end{aligned}$$

which is nothing but (2.13). Repeating the same process for $\nabla(Q - Q_\mu)$ gives (2.14).

Finally, using the equations for P^μ_ε and P^μ_0, we have

$$\Delta(P^\mu_\varepsilon - P^\mu_0) = -(Q^\mu_\varepsilon - Q^\mu_0) - \mu\varepsilon x\cdot\nabla P^\mu_\varepsilon.$$

We conclude to (2.12) thanks to (2.13) for the first term and thanks to (2.2) for the second one. □

3. Functional inequalities

We gather in this section some functional inequalities that we shall use through the paper. First, we provide some estimates over the solution for the Poisson problem.

Lemma 3.1. *There holds*

$$\|D^2\kappa * g\|_{L^2} \lesssim \|g\|_{L^2}, \quad \forall\, g\in L^2, \tag{3.1}$$

and, for $k>2$, there holds

$$\|\nabla\kappa * g\|_{L^2} \lesssim \|g\|_{L^2_k}, \quad \forall\, g\in L^2_{k,0}. \tag{3.2}$$

Proof of Lemma 3.1. Thanks to Plancherel identity and because $\hat\kappa \simeq |\xi|^{-2}$, we have

$$\|\partial^2_{ij}\kappa * g\|_{L^2} \simeq \|\xi_i\xi_j|\xi|^{-2}\hat g\|_{L^2} \le \|\hat g\|_{L^2} \simeq \|g\|_{L^2},$$

what establishes (3.1). We now prove (3.2). We similarly have

$$\|\nabla\kappa * g\|^2_{L^2} \simeq \int \mathbf{1}_{|\xi|\le 1}\frac{|\widehat g(\xi)|^2}{|\xi|^2}\,\mathrm{d}\xi + \int \mathbf{1}_{|\xi|>1}\frac{|\widehat g(\xi)|^2}{|\xi|^2}\,\mathrm{d}\xi =: I_1 + I_2.$$

For the second term we have

$$I_2 \le \int |\widehat g(\xi)|^2\,\mathrm{d}\xi = \|g\|^2_{L^2}.$$

For the first term, using that $\widehat g(0)=0$ because $\langle\!\langle g\rangle\!\rangle = 0$, we have

$$\widehat g(\xi) = \xi\cdot\int_0^1 D_\xi \widehat g(\theta\xi)\,\mathrm{d}\theta,$$

and we thus obtain

$$I_1 \le \left(\sup_{|\xi|\le 1}|D_\xi\widehat g(\xi)|^2\right)\int \mathbf{1}_{|\xi|\le 1}\,\mathrm{d}\xi \lesssim \|\widehat{xg}\|^2_{L^\infty} \lesssim \|xg\|^2_{L^1} \lesssim \|g\|^2_{L^2_k},$$

by using some classical and elementary Fourier identity and estimate as well as the continuous embedding $L^2_k \subset L^1_1$. □

Lemma 3.2. *For $k > 1$, $p > 2$ and $2 \le q \le p$, we have*

$$\|\nabla\kappa * g\|_{L^p} \lesssim \|g\|_{L^q_k}, \quad \forall\, g \in L^q_k. \tag{3.3}$$

Proof of Lemma 3.2. We split $|\nabla\kappa| := K_1 + K_2$, with

$$K_1 := \frac{1}{|x|}\mathbf{1}_{|x|\le 1} \in L^{r_1}, \forall\, r_1 < 2, \quad K_2 := \frac{1}{|x|}\mathbf{1}_{|x|\ge 1} \in L^{r_2}, \forall\, r_2 > 2,$$

that we use with $r_1 := (1 + \frac{1}{p} - \frac{1}{q})^{-1}$ and $r_2 := p$. We then have

$$\begin{aligned} \|\nabla\kappa * f\|_{L^p} &\le \|K_1 * f\|_{L^p} + \|K_2 * f\|_{L^p} \\ &\le \|K_1\|_{L^{r_1}}\|f\|_{L^q} + \|K_2\|_{L^p}\|f\|_{L^1} \\ &\lesssim \|f\|_{L^q_k}, \end{aligned}$$

where we have used the convolution embeddings $L^{r_1} * L^q \subset L^p$ and $L^1 * L^p \subset L^p$ in the second line as well as the Cauchy-Schwarz inequality in the last line in order to prove $L^q_k \subset L^1$. □

We recall the following two particular cases of the Gagliardo-Nirenberg interpolation Theorem in dimension 2.

Lemma 3.3. (1) *For any $p > 2$ and setting $\theta = \frac{p}{2+p}$, we have*

$$\|\nabla w\|_{L^p} \lesssim \|w\|^{1-\theta}_{L^p}\|\nabla^2 w\|^{\theta}_{L^2}, \quad \forall\, w \in L^p \cap \dot H^2. \tag{3.4}$$

(2) *The following Ladyzhenskaya's inequality holds*

$$\|f\|_{L^4} \lesssim \|f\|^{1/2}_{L^2}\|\nabla f\|^{1/2}_{L^2}, \quad \forall\, f \in H^1. \tag{3.5}$$

We will also need the following non classical Gagliardo-Nirenberg type interpolation inequality.

Lemma 3.4. *Let $p \in (2,\infty)$. For any $\beta > 0$, there is $C_\beta > 0$ such that*

$$\|\nabla w\|^2_{L^2} \le \beta\|w\|^2_{L^p} + C_\beta\|\nabla^2 w\|^2_{L^2}, \quad \forall\, w \in L^p \cap \dot H^2.$$

Proof of Lemma 3.4. We compute

$$\begin{aligned} \|\nabla w\|^2_{L^2} &= \int_{|\xi|\le 1}(\langle\xi\rangle^{-1}|\widehat w|)(|\xi|^2|\widehat w|) + \int_{|\xi|>1}(\langle\xi\rangle^{-2/3}|\widehat w|^{2/3})(|\xi|^{8/3}|\widehat w|^{4/3}) \\ &\lesssim \|\langle\xi\rangle^{-1}\widehat w\|_{L^2}\||\xi|^2\widehat w\|_{L^2} + \|\langle\xi\rangle^{-1}\widehat w\|^{2/3}_{L^2}\||\xi|^2\widehat w\|^{4/3}_{L^2} \\ &\lesssim \|w\|_{H^{-1}}\|\nabla^2 w\|_{L^2} + \|w\|^{2/3}_{H^{-1}}\|\nabla^2 w\|^{4/3}_{L^2} \\ &\lesssim \|w\|_{L^p}\|\nabla^2 w\|_{L^2} + \|w\|^{2/3}_{L^p}\|\nabla^2 w\|^{4/3}_{L^2}, \end{aligned}$$

where we have used the Cauchy-Schwarz inequality in the second line and the continuous embedding $L^p(\mathbb{R}^2) \subset H^{-1}(\mathbb{R}^2)$ (consequence of the embedding $H^1(\mathbb{R}^2) \subset L^{p'}(\mathbb{R}^2)$) in the last line. We then conclude by applying Young's inequality. □

4. Estimates for $\mathcal{L}_{1,1}$

In this section, we establish some dissipativity estimates and related semigroup decay estimates successively on the operators $\mathcal{L}_{1,1}$ and related operators.

4.1. **Dissipativity estimates related to $\mathcal{L}_{1,1}$.** In order to keep track of the $\varepsilon \ge 0$ dependence, let us denote

$$\Lambda_\varepsilon := \mathcal{L}_{1,1},$$

where we recall that this one is defined by

$$\mathcal{L}_{1,1} g := \Delta g + \mathrm{div}(\mu x g - g \nabla P - Q \nabla \kappa * g).$$

We start with a first fundamental dissipativity estimate.

Lemma 4.1. *For any $k > 3$, there some constants $\varepsilon_0 > 0$, small enough, and $C_0, \varrho_0 > 0$, large enough, such that*

$$\langle \Lambda_\varepsilon g, g \rangle_{L^2_k} \le -\mu(k-2)\|g\|^2_{L^2_k} - \frac{1}{2}\|\nabla g\|^2_{L^2_k} + C_0 \|g\|^2_{L^2(B_{\varrho_0})}, \tag{4.1}$$

for any $\varepsilon \in (0, \varepsilon_0)$ and $g \in H^2_k$.

Proof of Lemma 4.1. We briefly repeat the proof of [8, Lemma 4.4]. We compute

$$\begin{aligned}\langle \Lambda_\varepsilon g, g \rangle_{L^2_k} &= \int \Delta g g \langle x \rangle^{2k} + \int \mathrm{div}(\mu x g) g \langle x \rangle^{2k} - \int \mathrm{div}(g \nabla P) g \langle x \rangle^{2k} - \int \mathrm{div}(Q \nabla \kappa * g) g \langle x \rangle^{2k} \\ &= I_1 + I_2 + I_3 + I_4,\end{aligned}$$

and estimate each term separately. For the two first terms we have

$$I_1 + I_2 = -\int |\nabla g|^2 \langle x \rangle^{2k} + \int \psi_1 g^2 \langle x \rangle^{2k}$$

where

$$\begin{aligned}\psi_1 &= \frac{|\nabla \langle x \rangle^k|^2}{\langle x \rangle^{2k}} + \frac{\Delta \langle x \rangle^k}{\langle x \rangle^k} + \mu - \mu x \cdot \frac{\nabla \langle x \rangle^k}{\langle x \rangle^k} \\ &= k(2k+\mu)\langle x \rangle^{-2} - \mu(k-1) - 2k\langle x \rangle^{-4}.\end{aligned}$$

Moreover, for the third term we compute

$$\begin{aligned} I_3 &= \int g \nabla P \cdot \nabla g \langle x \rangle^{2k} + 2 \int g^2 \langle x \rangle^{2k} \nabla P \cdot \frac{\nabla \langle x \rangle^k}{\langle x \rangle^k} \\ &= -\frac{1}{2} \int \Delta P g^2 \langle x \rangle^{2k} + \int \nabla P \cdot \frac{\nabla \langle x \rangle^k}{\langle x \rangle^k} g^2 \langle x \rangle^{2k}.\end{aligned}$$

Thanks to the uniform estimates (2.9) and (2.10) on P, we observe that

$$\left| \nabla P \cdot \frac{\nabla \langle x \rangle^k}{\langle x \rangle^k} \right| \le \|x \cdot \nabla P\|_{L^\infty} \langle x \rangle^{-1} \le C_1 \langle x \rangle^{-1}$$

and

$$|\Delta P| \le C_2 \mu \varepsilon + C_3 \langle x \rangle^{-1}$$

for some constants $C_i > 0$, which imply

$$I_3 \le \frac{\varepsilon \mu C_2}{2} \|g\|^2_{L^2_k} + \left(C_1 + \frac{C_3}{2} \right) \|\langle x \rangle^{-\frac{1}{2}} g\|^2_{L^2_k}.$$

For the last term we write

$$I_4 = \int Q(\nabla \kappa * g) \nabla g \langle x \rangle^{2k} + 2 \int Q (\nabla \kappa * g) \frac{\nabla \langle x \rangle^k}{\langle x \rangle^k} g \langle x \rangle^{2k}.$$

Since $\|Q\langle x \rangle^{2k}\|_{L^\infty} \le C_4$ and $\|Q \langle x \rangle^k \nabla \langle x \rangle^k\|_{L^\infty} \le C_4$ thanks to estimate (2.8), we obtain

$$\begin{aligned} I_4 &\le C_4 \|\nabla \kappa * g\|_{L^2} \|\nabla g\|_{L^2} + 2C_4 \|\nabla \kappa * g\|_{L^2} \|g\|_{L^2} \\ &\le C_4' \|\nabla \kappa * g\|^2_{L^2} + \frac{1}{2} \|\nabla g\|^2_{L^2} + C_4' \|g\|^2_{L^2} \\ &\le C_4'' \|\langle x \rangle^{-1} g\|^2_{L^2_k} + \frac{1}{2} \|\nabla g\|^2_{L^2},\end{aligned}$$

where we have used Lemma 3.1 and Young's inequality. Gathering the previous estimates, we get

$$\langle \Lambda_\varepsilon g, g \rangle_{L^2_k} \le -\frac{1}{2} \int |\nabla g|^2 \langle x \rangle^{2k} + \int \bar{\psi}_1 g^2 \langle x \rangle^{2k}$$

with

$$\begin{aligned}\bar{\psi}_1 &= -\mu\left(k-1-\frac{\varepsilon C_2}{2}\right)+\left(C_4''+C_1+\frac{C_3}{2}\right)\langle x\rangle^{-1}+k(2k+\mu)\langle x\rangle^{-2}-k^2\langle x\rangle^{-4}\\ &\le -\mu\left(k-1-\frac{\varepsilon C_2}{2}\right)+C_5\langle x\rangle^{-1}.\end{aligned}$$

We remark that, for any $\varrho_0\ge 1$, we have

$$\langle x\rangle^{-1}\langle x\rangle^{2k}\le \varrho_0^{2k-1}\mathbf{1}_{\langle x\rangle\le\varrho_0}+\frac{1}{\varrho_0}\langle x\rangle^{2k},$$

thus we obtain

$$\langle\Lambda_\varepsilon g,g\rangle_{L^2_k}\le -\frac12\|\nabla g\|^2_{L^2_k}-\mu\left(k-1-\frac{\varepsilon C_2}{2}-\frac{C_5}{\mu\varrho_0}\right)\|g\|^2_{L^2_k}+C_0\|g\|^2_{L^2(B_{\varrho_0})} \tag{4.2}$$

where $C_0=C_5\varrho_0^{2k-1}$. We therefore choose $\varepsilon_0>0$ small enough such that $\varepsilon_0C_2\le 1$ and $\varrho_0\ge 1$ large enough such that $\frac{C_5}{\mu\varrho_0}\le 1/2$, which concludes the proof. ☐

4.2. **Splitting of the operator $\mathcal{L}_{1,1}$.** We introduce the splitting

$$\Lambda_\varepsilon=\mathcal{A}+\mathcal{B}_\varepsilon,\quad \mathcal{A}:=M\chi_\varrho,\quad \mathcal{B}_\varepsilon:=\Lambda_\varepsilon-\mathcal{A},$$

with $\chi_\varrho(x):=\chi(x/\varrho)$, $\chi\in\mathcal{D}(\mathbb{R}^2)$, $\mathbf{1}_{B(0,1)}\le\chi\le\mathbf{1}_{B(0,2)}$, and constants $M,\varrho>0$. We immediately deduce from Lemma 4.1 that $\mathcal{B}_\varepsilon$ is dissipative, more precisely:

Corollary 4.2. *For any $k>3$, any $\varepsilon\in(0,\varepsilon_0)$ and any constants $M\ge C_0$ and $\varrho\ge\varrho_0$, there holds*

$$\langle\mathcal{B}_\varepsilon g,g\rangle_{L^2_k}\le-\mu(k-2)\|g\|^2_{L^2_k}-\frac12\|\nabla g\|^2_{L^2_k}\le-\lambda\|g\|^2_{L^2_k}-\sigma\|g\|^2_{H^1_k} \tag{4.3}$$

for any $0\le\lambda<\mu(k-2)$ with $\sigma=\min(1/2,\mu-\lambda)$, and where $\varepsilon_0,C_0,\varrho_0>0$ are taken from Lemma 4.1.

Remark 4.3. We shall fix hereafter the parameters $M\ge C_0$ and $\varrho\ge\varrho_0$ in the definition of $\mathcal{B}_\varepsilon$ such that Corollary 4.2 holds.

In order to work at the level of the semigroup, we reformulate (4.3) in the following way.

Lemma 4.4. *For any $k>3$, $\varepsilon\in(0,\varepsilon_0)$, $M\ge C_0$ and $\varrho\ge\varrho_0$, there holds*

(1) *For all $0\le\lambda<\mu(k-2)$ and all $g\in L^2_k$, we have*

$$\|\mathbf{e}_\lambda S_{\mathcal{B}_\varepsilon}(\cdot)g\|_{L^\infty_tL^2_k}+\|\mathbf{e}_\lambda S_{\mathcal{B}_\varepsilon}(\cdot)g\|_{L^2_tH^1_k}\lesssim\|g\|_{L^2_k}.$$

(2) *For all $0\le\lambda<\mu(k-2)$ and all $\mathbf{e}_\lambda R\in L^2_tH^{-1}_k$, we have*

$$\|\mathbf{e}_\lambda(S_{\mathcal{B}_\varepsilon}*R)\|_{L^\infty_tL^2_k}+\|\mathbf{e}_\lambda(S_{\mathcal{B}_\varepsilon}*R)\|_{L^2_tH^1_k}\lesssim\|\mathbf{e}_\lambda R\|_{L^2_tH^{-1}_k}.$$

Proof of Lemma 4.4. Let $0\le\lambda<\mu(k-2)$. For $g\in L^2_k$, we first consider $f:=\mathbf{e}_\lambda S_{\mathcal{B}_\varepsilon}(\cdot)g$ the solution to the evolution equation

$$\partial_t f=\mathcal{B}_\varepsilon f+\lambda f,\quad f(0)=g.$$

Because of (4.3), we have

$$\frac12\frac{d}{dt}\|f\|^2_{L^2_k}=\langle\mathcal{B}_\varepsilon f,f\rangle_{L^2_k}+\lambda\|f\|^2_{L^2_k}\le-\sigma\|f\|^2_{H^1_k}$$

from which we deduce (1) thanks to the Grönwall's lemma.

For R such that $\mathbf{e}_\lambda R\in L^2_tH^{-1}_k$, we next consider $f:=\mathbf{e}_\lambda(S_{\mathcal{B}_\varepsilon}*R)$ the solution to the evolution equation

$$\partial_t f=\mathcal{B}_\varepsilon f+\lambda f+\mathbf{e}_\lambda R,\quad f(0)=0.$$

Because of (4.3) and the Young inequality, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|f\|_{L^2_k}^2 &= \langle \mathcal{B}_\varepsilon f, f\rangle_{L^2_k} + \lambda\|f\|_{L^2_k}^2 + \langle \mathbf{e}_\lambda R, f\rangle_{L^2_k}\\ &\le -\sigma\|f\|_{H^1_k}^2 + \|\mathbf{e}_\lambda R\|_{H^{-1}_k}\|f\|_{H^1_k}\\ &\le -\frac{\sigma}{2}\|f\|_{H^1_k}^2 + C\|\mathbf{e}_\lambda R\|_{H^{-1}_k}^2,\end{aligned}$$

for some constant $C = C(\mu,\lambda) > 0$. We deduce (2) thanks to the Grönwall's lemma again. □

4.3. **Spectral analysis of $\mathcal{L}_{1,1}$.** We deduce a nice localization of the spectrum of $\mathcal{L}_{1,1}$ from the previous estimates and a perturbation argument. Let us denote by Λ_0 the linearized operator of the parabolic-elliptic Keller-Segel equation which is given by

$$\Lambda_0 g = \Delta g + \mathrm{div}(\mu x g - g\nabla P_0 - Q_0\nabla\kappa * g),$$

where (Q_0, P_0) is a solution to (1.3) with $\varepsilon = 0$. From [7, 11], we know that for $k > 3$ and $0 < \lambda < \mu$, there exists a constant $C = C(\lambda,\mu,k) \ge 1$ such that

$$\|S_{\Lambda_0}(t)f\|_{L^2_k} \le Ce^{-\lambda t}\|f\|_{L^2_k}, \quad \forall\, f \in L^2_{k,0},$$

and the spectrum verifies

$$\Sigma(\Lambda_0) \cap \Delta_{-\mu} = \{0\} \tag{4.4}$$

where $\Delta_{-\mu} := \{z \in \mathbb{C} : \Re e z > -\mu\}$.

By a perturbation argument similar to the one used in [18] (see also [27, 16]), we are able to obtain a similar picture for the operator $\mathcal{L}_{1,1} = \Lambda_\varepsilon$.

Proposition 4.5. *Let $k > 3$. For any $0 < \lambda < \mu$, there is $\varepsilon_* > 0$ small enough, such that the operator Λ_ε on L^2_k satisfies*

$$\Sigma(\Lambda_\varepsilon) \cap \Delta_{-\mu} = \{0\}, \quad \forall\, \varepsilon \in (0, \varepsilon^*).$$

Proof of Proposition 4.5. We split the proof into several steps.

Step 1. We claim that

$$\mathcal{U}_\varepsilon(z) := \mathcal{R}_{\mathcal{B}_\varepsilon}(z) - \mathcal{R}_{\Lambda_0}(z)\mathcal{A}\mathcal{R}_{\mathcal{B}_\varepsilon}(z)$$

is uniformly bounded in $\mathscr{B}(L^2_k)$ and $\mathscr{B}(H^{-1}_k, H^1_k)$ for any $z \in \Omega := \Delta_{-\lambda}\backslash B(0, r/2)$ any $\varepsilon \ge 0$ and $0 < r < \lambda < \mu$. On the one hand, $\mathcal{R}_{\mathcal{B}_\varepsilon}(z) \in \mathscr{B}(L^2_k)$ is just an immediate consequence of the growth estimate on $S_{\mathcal{B}_\varepsilon}$ established in Lemma 4.4-(1). For proving $\mathcal{R}_{\mathcal{B}_\varepsilon}(z) \in \mathscr{B}(H^{-1}_k, H^1_k)$, we consider first $g \in L^2_k$, $z \in \Delta_{-\mu}$ and we define $f := \mathcal{R}_{\mathcal{B}_\varepsilon}(z)g$, so that $(z - \mathcal{B}_\varepsilon)f = g$. Using (4.3) and the fact that $\mu(k-2) \ge \mu$, we deduce

$$\frac{1}{2}\|\nabla f\|_{L^2_k}^2 + (\Re e z + \mu)\|f\|_{L^2_k}^2 \le \langle (z - \mathcal{B}_\varepsilon)f, f\rangle_{L^2_k} = \langle f, g\rangle_{L^2_k} \le \|f\|_{H^1_k}\|g\|_{H^{-1}_k}$$

and thus

$$\|\nabla f\|_{L^2_k} \le \max(2, \mu^{-1})\|g\|_{H^{-1}_k}. \tag{4.5}$$

By a density argument, the same holds for any $g \in H^{-1}_k$. From (4.4), we also have $\mathcal{R}_{\Lambda_0}(z) \in \mathscr{B}(L^2_k)$ uniformly bounded in $\mathscr{B}(L^2_k)$ for any $z \in \Omega := \Delta_{-\lambda}\backslash B(0, r/2)$. Moreover, the proof of the bound in $\mathscr{B}(L^2_k, H^1_k)$ is exactly the same as for $\mathcal{R}_{\mathcal{B}_\varepsilon}(z)$. Indeed arguing as in Lemma 4.1, we first obtain

$$\langle \Lambda_0 f, f\rangle_{L^2_k} \le -\mu\|f\|_{L^2_k}^2 - \frac{1}{2}\|\nabla f\|_{L^2_k}^2 + C_0\|f\|_{L^2(B_{\varrho_0})}^2.$$

Defining $f := \mathcal{R}_{\Lambda_0}(z)g$, we deduce

$$(\Re e z + \mu)\|f\|_{L^2_k}^2 + \frac{1}{2}\|\nabla f\|_{L^2_k}^2 - C_0\|f\|_{L^2(B_{\varrho_0})}^2 \le \langle (z - \Lambda_0)f, f\rangle_{L^2_k} = \langle g, f\rangle_{L^2_k} \le C\|g\|_{L^2_k}^2.$$

Step 2. We claim that the family of operators (Λ_ε) converges in the sense

$$\|\Lambda_\varepsilon - \Lambda_0\|_{\mathscr{B}(H^1_k, L^2_k)} \le \eta_1(\varepsilon) \to 0.$$

We may indeed write

$$\begin{aligned}(\Lambda_\varepsilon - \Lambda_0)g &= (\mathcal{B}_\varepsilon - \mathcal{B}_0)g \\ &= -\operatorname{div}(g\nabla(P_\varepsilon - P_0)) - \operatorname{div}((Q_\varepsilon - Q_0)\nabla\kappa * g) \\ &= -\nabla g \cdot \nabla(P_\varepsilon - P_0) + g\Delta(P_\varepsilon - P_0) \\ &\quad + \nabla(Q_\varepsilon - Q_0) \cdot \nabla\kappa * g - (Q_\varepsilon - Q_0)g,\end{aligned}$$

so that

$$\begin{aligned}\|(\Lambda_\varepsilon - \Lambda_0)g\|_{L^2_k} &\le \|\nabla(P_\varepsilon - P_0)\|_{L^\infty}\|\nabla g\|_{L^2_k} + \|\Delta(P_\varepsilon - P_0)\|_{L^\infty}\|g\|_{L^2_k} \\ &\quad + \|\langle x\rangle^k\nabla(Q_\varepsilon - Q_0)\|_{L^\infty}\|g\|_{L^2_{1+0}} + \|Q_\varepsilon - Q_0\|_{L^\infty}\|g\|_{L^2_k}.\end{aligned}$$

We immediately conclude since we are able to prove (see Lemma 2.3)

$$\nabla(P_\varepsilon - P_0) \to 0, \quad \Delta(P_\varepsilon - P_0) \to 0, \quad \langle x\rangle^k\nabla(Q_\varepsilon - Q_0) \to 0, \quad Q_\varepsilon - Q_0 \to 0$$

uniformly in $L^\infty(\mathbb{R}^2)$.

Step 3. We claim that $\Sigma(\Lambda_\varepsilon) \cap \Delta_a \subset B(0, r/2)$ for any $\varepsilon \in (0, \varepsilon_0)$, choosing $\varepsilon_0 > 0$ small enough. On the one hand, we write the two resolvent equations

$$\begin{aligned}\mathcal{R}_{\Lambda_\varepsilon} &= \mathcal{R}_{\mathcal{B}_\varepsilon} - \mathcal{R}_{\Lambda_\varepsilon}\mathcal{A}\mathcal{R}_{\mathcal{B}_\varepsilon}, \\ \mathcal{R}_{\Lambda_\varepsilon} &= \mathcal{R}_{\Lambda_0} - \mathcal{R}_{\Lambda_\varepsilon}(\Lambda_\varepsilon - \Lambda_0)\mathcal{R}_{\Lambda_0},\end{aligned}$$

from what we deduce

$$\mathcal{R}_{\Lambda_\varepsilon} = \mathcal{R}_{\mathcal{B}_\varepsilon} - \mathcal{R}_{\Lambda_0}\mathcal{A}\mathcal{R}_{\mathcal{B}_\varepsilon} + \mathcal{R}_{\Lambda_\varepsilon}(\Lambda_\varepsilon - \Lambda_0)\mathcal{R}_{\Lambda_0}\mathcal{A}\mathcal{R}_{\mathcal{B}_\varepsilon},$$

or equivalently

$$\mathcal{R}_{\Lambda_\varepsilon}(I + \mathcal{K}_\varepsilon) = \mathcal{U}_\varepsilon,$$

with

$$\mathcal{K}_\varepsilon := (\Lambda_0 - \Lambda_\varepsilon)\mathcal{R}_{\Lambda_0}\mathcal{A}\mathcal{R}_{\mathcal{B}_\varepsilon}.$$

On the other hand, from *Step 1*, we have $\mathcal{R}_{\Lambda_0}(z)\mathcal{A}\mathcal{R}_{\mathcal{B}_\varepsilon}(z)$ is bounded in $\mathscr{B}(L^2_k, H^1_k)$ uniformly in $z \in \Omega := \Delta_{-\lambda}\backslash B(0, r/2)$ and $\Lambda_0 - \Lambda_\varepsilon$ is small in $\mathscr{B}(H^1_k, L^2_k)$ for $\varepsilon > 0$ small, so that both estimates together imply

$$\sup_{z\in\Delta_{-\lambda}\backslash B(0,r/2)} \|\mathcal{K}_\varepsilon(z)\|_{\mathscr{B}(L^2)} < 1,$$

for any $0 < r < \lambda < \mu$ and $\varepsilon \in (0, \varepsilon_0)$, with $\varepsilon_0 = \varepsilon_0(r, \lambda) > 0$ small enough. This implies that $I + \mathcal{K}_\varepsilon$ is invertible on $\Omega := \Delta_{-\lambda}\backslash B(0, r/2)$ so that

$$\mathcal{R}_{\Lambda_\varepsilon} = \mathcal{U}_\varepsilon(I + \mathcal{K}_\varepsilon)^{-1}$$

is bounded on Ω, which ends the proof.

Step 4. We define now

$$\Pi_\varepsilon := \frac{i}{2\pi}\int_\Gamma \mathcal{R}_{\Lambda_\varepsilon}(z)\,dz, \quad \Gamma := \{z \in \mathbb{C}; |z| = r\},$$

the Dunford projector on the eigenspace associated to eigenvalues of Λ_ε which belong to the ball $B(0, r)$. We write

$$\begin{aligned}\Pi_\varepsilon &= \frac{i}{2\pi}\int_\Gamma \mathcal{U}_\varepsilon\,(I + \mathcal{K}_\varepsilon)^{-1}\,dz \\ &= \frac{i}{2\pi}\int_\Gamma \mathcal{R}_{\mathcal{B}_\varepsilon}\{I - \mathcal{K}_\varepsilon\,(I + \mathcal{K}_\varepsilon)^{-1}\}\,dz - \frac{i}{2\pi}\int_\Gamma \mathcal{R}_{\Lambda_0}\,\mathcal{A}\,\mathcal{R}_{\mathcal{B}_\varepsilon}\,(I + \mathcal{K}_\varepsilon)^{-1}\,dz \\ &= -\frac{i}{2\pi}\int_\Gamma \mathcal{R}_{\mathcal{B}_\varepsilon}\mathcal{K}_\varepsilon\,(I + \mathcal{K}_\varepsilon)^{-1}\,dz - \frac{i}{2\pi}\int_\Gamma \mathcal{R}_{\Lambda_0}\,\mathcal{A}\,\mathcal{R}_{\mathcal{B}_\varepsilon}\,(I + \mathcal{K}_\varepsilon)^{-1}\,dz,\end{aligned}$$

and

$$\begin{aligned}\Pi_0 &= \frac{i}{2\pi}\int_\Gamma \{\mathcal{R}_{\mathcal{B}_0} - \mathcal{R}_{\Lambda_0}\mathcal{A}\mathcal{R}_{\mathcal{B}_0}\}\,dz \\ &= -\frac{i}{2\pi}\int_\Gamma \mathcal{R}_{\Lambda_0}\mathcal{A}\mathcal{R}_{\mathcal{B}_0}\{(I + \mathcal{K}_\varepsilon)^{-1} + \mathcal{K}_\varepsilon(I + \mathcal{K}_\varepsilon)^{-1}\}\,dz.\end{aligned}$$

We deduce

$$\begin{aligned}\Pi_\varepsilon - \Pi_0 &= \frac{i}{2\pi}\int_\Gamma (\mathcal{R}_{\Lambda_0}\mathcal{A}\mathcal{R}_{\mathcal{B}_0} - \mathcal{R}_{\mathcal{B}_\varepsilon})\mathcal{K}_\varepsilon\,(I+\mathcal{K}_\varepsilon)^{-1}\,dz \\ &\quad + \frac{i}{2\pi}\int_\Gamma \mathcal{R}_{\Lambda_0}\mathcal{A}\{\mathcal{R}_{\mathcal{B}_0} - \mathcal{R}_{\mathcal{B}_\varepsilon}\}(I+\mathcal{K}_\varepsilon)^{-1}\,dz \\ &= -\frac{i}{2\pi}\int_\Gamma \mathcal{U}_\varepsilon\mathcal{K}_\varepsilon\,(I+\mathcal{K}_\varepsilon)^{-1}\,dz \\ &\quad + \frac{i}{2\pi}\int_\Gamma \mathcal{R}_{\Lambda_0}\mathcal{A}\mathcal{R}_{\mathcal{B}_0}\{\mathcal{B}_0 - \mathcal{B}_\varepsilon\}\mathcal{R}_{\mathcal{B}_\varepsilon}(I+\mathcal{K}_\varepsilon)^{-1}\,dz.\end{aligned}$$

We conclude that $\|\Pi_\varepsilon - \Pi_0\|_{\mathscr{B}(L^2)} = \mathcal{O}(\varepsilon) < 1$ for $\varepsilon > 0$ small enough by taking advantage of the estimates established in *Step 1* and *Step 2*. By classical operator theory (see for instance the arguments presented in [16] in order to prove [16, Chap 1, (4.43)]) one deduces that $\dim \Pi_\varepsilon = \dim \Pi_0 = 1$. On the other hand, at first glance we have $\Lambda_\varepsilon^* 1 = 0$ and $1 \in (L^2_k)'$ so that $0 \in \Sigma(\Lambda_\varepsilon^*) = \Sigma(\Lambda_\varepsilon)$, and 0 is the only spectral value of Λ_ε in the ball $B(0,r)$. □

4.4. **Semigroup decay estimates for $\mathcal{L}_{1,1}$.** We are now able to deduce a nice semigroup decay estimate on $S_{\mathcal{L}_{1,1}}$ from the previous estimates on the resolvent.

Proposition 4.6. *With the notation of Proposition 4.5, for $k > 3$, all $0 < \lambda < \mu$ and all $\varepsilon \in (0, \varepsilon_*)$ there holds, for any $g \in L^2_{k,0}$,*

$$\|S_{\mathcal{L}_{1,1}}(t)g\|_{L^2_k} \lesssim e^{-\lambda t}\|g\|_{L^2_k}.$$

Proof of Proposition 4.6. It is a consequence of Proposition 4.5 and of the splitting structure of the operator Λ_ε. More precisely, we may for instance apply the quantitative mapping theorem [19, Theorem 2.1], where it is worth emphasizing that $\mathcal{R}_{\mathcal{B}_\varepsilon}(z) : L^2_k \to H^1_k \subset D(\Lambda_\varepsilon^{1/2})$ with uniformly bound in $z \in \Delta_{-\lambda}$, which is a strong enough information in order to establish [19, (2.23)] without checking [19, **(H2)**]. Alternatively, one can use the Gearhart-Prüss-Greiner theorem [13, 25, 1] in order to get the same conclusion. □

Thanks to the previous estimate for $S_{\mathcal{L}_{1,1}}$ and the estimates for $S_{\mathcal{B}_\varepsilon}$ in Lemma 4.4, we are able to deduce semigroup estimates for $S_{\mathcal{L}_{1,1}}$ (in Propositions 4.7 below) similar to those satisfied by $S_{\mathcal{B}_\varepsilon}$.

We start observing that, thanks to Duhamel's formula, we have

$$S_{\mathcal{L}_{1,1}} = S_{\mathcal{B}_\varepsilon} + S_{\mathcal{B}_\varepsilon}\mathcal{A} * S_{\mathcal{L}_{1,1}} \quad \text{and} \quad S_{\mathcal{L}_{1,1}} = S_{\mathcal{B}_\varepsilon} + S_{\mathcal{L}_{1,1}} * \mathcal{A}S_{\mathcal{B}_\varepsilon}. \tag{4.6}$$

Denoting $\Pi^\perp g = g - \Pi g$, where Π is the projection onto $\mathrm{Ker}(\Lambda_\varepsilon)$, we obtain

$$S_{\mathcal{L}_{1,1}}\Pi^\perp = S_{\mathcal{B}_\varepsilon}\Pi^\perp + (S_{\mathcal{B}_\varepsilon}\mathcal{A} * S_{\mathcal{L}_{1,1}}\Pi^\perp) \quad \text{and} \quad S_{\mathcal{L}_{1,1}}\Pi^\perp = \Pi^\perp S_{\mathcal{B}_\varepsilon} + (S_{\mathcal{L}_{1,1}}\Pi^\perp * \mathcal{A}S_{\mathcal{B}_\varepsilon}). \tag{4.7}$$

Using that $S_{\mathcal{L}_{1,1}}\Pi^\perp = \Pi^\perp S_{\mathcal{L}_{1,1}}$, and iterating the formulas yields

$$S_{\mathcal{L}_{1,1}}\Pi^\perp = S_{\mathcal{B}_\varepsilon}\Pi^\perp + S_{\mathcal{B}_\varepsilon}\mathcal{A} * \Pi^\perp S_{\mathcal{B}_\varepsilon} + S_{\mathcal{B}_\varepsilon}\mathcal{A} * S_{\mathcal{L}_{1,1}}\Pi^\perp * \mathcal{A}S_{\mathcal{B}_\varepsilon}$$

and

$$S_{\mathcal{L}_{1,1}}\Pi^\perp = \Pi^\perp S_{\mathcal{B}_\varepsilon} + S_{\mathcal{B}_\varepsilon}\Pi^\perp * \mathcal{A}S_{\mathcal{B}_\varepsilon} + S_{\mathcal{B}_\varepsilon}\mathcal{A} * S_{\mathcal{L}_{1,1}}\Pi^\perp * \mathcal{A}S_{\mathcal{B}_\varepsilon}. \tag{4.8}$$

We deduce by combining some results of section 4.2 and the above Proposition 4.6, some additional estimates on the semigroup $S_{\mathcal{L}_{1,1}}$.

Proposition 4.7. *Let $0 \le \lambda < \mu$ and $k > 3$. There is $\varepsilon_* > 0$ small enough such that for any $\varepsilon \in (0, \varepsilon_*)$ the following holds:*

(1) *For all $g \in L^2_{k,0}$ we have*

$$\|\mathbf{e}_\lambda S_{\mathcal{L}_{1,1}}(\cdot)g\|_{L^\infty_t L^2_k} + \|\mathbf{e}_\lambda S_{\mathcal{L}_{1,1}}(\cdot)g\|_{L^2_t H^1_k} \lesssim \|g\|_{L^2_k}.$$

(2) *For all $\mathbf{e}_\lambda R \in L^2_t H^{-1}_k$ with $\Pi R = 0$, we have*

$$\left\|\mathbf{e}_\lambda(S_{\mathcal{L}_{1,1}} * R)\right\|_{L^\infty_t L^2_k} + \left\|\mathbf{e}_\lambda(S_{\mathcal{L}_{1,1}} * R)\right\|_{L^2_t H^1_k} \lesssim \|\mathbf{e}_\lambda R\|_{L^2_t H^{-1}_k}.$$

Proof of Proposition 4.7. Proof of (1). Remark that $g = \Pi^\perp g$, since $g \in L^2_{k,0}$. The first estimate is nothing but Proposition 4.6. In particular, we deduce from this one that

$$\|\mathbf{e}_\lambda S_{\mathcal{L}_{1,1}} \Pi^\perp\|_{L^1_t \mathscr{B}(L^2_k)} \lesssim \|\mathbf{e}_{\lambda'} S_{\mathcal{L}_{1,1}} \Pi^\perp\|_{L^\infty_t \mathscr{B}(L^2_k)} \lesssim 1, \tag{4.9}$$

by choosing $\lambda < \lambda' < 1$. On the other hand, thanks to Lemma 4.4-(1) and the first estimate, we have

$$\|\mathbf{e}_\lambda S_{\mathcal{B}_\varepsilon}(\cdot) g\|_{L^2_t H^1_k} \lesssim \|g\|_{L^2_k}$$

and

$$\begin{aligned} \|\mathbf{e}_\lambda (S_{\mathcal{B}_\varepsilon} * \mathcal{A} S_{\mathcal{L}_{1,1}})(\cdot) g\|_{L^2_t H^1_k} &\lesssim \|\mathbf{e}_\lambda \mathcal{A} S_{\mathcal{L}_{1,1}}(\cdot) g\|_{L^2_t H^{-1}_k} \\ &\lesssim \|\mathbf{e}_\lambda S_{\mathcal{L}_{1,1}}(\cdot) g\|_{L^2_t L^2_k} \\ &\lesssim \|g\|_{L^2_k}, \end{aligned}$$

where we have used that $\mathcal{A} \in \mathscr{B}(L^2_k)$, from which together with the first identity in (4.6), we immediately obtain the second estimate.

Proof of (2). Remarking that $R(t) = \Pi^\perp R(t)$, for all $t \geq 0$, and using the second identity in (4.7), we may write

$$S_{\mathcal{L}_{1,1}} * R = S_{\mathcal{L}_{1,1}} \Pi^\perp * R = \Pi^\perp (S_{\mathcal{B}_\varepsilon} * R) + S_{\mathcal{L}_{1,1}} \Pi^\perp * \mathcal{A} S_{\mathcal{B}_\varepsilon} * R, \tag{4.10}$$

and thus

$$\mathbf{e}_\lambda (S_{\mathcal{L}_{1,1}} * R) = \mathbf{e}_\lambda \Pi^\perp (S_{\mathcal{B}_\varepsilon} * R) + (\mathbf{e}_\lambda S_{\mathcal{L}_{1,1}} \Pi^\perp) * \mathcal{A}[\mathbf{e}_\lambda (S_{\mathcal{B}_\varepsilon} * R)].$$

We deduce

$$\begin{aligned} \|\mathbf{e}_\lambda (S_{\mathcal{L}_{1,1}} * R)\|_{L^\infty_t L^2_k} &\leq \|\mathbf{e}_\lambda (S_{\mathcal{B}_\varepsilon} * R)\|_{L^\infty_t L^2_k} + \|(\mathbf{e}_\lambda S_{\mathcal{L}_{1,1}} \Pi^\perp) * \mathcal{A}[\mathbf{e}_\lambda (S_{\mathcal{B}_\varepsilon} * R)]\|_{L^\infty_t L^2_k} \\ &\leq \|\mathbf{e}_\lambda (S_{\mathcal{B}_\varepsilon} * R)\|_{L^\infty_t L^2_k} \\ &\quad + \|\mathbf{e}_\lambda S_{\mathcal{L}_{1,1}} \Pi^\perp\|_{L^1_t(\mathscr{B}(L^2_k))} \|\mathcal{A}\|_{\mathscr{B}(L^2_k)} \|\mathbf{e}_\lambda (S_{\mathcal{B}_\varepsilon} * R)\|_{L^\infty_t L^2_k} \\ &\lesssim \|\mathbf{e}_\lambda R\|_{L^2_t H^{-1}_k}, \end{aligned}$$

where we have used Lemma 4.4-(2) and (4.9) in last line. We have established the first estimate in *(2)*.

For the second term, using (4.8), we may write

$$S_{\mathcal{L}_{1,1}} * R = \Pi^\perp (S_{\mathcal{B}_\varepsilon} * R) + S_{\mathcal{B}_\varepsilon} \Pi^\perp * \mathcal{A} S_{\mathcal{B}_\varepsilon} * R + S_{\mathcal{B}_\varepsilon} \mathcal{A} * S_{\mathcal{L}_{1,1}} \Pi^\perp * \mathcal{A} S_{\mathcal{B}_\varepsilon} * R,$$

and thus

$$\mathbf{e}_\lambda (S_{\mathcal{L}_{1,1}} * R) = \Pi^\perp \mathbf{e}_\lambda (S_{\mathcal{B}_\varepsilon} * R) + \mathbf{e}_\lambda [S_{\mathcal{B}_\varepsilon} * (\Pi^\perp \mathcal{A} S_{\mathcal{B}_\varepsilon} * R)] + \mathbf{e}_\lambda [(S_{\mathcal{B}_\varepsilon} \mathcal{A}) * (S_{\mathcal{L}_{1,1}} \Pi^\perp * \mathcal{A} S_{\mathcal{B}_\varepsilon} * R)].$$

We now estimate each term separately. From Lemma 4.4-(2), we have

$$\|\mathbf{e}_\lambda (S_{\mathcal{B}_\varepsilon} * R)\|_{L^2_t H^1_k} \lesssim \|\mathbf{e}_\lambda R\|_{L^2_t H^{-1}_k}.$$

From Lemma 4.4-(2) again, we also have

$$\begin{aligned} \|\mathbf{e}_\lambda [S_{\mathcal{B}_\varepsilon} * (\Pi^\perp \mathcal{A} S_{\mathcal{B}_\varepsilon} * R)]\|_{L^2_t H^1_k} &\lesssim \|\mathbf{e}_\lambda (\Pi^\perp \mathcal{A} S_{\mathcal{B}_\varepsilon} * R)\|_{L^2_t H^{-1}_k} \\ &\lesssim \|\mathbf{e}_\lambda (\Pi^\perp \mathcal{A} S_{\mathcal{B}_\varepsilon} * R)\|_{L^2_t L^2_k} \\ &\lesssim \|\mathbf{e}_\lambda (S_{\mathcal{B}_\varepsilon} * R)\|_{L^2_t L^2_k} \\ &\lesssim \|\mathbf{e}_\lambda R\|_{L^2_t H^{-1}_k}. \end{aligned}$$

We finally have

$$\begin{aligned} &\|\mathbf{e}_\lambda [(S_{\mathcal{B}_\varepsilon} \mathcal{A}) * (S_{\mathcal{L}_{1,1}} \Pi^\perp * \mathcal{A} S_{\mathcal{B}_\varepsilon} * R)]\|_{L^2_t H^1_k} \\ &\quad \lesssim \|(\mathbf{e}_\lambda S_{\mathcal{L}_{1,1}} \Pi^\perp) * [\mathcal{A} \mathbf{e}_\lambda (S_{\mathcal{B}_\varepsilon} * R)]\|_{L^2_t H^{-1}_k} \\ &\quad \lesssim \|(\mathbf{e}_\lambda S_{\mathcal{L}_{1,1}} \Pi^\perp) * [\mathcal{A} \mathbf{e}_\lambda (S_{\mathcal{B}_\varepsilon} * R)]\|_{L^2_t L^2_k} \\ &\quad \lesssim \|\mathbf{e}_\lambda S_{\mathcal{L}_{1,1}} \Pi^\perp\|_{L^1_t(\mathscr{B}(L^2_k))} \|\mathcal{A}\|_{\mathscr{B}(L^2_k)} \|[\mathbf{e}_\lambda (S_{\mathcal{B}_\varepsilon} * R)]\|_{L^2_t L^2_k} \\ &\quad \lesssim \|\mathbf{e}_\lambda R\|_{L^2_t H^{-1}_k}, \end{aligned}$$

where we have used Lemma 4.4-(2) in the second line as well as Proposition 4.6 and Lemma 4.4-(2) in the last line. Putting the three last estimates together, we conclude to the second estimate in (2). ☐

5. Estimates for $\mathcal{L}_{2,2}$

In this section, we are concerned with establishing estimates on $S_{\mathcal{L}_{2,2}}$, where we recall that $\mathcal{L}_{2,2}$ is defined

$$\mathcal{L}_{2,2}w = \frac{1}{\varepsilon}\Delta w + \mu x \cdot \nabla w + \nabla\kappa * [Q\nabla w].$$

Lemma 5.1. *For any $p > 2$, there exist $c_p, C_p > 0$ such that*

$$\int (\mathcal{L}_{2,2}w)w^{p-1} \leq -\frac{c_p}{\varepsilon}\|\nabla w^{p/2}\|_{L^2}^2 - \frac{2\mu}{p}\|w\|_{L^p}^p + C\|w\|_{L^p}^{p/p'}\|\nabla w\|_{L^p},$$

for any $w \in W^{2,p}$.

Proof of Lemma 5.1. We classically compute

$$\int (\mathcal{L}_{2,2}w)w^{p-1} = -\frac{c_p}{\varepsilon}\int |\nabla w^{p/2}|^2 - 2\frac{\mu}{p}\int w^p + \int w^{p-1}\nabla\kappa * (Q\nabla w).$$

We conclude by observing that

$$\begin{aligned} \int w^{p-1}\nabla\kappa * (Q\nabla w) &\leq \|w\|_{L^p}^{p/p'}\|\nabla\kappa * (Q\nabla w)\|_{L^p} \\ &\lesssim \|w\|_{L^p}^{p/p'}\|Q\nabla w\|_{L^p_2} \\ &\lesssim \|w\|_{L^p}^{p/p'}\|\nabla w\|_{L^p}, \end{aligned}$$

where we have used Hölder's inequality in the first line, Lemma 3.2 in the third line, and Lemma 2.2 in the fourth one. ☐

We observe that

$$\nabla\mathcal{L}_{2,2}w = \frac{1}{\varepsilon}\Delta(\nabla w) + \mu x \cdot \nabla(\nabla w) + \mu\nabla w + \nabla^2\kappa * [Q\nabla w],$$

where we denote

$$(x \cdot \nabla\Phi)_i = x_\ell\partial_\ell\Phi_i, \quad (\nabla^2\kappa * \Phi)_i = \partial_{i\ell}\kappa * \Phi_\ell,$$

for any vector Φ.

Lemma 5.2. *For any $w \in H^2$, there holds*

$$\langle\mathcal{L}_{2,2}w, w\rangle_{\dot{H}^1} = -\frac{1}{\varepsilon}\|\nabla^2 w\|_{L^2}^2 - \|Q^{1/2}\nabla w\|_{L^2}^2.$$

Proof of Lemma 5.2. A straightforward computation gives

$$\begin{aligned} \langle\nabla\mathcal{L}_{2,2}w, \nabla w\rangle_{L^2} &= \int (\frac{1}{\varepsilon}\Delta\nabla w + \mu\nabla w + \mu x \cdot \nabla^2 w + \nabla^2\kappa * [Q\nabla w])\nabla w \\ &= -\frac{1}{\varepsilon}\int |\nabla^2 w|^2 - \int Q|\nabla w|^2, \end{aligned}$$

where we have performed two integrations by parts for the last term and we have used the identity $\Delta\kappa = -\delta$. ☐

As a consequence of previous estimates, we obtain the following decay and regularization estimates for the semigroup $S_{\mathcal{L}_{2,2}}$.

Lemma 5.3. *Let $p \in (2, \infty)$ and $0 \leq \vartheta < \frac{2\mu}{p}$. There is $\varepsilon_2 > 0$ small enough such that for any $\varepsilon \in (0, \varepsilon_2)$ the following holds:*

(1) *For all $w \in L^p \cap \dot{H}^1$, we have*

$$\|\mathbf{e}_\vartheta S_{\mathcal{L}_{2,2}}(\cdot)w\|_{L^\infty_t(L^p\cap\dot{H}^1)} + \|\mathbf{e}_\vartheta S_{\mathcal{L}_{2,2}}(\cdot)w\|_{L^2_tL^p} + \frac{1}{\sqrt{\varepsilon}}\|\mathbf{e}_\vartheta S_{\mathcal{L}_{2,2}}(\cdot)w\|_{L^2_t\dot{H}^2} \lesssim \|w\|_{L^p\cap\dot{H}^1}.$$

(2) *For all* $\mathbf{e}_\vartheta S \in L^2_t(L^p \cap \dot H^1)$, *we have*

$$\left\|\mathbf{e}_\vartheta(S_{\mathcal{L}_{2,2}} * S)\right\|_{L^\infty_t(L^p\cap \dot H^1)} + \left\|\mathbf{e}_\vartheta(S_{\mathcal{L}_{2,2}} * S)\right\|_{L^2_t L^p} + \frac{1}{\sqrt{\varepsilon}}\left\|\mathbf{e}_\vartheta(S_{\mathcal{L}_{2,2}} * S)\right\|_{L^2_t \dot H^2} \lesssim \|\mathbf{e}_\vartheta S\|_{L^2_t L^p} + \|\mathbf{e}_\vartheta S\|_{L^2_t \dot H^1}.$$

Proof of Lemma 5.3. Let us denote $\phi = \mathbf{e}_\vartheta S_{\mathcal{L}_{2,2}}(\cdot) w$ the solution to the evolution equation

$$\partial_t \phi = \mathcal{L}_{2,2}\phi + \vartheta\phi, \quad \phi(0) = w.$$

Thanks to Lemma 5.1, we have

$$\frac{1}{p}\frac{d}{dt}\|\phi\|^p_{L^p} \le \left(\vartheta - \frac{2\mu}{p}\right)\|\phi\|^p_{L^p} + C\|\phi\|^{p/p'}_{L^p}\|\nabla\phi\|_{L^p},$$

and because

$$\frac{1}{2}\frac{d}{dt}\|\phi\|^2_{L^p} = \|\phi\|^{2-p}_{L^p}\left(\frac{1}{p}\frac{d}{dt}\|\phi\|^p_{L^p}\right),$$

we get

$$\frac{1}{2}\frac{d}{dt}\|\phi\|^2_{L^p} \le \left(\vartheta - \frac{2\mu}{p}\right)\|\phi\|^2_{L^p} + C\|\phi\|_{L^p}\|\nabla\phi\|_{L^p}.$$

Combining that last estimate with Lemma 5.2 yields

$$\frac{1}{2}\frac{d}{dt}\left\{\|\phi\|^2_{L^p} + \|\nabla\phi\|^2_{L^2}\right\} \le \left(\vartheta - \frac{2\mu}{p}\right)\|\phi\|^2_{L^p} + \vartheta\|\nabla\phi\|^2_{L^2} - \frac{1}{\varepsilon}\|\nabla^2\phi\|^2_{L^2} + C\|\phi\|_{L^p}\|\nabla\phi\|_{L^p}.$$

Observing that thanks to (3.4) and Young's inequality, we have

$$C\|\phi\|_{L^p}\|\nabla\phi\|_{L^p} \le C\|\phi\|^{2-\theta}_{L^p}\|\nabla^2\phi\|^\theta_{L^2} \le C\varepsilon^{\frac{\theta}{2-\theta}}\|\phi\|^2_{L^p} + \frac{1}{2\varepsilon}\|\nabla^2\phi\|^2_{L^2},$$

with $\theta = p/(2+p)$, we thus obtain

$$\frac{1}{2}\frac{d}{dt}\left\{\|\phi\|^2_{L^p} + \|\nabla\phi\|^2_{L^2}\right\} \le -\left(\frac{2\mu}{p} - \vartheta - C\varepsilon^{\frac{\theta}{2-\theta}}\right)\|\phi\|^2_{L^p} + \vartheta\|\nabla\phi\|^2_{L^2} - \frac{1}{2\varepsilon}\|\nabla^2\phi\|^2_{L^2}.$$

Finally, taking $\varepsilon > 0$ small enough and using Lemma 3.4, we hence deduce

$$\frac{1}{2}\frac{d}{dt}\left\{\|\phi\|^2_{L^p} + \|\nabla\phi\|^2_{L^2}\right\} \le -\sigma\|\phi\|^2_{L^p} - \frac{1}{4\varepsilon}\|\nabla^2\phi\|^2_{L^2},$$

for some $\sigma > 0$, from which (1) follows by Grönwall's lemma.

We now consider $\phi = \mathbf{e}_\vartheta(S_{\mathcal{L}_{2,2}} * S)$ the solution to the evolution equation

$$\partial_t\phi = \mathcal{L}_{2,2}\phi + \vartheta\phi + \mathbf{e}_\vartheta S, \quad \phi(0) = 0.$$

Arguing as above we have

$$\frac{1}{2}\frac{d}{dt}\left\{\|\phi\|^2_{L^p} + \|\nabla\phi\|^2_{L^2}\right\} \le -\left(\frac{2\mu}{p} - \vartheta - C\varepsilon^{\frac{\theta}{2-\theta}}\right)\|\phi\|^2_{L^p} + \vartheta\|\nabla\phi\|^2_{L^2} - \frac{1}{2\varepsilon}\|\nabla^2\phi\|^2_{L^2} + \|\phi\|_{L^p}\|\mathbf{e}_\vartheta S\|_{L^p} + \|\nabla\phi\|_{L^2}\|\mathbf{e}_\vartheta\nabla S\|_{L^2}.$$

By Young's inequality for any $\beta > 0$ and some $C_\beta > 0$, we get

$$\frac{1}{2}\frac{d}{dt}\left\{\|\phi\|^2_{L^p} + \|\nabla\phi\|^2_{L^2}\right\} \le -\left(\frac{2\mu}{p} - \vartheta - C\varepsilon^{\frac{\theta}{2-\theta}} - \beta\right)\|\phi\|^2_{L^p} + (\vartheta+\beta)\|\nabla\phi\|^2_{L^2} - \frac{1}{2\varepsilon}\|\nabla^2\phi\|^2_{L^2} + C_\beta\|\mathbf{e}_\vartheta S\|^2_{L^p} + C_\beta\|\mathbf{e}_\vartheta\nabla S\|^2_{L^2}.$$

We then conclude to (2) arguing as before by taking $\varepsilon, \beta > 0$ small enough, using Lemma 3.4 and applying Grönwall's lemma again. ∎

6. Semigroup estimates for the linearized system

We start with some estimates on the out of the diagonal operators $\mathcal{L}_{1,2}$ and $\mathcal{L}_{2,1}$ which we recall that they are defined in (1.9) by

$$\mathcal{L}_{1,2}w = -\operatorname{div}(Q\nabla w), \quad \mathcal{L}_{2,1}g = g + \nabla\kappa * [g\nabla P + Q\nabla\kappa * g] .$$

Lemma 6.1. *For $k > 3$ and $p \in (2,\infty)$, there hold*

$$\|\mathcal{L}_{1,2}w\|_{H^{-1}_k} \lesssim \|w\|_{L^p}^{1-\theta}\|\nabla^2 w\|_{L^2}^{\theta} \quad \forall w \in L^p \cap \dot H^2,$$

with $\theta := p/(2+p)$, and

$$\|\mathcal{L}_{2,1}g\|_{L^p} + \|\mathcal{L}_{2,1}g\|_{\dot H^1} \lesssim \|g\|_{H^1_k}, \quad \forall\, g \in H^1_k.$$

Proof of Lemma 6.1. For the first estimate, we write

$$\begin{aligned} \|\operatorname{div}(\nabla Q\cdot\nabla w)\|_{H^{-1}_k} &\lesssim \|\nabla Q\cdot\nabla w\|_{L^2_k} \\ &\lesssim \|\nabla w\|_{L^p} \\ &\lesssim \|w\|_{L^p}^{1-\theta}\|\nabla^2 w\|_{L^2}^{\theta}, \end{aligned}$$

where we have used the exponential decay of Q (see Lemma 2.2) in the third line together with Hölder's inequality, and also (3.4) in the last one.

For the second estimate, we write

$$\|\mathcal{L}_{2,1}g\|_{L^p} \le \|g\|_{L^p} + \|\nabla\kappa * [g\nabla P]\|_{L^p} + \|\nabla\kappa * [Q\nabla\kappa * g]\|_{L^p},$$

and we estimate the three terms at the RHS separately. On the one hand, we have

$$\begin{aligned} \|\nabla\kappa * [g\nabla P]\|_{L^p} &\lesssim \|g\nabla P\|_{L^p_2} \\ &\lesssim \|g\|_{L^p_1} \end{aligned}$$

where we have used Lemma 3.2 in the first line and Lemma 2.2 in the second one. Similarly, we have

$$\begin{aligned} \|\nabla\kappa * [Q\nabla\kappa * g]\|_{L^p} &\lesssim \|Q\nabla\kappa * g\|_{L^p_2} \\ &\lesssim \|\nabla\kappa * g\|_{L^p} \\ &\lesssim \|g\|_{L^p_2} \end{aligned}$$

by using successively Lemma 3.2, Lemma 2.2 and Lemma 3.2 again. Therefore we get

$$\|\mathcal{L}_{2,1}g\|_{L^p} \lesssim \|g\|_{L^p_2} \lesssim \|g\|_{H^1_k}$$

thanks to the Sobolev embedding $H^1_k(\mathbb{R}^2) \subset L^p_k(\mathbb{R}^2)$.

For the third estimate, we write

$$\begin{aligned} \|\mathcal{L}_{2,1}g\|_{\dot H^1} &\le \|\nabla g\|_{L^2} + \|\nabla^2\kappa * [g\nabla P]\|_{L^2} + \|\nabla^2\kappa * [Q\nabla\kappa * g]\|_{L^2} \\ &\lesssim \|\nabla g\|_{L^2} + \|g\nabla P\|_{L^2} + \|Q\nabla\kappa * g\|_{L^2}, \end{aligned}$$

where we have used (3.1) in order to handle the two last terms. Thanks to Lemma 2.2, we have

$$\|g\nabla P\|_{L^2} \lesssim \|g\|_{L^2}.$$

We also have

$$\|Q\nabla\kappa * g\|_{L^2} \lesssim \|\nabla\kappa * g\|_{L^p} \lesssim \|g\|_{L^2_2},$$

where we have used Hölder's inequality and then Lemma 3.2. To conclude, we put together the three last inequalities. ☐

As a consequence of Proposition 4.7, Lemma 5.3 and Lemma 6.1, we obtain the following semigroup estimate on the linearized problem.

Proposition 6.2. *Let $0 \le \lambda < \mu$ and $k > 3$. There is $\varepsilon_* > 0$ small enough such that for any $\varepsilon \in (0,\varepsilon_*)$ there holds:*

(1) *For any $(g_0,w_0) \in L^2_{k,0} \times (L^p \cap \dot H^1)$, with $2 < p < \frac{2\mu}{\lambda}$, we have*

$$\|\mathbf{e}_\lambda S_{\mathcal{L}}(\cdot)(g_0,w_0)\|_{L^\infty_t(\mathcal{X})} + \|\mathbf{e}_\lambda S_{\mathcal{L}}(\cdot)(g_0,w_0)\|_{L^2_t(\mathcal{Y})} \lesssim \|(g_0,w_0)\|_{\mathcal{X}}.$$

(2) *For any* $\mathbf{e}_\lambda \mathcal{R} = \mathbf{e}_\lambda(\mathcal{R}_1, \mathcal{R}_2) \in L^2_t(H^{-1}_k \times (L^p \cap \dot H^1))$ *with* $\Pi \mathcal{R}_1 = 0$ *we have*

$$\|\mathbf{e}_\lambda (S_\mathcal{L} * \mathcal{R})\|_{L^\infty_t(\mathcal{X})} + \|\mathbf{e}_\lambda (S_\mathcal{L} * \mathcal{R})\|_{L^2_t(\mathcal{Y})} \lesssim \|\mathbf{e}_\lambda \mathcal{R}\|_{L^2_t(H^{-1}_k \times (L^p \cap \dot H^1))}.$$

Proof of Proposition 6.2. We split the proof into two steps.

Proof of (1). Let us denote

$$(g(t), w(t)) = S_\mathcal{L}(t)(g_0, w_0),$$

so that

$$g(t) = S_{\mathcal{L}_{1,1}}(t) g_0 + (S_{\mathcal{L}_{1,1}} * \mathcal{L}_{1,2} w)(t)$$

and

$$w(t) = S_{\mathcal{L}_{2,2}}(t) w_0 + (S_{\mathcal{L}_{2,2}} * \mathcal{L}_{2,1} g)(t).$$

We observe that $\langle\!\langle \mathcal{L}_{1,2} w \rangle\!\rangle = 0$ so that $\Pi(\mathcal{L}_{1,2} w) = 0$, and we can then hereafter apply the results of Proposition 4.7 to $S_{\mathcal{L}_{1,1}} * \mathcal{L}_{1,2} w$. From Proposition 4.7-(1), we have

$$\|\mathbf{e}_\lambda S_{\mathcal{L}_{1,1}}(\cdot) g_0\|_{L^\infty_t L^2_k} + \|\mathbf{e}_\lambda S_{\mathcal{L}_{1,1}}(\cdot) g_0\|_{L^2_t H^1_k} \lesssim \|g_0\|_{L^2_k}.$$

On the other hand, from Proposition 4.7-(2) , we have

$$\begin{aligned}\|\mathbf{e}_\lambda (S_{\mathcal{L}_{1,1}} * \mathcal{L}_{1,2} w)\|_{L^\infty_t L^2_k} + \|\mathbf{e}_\lambda (S_{\mathcal{L}_{1,1}} * \mathcal{L}_{1,2} w)\|_{L^2_t H^1_k} &\lesssim \|\mathbf{e}_\lambda \mathcal{L}_{1,2} w\|_{L^2_t H^{-1}_k} \\ &\lesssim \|\mathbf{e}_\lambda w\|^{1-\theta}_{L^2_t L^p} \|\mathbf{e}_\lambda w\|^{\theta}_{L^2_t \dot H^2}\end{aligned}$$

with $\theta = p/(2+p)$, where we have used the first estimate in Lemma 6.1 and the Hölder inequality in the second line. We have thus established

$$\|\mathbf{e}_\lambda g\|_{L^\infty_t L^2_k} + \|\mathbf{e}_\lambda g\|_{L^2_t H^1_k} \le C_1 \|g_0\|_{L^2_k} + C_2 \|\mathbf{e}_\lambda w\|^{1-\theta}_{L^2_t L^p} \|\mathbf{e}_\lambda w\|^{\theta}_{L^2_t \dot H^2}, \tag{6.1}$$

for some constant $C_1, C_2 > 0$.

We now come to the estimate of w. We recall that from Lemma 5.3-(1), we have

$$\|\mathbf{e}_\lambda S_{\mathcal{L}_{2,2}}(\cdot) w_0\|_{L^\infty_t(L^p \cap \dot H^1)} + \|\mathbf{e}_\lambda S_{\mathcal{L}_{2,2}}(\cdot) w_0\|_{L^2_t L^p} + \frac{1}{\sqrt{\varepsilon}} \|\mathbf{e}_\lambda S_{\mathcal{L}_{2,2}}(\cdot) w_0\|_{L^2_t \dot H^2} \lesssim \|w_0\|_{L^p \cap \dot H^1}.$$

We recall that from Lemma 5.3-(2), we have

$$\begin{aligned}\|\mathbf{e}_\lambda (S_{\mathcal{L}_{2,2}} * \mathcal{L}_{2,1} g)\|_{L^\infty_t(L^p \cap \dot H^1)} + \|\mathbf{e}_\lambda (S_{\mathcal{L}_{2,2}} * \mathcal{L}_{2,1} g)\|_{L^2_t L^p} + \frac{1}{\sqrt{\varepsilon}} \|\mathbf{e}_\lambda (S_{\mathcal{L}_{2,2}} * \mathcal{L}_{2,1} g)\|_{L^2_t \dot H^2} \\ \lesssim \|\mathbf{e}_\lambda \mathcal{L}_{2,1} g\|_{L^2_t L^p} + \|\mathbf{e}_\lambda \mathcal{L}_{2,1} g\|_{L^2_t \dot H^1} \\ \lesssim \|\mathbf{e}_\lambda g\|_{L^2_t H^1_k},\end{aligned}$$

where we have used the second estimate in Lemma 6.1 for obtaining the last inequality. The two last estimates together, we have thus established

$$\|\mathbf{e}_\lambda w\|_{L^\infty_t(L^p \cap \dot H^1)} + \|\mathbf{e}_\lambda w\|_{L^2_t L^p} + \frac{1}{\sqrt{\varepsilon}} \|\mathbf{e}_\lambda w\|_{L^2_t \dot H^2} \le C_3 \|w_0\|_{L^2_k} + C_4 \|\mathbf{e}_\lambda g\|_{L^2_t H^1_k}, \tag{6.2}$$

for some constants $C_3, C_4 > 0$.

Coming back to (6.1) and using Young's inequality, we deduce that for any $\beta > 0$, there is some $C_\beta > 0$ such that

$$\|\mathbf{e}_\lambda g\|_{L^\infty_t L^2_k} + \|\mathbf{e}_\lambda g\|_{L^2_t H^1_k} \le C_1 \|g_0\|_{L^2_k} + \beta \|\mathbf{e}_\lambda w\|_{L^2_t L^p} + C_\beta \|\mathbf{e}_\lambda w\|_{L^2_t \dot H^2}.$$

Combining that last estimate with (6.2) yields

$$\begin{aligned}\|\mathbf{e}_\lambda g\|_{L^\infty_t L^2_k} + \|\mathbf{e}_\lambda g\|_{L^2_t H^1_k} \le C_1 \|g_0\|_{L^2_k} + \beta C_3 \|w_0\|_{L^2_k} + \beta C_4 \|\mathbf{e}_\lambda g\|_{L^2_t H^1_k} \\ + \sqrt{\varepsilon} C_\beta C_3 \|w_0\|_{L^2_k} + \sqrt{\varepsilon} C_\beta C_4 \|\mathbf{e}_\lambda g\|_{L^2_t H^1_k}.\end{aligned}$$

Choosing first $\beta > 0$ small enough and then $\varepsilon > 0$ small enough, we obtain

$$\|\mathbf{e}_\lambda g\|_{L^\infty_t L^2_k} + \|\mathbf{e}_\lambda g\|_{L^2_t H^1_k} \le C_5 \|g_0\|_{L^2_k} + C_6 \|w_0\|_{L^2_k}$$

for some constants $C_5, C_6 > 0$. We then conclude part (1) by gathering this last estimate with (6.2).

Step 2. Let us denote $(G, W)(t) = (S_{\mathcal{L}} * \mathcal{R})(t)$, so that

$$G(t) = (S_{\mathcal{L}_{1,1}} * \mathcal{L}_{1,2} W)(t) + (S_{\mathcal{L}_{1,1}} * \mathcal{R}_1)(t)$$

and

$$W(t) = (S_{\mathcal{L}_{2,2}} * \mathcal{L}_{2,1} G)(t) + (S_{\mathcal{L}_{2,2}} * \mathcal{R}_2)(t).$$

Observing that $\Pi \mathcal{R}_1 = \Pi(\mathcal{L}_{1,2} W) = 0$, we may use Proposition 4.7–(2) as well as additionally Lemma 6.1 for handling $S_{\mathcal{L}_{1,1}} * \mathcal{L}_{1,2} W$, and we obtain

$$\|\mathbf{e}_\lambda G\|_{L^\infty_t L^2_k} + \|\mathbf{e}_\lambda G\|_{L^2_t H^1_k} \le C_2 \|\mathbf{e}_\lambda W\|^{1-\theta}_{L^2_t L^p} \|\mathbf{e}_\lambda W\|^{\theta}_{L^2_t \dot{H}^2} + C_1' \|\mathbf{e}_\lambda \mathcal{R}_1\|_{L^2_t H^{-1}_k},$$

for some constant $C_1', C_2 > 0$ and with $\theta = p/(2+p)$.

Similarly, using Lemma 5.3–(2) and additionally Lemma 6.1 for handling $S_{\mathcal{L}_{2,2}} * \mathcal{L}_{2,1} G$, we have

$$\|\mathbf{e}_\lambda W\|_{L^\infty_t (L^p \cap \dot{H}^1)} + \|\mathbf{e}_\lambda W\|_{L^2_t L^p} + \frac{1}{\sqrt{\varepsilon}} \|\mathbf{e}_\lambda W\|_{L^2_t \dot{H}^2} \le C_4 \|\mathbf{e}_\lambda G\|_{L^2_t H^1_k} + C_3' \|\mathbf{e}_\lambda \mathcal{R}_2\|_{L^2_t (L^p \cap \dot{H}^1)},$$

for some constants $C_3', C_4 > 0$. We can then conclude to (2) by arguing as in Step 1. $\square$

7. Proof of the nonlinear stability theorem

This section is devoted to the proof of Theorem 1.1. We fix $\lambda \ge 0$, $k > 3$ and $p > 2$ such that

$$\lambda < \frac{2\mu}{p}.$$

We next choose $\varepsilon_0 \in (0, \varepsilon_*)$, where $\varepsilon_* > 0$ is the small scale time provided by Proposition 6.2.

Consider the space

$$\mathcal{Z} = \left\{ (g, w) \in L^\infty_t (L^2_{k,0} \times (L^p \cap \dot{H}^1)) \cap L^2_t (H^1_k \times (L^p \cap \dot{H}^2)) \,\Big|\, \|(g, w)\|_{\mathcal{Z}} < \infty \right\}$$

with

$$\|(g, w)\|_{\mathcal{Z}} := \|\mathbf{e}_\lambda (g, w)\|_{L^\infty_t (\mathcal{X})} + \|\mathbf{e}_\lambda (g, w)\|_{L^2_t (\mathcal{Y})}.$$

For a fixed initial datum $(g_0, w_0) \in \mathcal{X}$, define next the mapping $\Phi : \mathcal{Z} \to \mathcal{Z}$, $(g, w) \mapsto \Phi[g, w]$ given by, for all $t \ge 0$,

$$\Phi[g, w](t) = S_{\mathcal{L}}(t)(g_0, w_0) + (S_{\mathcal{L}} * \mathcal{R}[(g, w), (g, w)])(t),$$

where

$$\begin{aligned} \mathcal{R}[(g, w), (g, w)] \;&=\; (R_1[(g, w), (g, w)] + S_1[(g, w), (g, w)], \\ &\qquad\qquad R_2[(g, w), (g, w)] + S_2[(g, w), (g, w)]) \,, \end{aligned}$$

with

$$\begin{aligned} R_1[(g, w), (\bar{g}, \bar{w})] &= -\operatorname{div}(g \nabla \bar{w}), \qquad & S_1[(g, w), (\bar{g}, \bar{w})] &= -\operatorname{div}(g \nabla \kappa * \bar{g}), \\ R_2[(g, w), (\bar{g}, \bar{w})] &= \nabla \kappa * [g \nabla \bar{w}], \qquad & S_2[(g, w), (\bar{g}, \bar{w})] &= \nabla \kappa * [g \nabla \kappa * \bar{g}]. \end{aligned}$$

We observe here that the first component of $\Phi[g, w](t)$ belongs to $L^2_{k,0}$ since

$$\Pi R_1[(g, w), (g, w)] = \Pi S_1[(g, w), (g, w)] = 0,$$

thus in the sequel we can apply the results of Proposition 6.2.

Thanks to Proposition 6.2, we have

$$\|S_{\mathcal{L}}(\cdot)(g_0, w_0)\|_{\mathcal{Z}} \lesssim \|(g_0, w_0)\|_{L^2_k \times (L^p \cap \dot{H}^1)},$$

as well as

$$\begin{aligned} \|S_{\mathcal{L}} * \mathcal{R}\|_{\mathcal{Z}} \lesssim \;& \|\mathbf{e}_\lambda R_1[(g, w), (g, w)]\|_{L^2_t H^{-1}_k} + \|\mathbf{e}_\lambda S_1[(g, w), (g, w)]\|_{L^2_t H^{-1}_k} \\ &+ \|\mathbf{e}_\lambda R_2[(g, w), (g, w)]\|_{L^2_t L^p} + \|\mathbf{e}_\lambda R_2[(g, w), (g, w)]\|_{L^2_t \dot{H}^1} \\ &+ \|\mathbf{e}_\lambda S_2[(g, w), (g, w)]\|_{L^2_t L^p} + \|\mathbf{e}_\lambda S_2[(g, w), (g, w)]\|_{L^2_t \dot{H}^1}, \end{aligned}$$

and we now estimate each term separately.

For the term associated to R_1, we first have

$$\begin{aligned}\|\operatorname{div}(g\nabla w)\|_{H^{-1}_k} &\lesssim \|g\nabla w\|_{L^2_k}\\ &\lesssim \|g\|_{L^4_k}\|\nabla w\|_{L^4}\\ &\lesssim \|g\|^{1/2}_{L^2_k}\|g\|^{1/2}_{H^1_k}\|\nabla w\|^{1/2}_{L^2}\|\nabla^2 w\|^{1/2}_{L^2},\end{aligned}$$

where we have used Hölder's inequality in the second line, and twice the Ladyzhenskaya's inequality (3.5) in the last one. We hence obtain

$$\begin{aligned}\|\mathbf{e}_\lambda R_1[(g,w),(g,w)]\|_{L^2_t H^{-1}_k} &\lesssim \|\mathbf{e}_\lambda g\|^{1/2}_{L^\infty_t L^2_k}\|\nabla w\|^{1/2}_{L^\infty_t L^2}\|\mathbf{e}_\lambda g\|^{1/2}_{L^2_t H^1_k}\|\nabla^2 w\|^{1/2}_{L^2_t L^2}\\ &\lesssim \|(g,w)\|^2_{\mathcal{Z}}.\end{aligned} \tag{7.1}$$

For the term associated to S_1, arguing similarly as above, we get

$$\begin{aligned}\|\operatorname{div}(g\nabla\kappa * g)\|_{H^{-1}_k} &\lesssim \|g\nabla\kappa * g\|_{L^2_k}\\ &\lesssim \|g\|_{L^4_k}\|\nabla\kappa * g\|_{L^4}\\ &\lesssim \|g\|^{1/2}_{L^2_k}\|g\|^{1/2}_{H^1_k}\|\nabla\kappa * g\|^{1/2}_{L^2}\|\nabla^2\kappa * g\|^{1/2}_{L^2}\\ &\lesssim \|g\|_{L^2_k}\|g\|_{H^1_k},\end{aligned}$$

where we have used Hölder's inequality in the second line, twice the Ladyzhenskaya's inequality (3.5) in the third line and finally Lemma 3.1 and (3.1) in the last line. We hence obtain

$$\begin{aligned}\|\mathbf{e}_\lambda S_1[(g,w),(g,w)]\|_{L^2_t H^{-1}_k} &\lesssim \|\mathbf{e}_\lambda g\|_{L^\infty_t L^2_k}\|g\|_{L^2_t H^1_k}\\ &\lesssim \|(g,w)\|^2_{\mathcal{Z}}.\end{aligned} \tag{7.2}$$

For the term associated to R_2, thanks to Lemma 3.2, we have

$$\|\nabla\kappa * (g\nabla w)\|_{L^p} \lesssim \|g\nabla w\|_{L^2_k}$$

and, because of (3.1), we have

$$\|\nabla\kappa * (g\nabla w)\|_{\dot H^1} \lesssim \|g\nabla w\|_{L^2}.$$

We can thus argue as above for obtaining (7.1), and we deduce

$$\|\mathbf{e}_\lambda R_2[(g,w),(g,w)]\|_{L^2_t L^p} + \|\mathbf{e}_\lambda R_2[(g,w),(g,w)]\|_{L^2_t \dot H^1} \lesssim \|(g,w)\|^2_{\mathcal{Z}}. \tag{7.3}$$

Finally, for the term associated to S_2, thanks to Lemma 3.2 and (3.1), we have similarly

$$\|\nabla\kappa * (g\nabla\kappa * g)\|_{L^p} \lesssim \|g\nabla\kappa * g\|_{L^2_k}$$

and

$$\|\nabla\kappa * (g\nabla\kappa * g)\|_{\dot H^1} \lesssim \|g\nabla\kappa * g\|_{L^2},$$

and therefore, arguing as for obtaining (7.2) yields

$$\|\mathbf{e}_\lambda S_2[(g,w),(g,w)]\|_{L^2_t L^p} + \|\mathbf{e}_\lambda S_2[(g,w),(g,w)]\|_{L^2_t \dot H^1} \lesssim \|(g,w)\|^2_{\mathcal{Z}}. \tag{7.4}$$

Putting together (7.1)–(7.4), we have hence obtained a first estimate

$$\|\Phi[g,w]\|_{\mathcal{Z}} \le C_0\|(g_0,w_0)\|_{H^1_k\times H^2} + C_1\|(g,w)\|^2_{\mathcal{Z}}. \tag{7.5}$$

Now, for $(g,w),(\bar g,\bar w)\in\mathcal{Z}$, we remark that

$$\Phi[g,w] - \Phi[\bar g,\bar w] = S_{\mathcal{L}} * (R_1^* + S_1^*, R_2^* + S_2^*)$$

with

$$\begin{aligned} R_1^* &= R_1[(g,w),(g,w)] - R_1[(\bar g,\bar w),(\bar g,\bar w)] \\ &= R_1[(g,w),(g,w)-(\bar g,\bar w)] + R_1[(g,w)-(\bar g,\bar w),(\bar g,\bar w)] \\ S_1^* &= S_1[(g,w),(g,w)] - S_1[(\bar g,\bar w),(\bar g,\bar w)] \\ &= S_1[(g,w),(g,w)-(\bar g,\bar w)] + S_1[(g,w)-(\bar g,\bar w),(\bar g,\bar w)] \\ R_2^* &= R_2[(g,w),(g,w)] - R_2[(\bar g,\bar w),(\bar g,\bar w)] \\ &= R_2[(g,w),(g,w)-(\bar g,\bar w)] + R_2[(g,w)-(\bar g,\bar w),(\bar g,\bar w)] \\ S_2^* &= S_2[(g,w),(g,w)] - S_2[(\bar g,\bar w),(\bar g,\bar w)] \\ &= S_2[(g,w),(g,w)-(\bar g,\bar w)] + S_2[(g,w)-(\bar g,\bar w),(\bar g,\bar w)]. \end{aligned}$$

Arguing exactly as above, we may establish a second estimate

$$\|\Phi(g,w)-\Phi(\bar g,\bar w)\|_{\mathcal Z} \le C_2\left(\|(g,w)\|_{\mathcal Z} + \|(\bar g,\bar w)\|_{\mathcal Z}\right)\|(g,w)-(\bar g,\bar w)\|_{\mathcal Z}. \tag{7.6}$$

As a consequence of the estimates (7.5) and (7.6), we can find $\eta_0,\eta_1>0$ small enough such that $C_0\eta_0 + C_1\eta_1^2 \le \eta_1$ and $2C_2\eta_1 < 1$ in such a way that Φ is a contraction on $B_{\mathcal Z}(0,\eta_1)$ for any $(g_0,w_0)\in B_{\mathcal X}(0,\eta_0)$. By a standard Banach fixed-point argument, one can construct a unique global mild solution $(g,w)\in\mathcal Z$ to (1.10) for any $(g_0,w_0)\in\mathcal X$ such that $\|(g_0,w_0)\|_{\mathcal X}\le\eta_0$. More specifically, choosing $\eta_1 := 2C_0\eta_0$ and $4C_0\max(C_1,C_2)\eta_0<1$, the above solution in particular verifies the energy estimate

$$\|(g,w)\|_{L^\infty_t(\mathcal X)} + \|(g,w)\|_{L^2_t(\mathcal Y)} \le 2C_0\|(g_0,w_0)\|_{\mathcal X}, \tag{7.7}$$

which is nothing but (1.7), as well as the decay estimate

$$\|\mathbf e_\lambda(g,w)\|_{L^\infty_t(\mathcal X)} + \|\mathbf e_\lambda(g,w)\|_{L^2_t(\mathcal Y)} \le 2C_0\|(g_0,w_0)\|_{\mathcal X}, \tag{7.8}$$

which is nothing but (1.8).

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