THE KINETIC FOKKER-PLANCK EQUATION IN A DOMAIN: ULTRACONTRACTIVITY, HYPOCOERCIVITY AND LONG-TIME ASYMPTOTIC BEHAVIOR

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Abstract. We consider the Kinetic Fokker-Planck (FKP) equation in a domain with Maxwell reflection condition on the boundary. We establish the ultracontractivity of the associated semigroup and the hypocoercivity of the associated operator. We deduce the convergence with constructive rate of the solution to the KFP equation towards the stationary state with same mass as the initial datum.

Contents

1. Introduction

In this paper, we consider the Kinetic Fokker-Planck (KFP) equation, also called the degenerated Kolmogorov or the ultraparabolic equation,

$$\partial_t f + v \cdot \nabla_x f - \Delta_v f - \mathrm{div}_v(vf) = 0 \quad \text{in} \quad \mathcal{U} \tag{1.1}$$

on the function $f := f_t = f(t,\cdot) = f(t,x,v)$, with $(t,x,v) \in \mathcal{U} := (0,T) \times \Omega \times \mathbb{R}^d$, $T \in (0,+\infty]$, $\Omega \subset \mathbb{R}^d$ a suitably smooth domain, $d \geq 3$, complemented with the Maxwell reflection condition on the boundary

$$\gamma_- f = \mathscr{R}\gamma_+ f = (1-\iota)\mathscr{S}\gamma_+ f + \iota \mathscr{D}\gamma_+ f \quad \text{on} \quad \Gamma_-, \tag{1.2}$$

and associated to an initial condition

$$f(0,x,v) = f_0(x,v) \quad \text{in} \quad \mathcal{O} := \Omega \times \mathbb{R}^d. \tag{1.3}$$

Here Γ_- denotes the incoming part of the boundary, $\mathscr{S}$ denotes the specular reflection operator, $\mathscr{D}$ denotes the diffusive reflection operator (see precise definitions below), and $\iota : \partial\Omega \to [0,1]$ denotes a (possibly space dependent) accommodation coefficient. More precisely, we assume that $\Omega := \{x \in \mathbb{R}^d;\, \delta(x) > 0\}$ for a $W^{2,\infty}(\mathbb{R}^d)$ function δ such that $|\delta(x)| := \mathrm{dist}(x,\partial\Omega)$ on a neighborhood of the boundary set $\partial\Omega$ and thus $n_x = n(x) := -\nabla\delta(x)$ coincides with the unit normal outward vector field on $\partial\Omega$. We next define $\Sigma^x_\pm := \{v \in \mathbb{R}^d; \pm v \cdot n_x > 0\}$ the sets of outgoing (Σ^x_+) and incoming (Σ^x_-) velocities at point $x \in \partial\Omega$, then the sets

$$\Sigma_\pm := \{(x,v);\, x \in \partial\Omega,\, v \in \Sigma^x_\pm\}, \quad \Gamma_\pm := (0,T) \times \Sigma_\pm,$$

and finally the outgoing and incoming trace functions $\gamma_\pm f := \mathbf{1}_{\Gamma_\pm}\gamma f$. The specular reflection operator $\mathscr{S}$ is defined by

$$(\mathscr{S}g)(x,v) := g(x,\mathcal{V}_x v), \quad \mathcal{V}_x v := v - 2n_x(n_x \cdot v), \tag{1.4}$$

and the diffusive operator $\mathscr{D}$ is defined by

$$(\mathscr{D}g)(x,v) := \mathscr{M}(v)\widetilde{g}(x), \quad \widetilde{g}(x) := \int_{\Sigma^x_+} g(x,w)\,(n_x \cdot w)\,dw, \tag{1.5}$$

Date: June 14, 2024.
2020 *Mathematics Subject Classification.* 35Q84, 35B40, 47D06.
Key words and phrases. Kinetic Fokker-Planck equation, ultracontractivity, hypocoercivity, long-time asymptotic behavior.

where $\mathscr{M}$ stands for the (conveniently normalized) Maxwellian function

$$\mathscr{M}(v) := (2\pi)^{-(d-1)/2} \exp(-|v|^2/2), \tag{1.6}$$

which is positive on $\mathbb{R}^d$ and verifies $\widetilde{\mathscr{M}} = 1$. We assume that the accomodation coefficient satisfies $\iota \in W^{1,\infty}(\partial\Omega)$. For further references, we also define the (differently normalized) Maxwellian function

$$f_\infty(x,v) = \frac{1}{|\Omega|}\,\mu(v) := \frac{1}{|\Omega|(2\pi)^{d/2}} \exp(-|v|^2/2), \tag{1.7}$$

which is positive on $\mathcal{O}$ and verifies $\|f_\infty\|_{L^1(\mathcal{O})} = 1$. The elementary (and well known at least at a formal level) properties of the Kinetic Fokker-Planck equation are that it is mass conservative, namely

$$\langle\langle f_t \rangle\rangle = \langle\langle f_0 \rangle\rangle, \quad \forall\, t \geq 0, \quad \text{with} \quad \langle\langle h \rangle\rangle := \int_{\mathcal{O}} h dx dv, \tag{1.8}$$

it is positivity preserving, namely $f_t \geq 0$ if $f_0 \geq 0$, and f_∞ is a stationary solution.

The aim of this paper is twofold:

(1) On the one hand, we prove the ultracontractivity of the semigroup associated to the evolution problem (1.1)–(1.2)–(1.3) by establishing some immediate gain of Lebesgue integrability and even immediate uniform bound estimate.

(2) On the other hand, we prove the convergence of the solution to the associated stationary state, namely $f_t \to \langle\langle f_0 \rangle\rangle f_\infty$ as $t \to \infty$, with constructive exponential rate in many weighted Lebesgue spaces.

These results extend some previous similar results known for other geometries or less general reflection conditions. For both problems, we adapt or modify some recent or forthcoming results established in [7, 12] for the Landau equation for the same geometry as considered here. In that sense, the techniques are not really new and the present contribution may rather be seen as a pedagogical illustration on one of the simplest models of the kinetic theory of some tools we develop in other papers for more elaborated kinetic models. We also refer to [10, 22, 11] for further developments of these techniques for related kinetic equations set in a domain with reflection conditions on the boundary.

For a weight function $\omega : \mathbb{R}^d \to (0,\infty)$ and a exponent $p \in [1,\infty]$, we define the associated weighted Lebesgue space

$$L^p_\omega := \{f \in L^1_{\mathrm{loc}}(\mathbb{R}^d);\ \|f\|_{L^p_\omega} := \|f\omega\|_{L^p} < \infty\}.$$

Our first main result is an ultracontractivity property.

Theorem 1.1. *There exist some constructive constants $\nu > 0$, $\theta \in (0,1)$, $C_1 \geq 1$, $C_2 \geq 0$ and a class of weight function $\mathfrak{W}_1$ such that for any exponents $p, q \in [1,\infty]$, $q > p$, any weight function $\omega \in \mathfrak{W}_1$ and any initial datum $f_0 \in L^p_\omega(\mathcal{O})$, the associated solution f to the Kinetic Fokker-Planck (KFP) equation (1.1)–(1.2)–(1.3) satisfies*

$$\|f(t)\|_{L^q_{\omega^\theta}} \leq C_1 \frac{e^{C_2 t}}{t^{\nu(1/p-1/q)}} \|f_0\|_{L^p_\omega}, \quad \forall\, t > 0. \tag{1.9}$$

We will show that the set $\mathfrak{W}_1$ contains at least some exponential functions. In the whole space $\Omega = \mathbb{R}^d$, such a kind of ultracontractivity property is a direct consequence of the representation of the solution thanks to the Kolmogorov kernel, see [33], as well as [31, 8] for related regularity estimates. Some local uniform estimate of a similar kind for a larger class of KFP equations in the whole space has been established [47, 13, 2] by using Moser iterative scheme introduced in [43, 44], from what some Gaussian upper bound on the fundamental solution may be derived, see [46, 34, 4]. In [23], the same local uniform estimates (as well as the Harnack inequality and the Holder regularity) has been shown for a still larger class of KFP equations in the whole space by using De Giorgi iterative scheme as introduced in [15]. We also refer to [1] for a general survey about these issues and to [50, 51, 35, 3, 32] for additional results on the KFP equations in the whole space. In [29], a gain of regularity estimate has been established by adapting Nash argument introduced in [45], see also [49, 24, 39] for further developments of the same technique.

In [19], an ultracontractivity result similar to ours is obtained for the KFP equation in a domain with specular reflection at the boundary by an extension argument to the whole space (used first in [26]) and then reduces the problem to the application of [47, 23]. In [52] some kind of regularity up to the boundary is proved for the KFP equation with inflow or specular reflection at the boundary using the extension argument of [19] and some appropriate change of coordinates. See also [48], where some similar results are established for the KFP equation with zero inflow.

We are next concerned with the longtime behavior estimate. We start by establishing a hypocoercivity result. For that purpose, we define the operator

$$\mathscr{L} f := -v \cdot \nabla_x f + \Delta_v f + \mathrm{div}_v(vf) \tag{1.10}$$

and we denote by $\mathrm{Dom}(\mathscr{L})$ its domain in the Hilbert space $\mathcal{H} := L^2(\mu^{-1}dxdv)$ endowed with the norm $\|f\|_{\mathcal{H}} = \|\mu^{-1/2} f\|_{L^2}$.

Theorem 1.2. *There exists a scalar product $(\!(\cdot,\cdot)\!)$ on the space $\mathcal{H}$ so that the associated norm $|\!|\!|\cdot|\!|\!|$ is equivalent to the usual norm $\|\cdot\|_{\mathcal{H}}$, and for which the linear operator $\mathscr{L}$ satisfies the following coercivity estimate: there is a positive constant $\lambda \in (0,1)$ such that*

$$(\!(-\mathscr{L} f, f)\!) \geq \lambda |\!|\!| f |\!|\!|^2 \tag{1.11}$$

for any $f \in \mathrm{Dom}(\mathscr{L})$ satisfying the boundary condition (1.2) and the mass condition $\langle\!\langle f \rangle\!\rangle = 0$.

The result and the proof is a mere adaptation and simplification of the same hypocoercivity estimate established in [7]. This last one is inspired, generalizes and simplifies some previous results established in [25, 9], see also [16, 20, 30, 27, 28, 49, 18] and the references therein for more material about the hypocoercivity theory.

We deduce from the two previous results the announced exponential convergence result.

Theorem 1.3. *There exists a class of weight functions $\mathfrak{W}_2$ such that for any weight function $\omega \in \mathfrak{W}_2$, any exponent $p \in [1,\infty]$ and any initial datum $f_0 \in L^p_\omega(\mathcal{O})$, the associated solution f to the KFP equation (1.1)–(1.2)–(1.3) satisfies*

$$\|f(t) - \langle\!\langle f_0 \rangle\!\rangle f_\infty\|_{L^p_\omega} \leq C e^{-\lambda t} \|f_0 - \langle\!\langle f_0 \rangle\!\rangle f_\infty\|_{L^p_\omega}, \quad \forall\, t \geq 0, \tag{1.12}$$

for the same constant $\lambda \in (0,1)$ as in Theorem 1.2 and for some constant $C = C(\omega)$.

It is worth emphasizing that the set $\mathfrak{W}_2$ contains some exponential functions and some polynomial (increasing fast enough) functions. The case $p = 2$ and $\omega = \mu^{-1/2}$ is an immediate consequence of Theorem 1.2. The general case is then deduced from this particular one thanks to Theorem 1.1 and some enlargement and shrinking techniques introduced and developed in [24, 39, 40].

Let us end the introduction by describing the organisation of the paper which is mainly dedicated to the proof of the above results.

In Section 2 we establish some growth estimates in many weighted Lebesgue spaces on the semigroup associated to the KFP equation (1.1)–(1.2)–(1.3). We do not discuss the existence and uniqueness issues about solutions to the KFP equation and the construction of the associated positive semigroup which will be discussed in detail in the companion paper [10]. We however emphasizes that solutions to the KFP equation must be understood in the renormalized sense as defined in [17, 38] so that the associated trace functions are well defined, see [38, 10, 12] and the references therein. We thus rather focus on the (a priori) estimates by exhibiting suitable twisted weight estimates for the solutions to the KFP equation (1.1)–(1.2)–(1.3) and its dual counterpart.

Section 3 is dedicated to the proof of Theorem 1.1. The strategy mixes Moser's gain of integrability argument of [44] and Nash's duality and interpolation arguments of [45]. It is also based on a twisted weight argument which is somehow slightly more elaborated than the one used in the previous sections. In Section 4, we prove Theorem 1.2 and Section 5 is dedicated to the proof of Theorem 1.3

2. Weighted L^p growth estimates

This section is devoted to the proof of a first and somehow rough set of growth estimates in some convenient weighted L^p spaces for solutions to the KFP equation (1.1)–(1.2)–(1.3) and the associated semigroup that we denote by the same letter $S_{\mathcal{L}}$ whatever is the space in which it is considered. It is classical that we may work at the level of the evolution equation and the associated generator or at the level of the associated semigroup. We will do the job at both levels.

As announced, we will not bother with too much rigorous justification but rather establish a priori weighted Lebesgue norm estimates from what we may very classically deduce the well-posedness of the Cauchy problem (1.1)–(1.2)–(1.3) and also deduce the existence of the associated semigroup. The solutions of the KFP equations would have to be understood in a appropriate renormalized sense, but again we will not bother about this important but technical point and we will freely make the computations as if the considered functions are smooth and fast enough decaying at infinity. Because the KFP equation conserves the positivity, the associate semigroup is positive and we may thus only handle with nonnegative functions. All these issues are discussed in the companion papers [21, 10, 12] for more general classes of KFP equations and we thus refer to these works for more details.

We now introduce the class of weight function we deal with. We denote by $\mathscr{C}$ the operator

$$\mathscr{C} f := \Delta_v f + \mathrm{div}_v(vf), \tag{2.1}$$

which is nothing but the collision part of the Kinetic Fokker-Planck operator involved in (1.1). We observe that for $f,\omega : \mathbb{R}^d \to \mathbb{R}_+$ and $p \in [1,\infty)$, we have

$$\int_{\mathbb{R}^d} (\mathscr{C} f) f^{p-1} \omega^p dv = -\frac{4(p-1)}{p^2} \int_{\mathbb{R}^d} |\nabla_v (f\omega)^{p/2}|^2 + \int |f|^p \omega^p \varpi, \tag{2.2}$$

with

$$\varpi = \varpi_{\omega,p}(v) := 2\left(1-\frac{1}{p}\right)\frac{|\nabla_v\omega|^2}{\omega^2} + \left(\frac{2}{p}-1\right)\frac{\Delta_v\omega}{\omega} + \left(1-\frac{1}{p}\right) d - v\cdot\frac{\nabla_v\omega}{\omega}, \tag{2.3}$$

see for instance [21, Lemma 7.7] and the references therein. We define $\mathfrak{W}$ as the set of radially symmetric nondecreasing weight functions $\omega : \mathbb{R}^d \to (0,\infty)$ such that

$$\kappa = \kappa_\omega := \max_{p=1,\infty} \sup_{v\in\mathbb{R}^d} \varpi_{\omega,p} < \infty.$$

It is worth noticing that $\omega := \langle v\rangle^k e^{\zeta|v|^s}$, with $k \in \mathbb{R}$ and $s \ge 0$, satisfies

$$\varpi(v) \underset{|v|\to\infty}{\sim} (s\zeta)^2|v|^{2s-2} - s\zeta|v|^s \quad \text{if} \quad s>0,$$

$$\varpi(v) \underset{|v|\to\infty}{\sim} \frac{d}{p'} - k \quad \text{if} \quad s=0,$$

so that $\omega \in \mathfrak{W}$ when

$$s\in(0,2), \quad \text{or} \quad s=2 \ \text{ and } \ \zeta<1/2, \quad \text{or} \quad s=0 \ \text{ and } \ k>0.$$

On the other hand, we may check

$$\varpi_{\mathscr{M}^{-1+1/q},p}(v) = -\frac{1}{q}\left(1-\frac{1}{q}\right)|v|^2 + \left(\frac{1}{p}+\frac{1}{q}-\frac{2}{pq}\right) d, \tag{2.4}$$

so that for the limit case $\omega = \mathscr{M}^{-1} \in \mathfrak{W}$, since then $\varpi_{\mathscr{M}^{-1},p} \equiv 2d/p$. We finally define

$$\mathfrak{W}_0 := \left\{\omega \in \mathfrak{W}\,;\, 1 \lesssim \omega \lesssim \mathscr{M}^{-1},\ \omega^{-1}|v|,\, \omega\mathscr{M}|v| \in L^1(\mathbb{R}^d)\right\}. \tag{2.5}$$

Proposition 2.1. *For any weight function $\omega \in \mathfrak{W}_0$, there exist $\kappa \ge 0$ and $C \ge 1$ such that for any exponent $p \in [1,\infty]$ and any solution f to the KFP equation (1.1)–(1.2)–(1.3), there holds*

$$\|f_t\|_{L^p_\omega} \le C e^{\kappa t}\|f_0\|_{L^p_\omega}, \quad \forall t\ge 0, \tag{2.6}$$

and we write equivalently

$$S_{\mathcal{L}}(t) : L^p_\omega \to L^p_\omega, \quad \text{with growth rate } \ \mathcal{O}(e^{\kappa t}), \quad \forall t \ge 0. \tag{2.7}$$

We start recalling the following classical estimate based on very specific choices of the weight functions, so that Darrozès-Guiraud type inequality [14] may be used.

Lemma 2.2. *For any $p\in[1,\infty]$, the semigroup $S_{\mathcal{L}}$ is a contraction on $L^p_{\mathscr{M}^{-1+1/p}}$.*

Proof of Lemma 2.2. We fix $p \in [1,\infty)$, $0 \le f_0 \in L^p_{\mathscr{M}^{-1+1/p}}$ and we denote by $f = f(t,x,v) \ge 0$ the solution to the Cauchy problem associated to (1.1)–(1.2)–(1.3). We compute

$$\begin{aligned}
\frac{1}{p}\frac{d}{dt}\int_{\mathcal{O}} f^p \mathscr{M}^{1-p} &= \int_{\mathcal{O}} (\mathscr{C} f) f^{p-1}\mathscr{M}^{1-p} - \frac{1}{p}\int_\Sigma (\gamma f)^p \mathscr{M}^{1-p}\, n_x\cdot v \\
&\le \int_{\mathcal{O}} \varpi_{\mathscr{M}^{-1+1/p},p} f^p \mathscr{M}^{1-p} - \frac{1}{p}\int_{\Sigma_+} (\gamma_+ f)^p \mathscr{M}^{1-p}|n_x\cdot v| \\
&\quad + \frac{1}{p}\int_{\Sigma_-} \left\{(1-\iota)\mathscr{S}\gamma_+ f + \iota\mathscr{D}\gamma_+ f\right\}^p \mathscr{M}^{1-p}|n_x\cdot v|,
\end{aligned}$$

where we have used the Green-Ostrogradski formula in the first line, we have thrown away the first term coming from (2.2) in the second line, we have split the boundary term into two pieces and we have used the boundary condition on its incoming part in the second and third lines. For the last term we have

$$\begin{aligned}
&\int_{\Sigma_-} \{(1-\iota)\mathscr{S}\gamma_+ f + \iota \mathscr{D}\gamma_+ f\}^p\, \mathscr{M}^{1-p} |n_x\cdot v| \\
&\le \int_{\Sigma_-} (1-\iota)(\mathscr{S}\gamma_+ f)^p \mathscr{M}^{1-p}|n_x\cdot v| + \int_{\Sigma_-} \iota(\widetilde{\gamma_+ f})^p \mathscr{M}|n_x\cdot v| \\
&\le \int_{\Sigma_+} (1-\iota)(\gamma_+ f)^p \mathscr{M}^{1-p}|n_x\cdot v| + \int_{\partial\Omega} \iota(\widetilde{\gamma_+ f})^p,
\end{aligned}$$

where we have used the convexity of the function $s\mapsto s^p$ in the second line and we have used both the change of variables $v\mapsto \mathcal{V}_x v$ in the last integral (which transforms Σ_- into Σ_+ with unit Jacobian) and the normalization condition on $\mathscr{M}$ (see (1.6)) in the third line. Observing next that

$$\begin{aligned}
(\widetilde{\gamma_+ f})^p &= \Big(\int_{\Sigma_+^x} (\gamma_+ f/\mathscr{M})\, \mathscr{M}|n_x\cdot v|\, dv\Big)^p \\
&\le \int_{\Sigma_+^x} (\gamma_+ f/\mathscr{M})^p\, \mathscr{M}|n_x\cdot v|\, dv,
\end{aligned}$$

thanks to the Jensen inequality (also called Darrozès-Guiraud's inequality in this context!), which is true because of the normalization condition on $\mathscr{M}$. We have thus established

$$\int_{\Sigma_-} \{(1-\iota)\mathscr{S}\gamma_+ f + \iota\mathscr{D}\gamma_+ f\}^p\, \mathscr{M}^{1-p}|n_x\cdot v| \le \int_{\Sigma_+} (\gamma_+ f)^p \mathscr{M}^{1-p}|n_x\cdot v|,$$

from which we obtain

$$\frac{d}{dt}\int_{\mathcal{O}} f^p \mathscr{M}^{1-p} \le p\int_{\mathcal{O}} \varpi_{\mathscr{M}^{-1+1/p},p} f^p \mathscr{M}^{1-p}.$$

Coming back to (2.4), we observe that

$$\varpi_{\mathscr{M}^{-1+1/p},p}(v) = -\frac{1}{p}\left(1-\frac{1}{p}\right)|v|^2 + \frac{2}{p}\left(1-\frac{1}{p}\right) d \le 0,$$

from what we immediately deduce that $S_{\mathcal{L}}$ is a contraction on $L^p_{\mathscr{M}^{-1+1/p}}$ when $p\in[1,\infty)$. We get the same conclusion in $L^\infty_{\mathscr{M}^{-1}}$ by letting $p\to\infty$. □

We extend the decay estimate to a general weight function in a L^1 framework by using an appropriate modification of the initial weight. That kind of moment estimate is reminiscent of L^1 hypodissipativity techniques, see e.g. [41, 24, 6]. Our mutiplicator is inspired from the usual multiplicator used in order to control the diffusive operator in previous works on the Boltzmann equation, see e.g. [5, 37, 38, 6]. For further references, we define the formal adjoints

$$\mathscr{L}^* := v\cdot\nabla_x + \mathscr{C}^*, \quad \mathscr{C}^* g := \Delta_v g - v\cdot\nabla_v g. \tag{2.8}$$

Lemma 2.3. *Let $\omega:\mathbb{R}^d\to(0,\infty)$ be a radially symmetric nondecreasing weight function such that $\omega\in\mathfrak{W}$ and $\mathscr{M}\omega|v|\in L^1(\mathbb{R}^d)$. There exists $\kappa\ge 0$ such that we have*

$$S_{\mathcal{L}}(t): L^1(\omega)\to L^1(\omega), \quad \forall\, t\ge 0,$$

with growth estimate $\mathcal{O}(e^{\kappa t})$.

It is worth emphasizing that with a very similar proof we may establish the same growth rate in L^p_ω for $p\in(1,\infty)$, but we were not able to reach the limit exponent $p=\infty$ because our estimates blow up as $p\to\infty$.

Proof of Lemma 2.3. Without loss of generality we may suppose that $\omega\ge 1$. We split the proof into two steps.

Step 1. For $0\le f_0\in L^1(\omega)$, we denote by $f=f(t,x,v)\ge 0$ the solution to the Cauchy problem (1.1)–(1.2)–(1.3), so that $f(t)=S_{\mathcal{L}}(t)f_0$.

We introduce the weight functions

$$\omega_A(v) := \chi_A(v) + (1-\chi_A(v))\omega(v),$$

with $\chi_A(v):=\chi(|v|/A)$, $A\ge 1$ to be chosen later and $\chi\in C^2(\mathbb{R}_+)$, $\mathbf{1}_{[0,1]}\le\chi\le\mathbf{1}_{[0,2]}$, and next

$$\widetilde{\omega}(x,v) := \omega_A(v) + \frac{1}{2} n_x\cdot\widetilde{v},$$

with $\hat v := v/\langle v\rangle$ and $\widetilde v := \hat v/\langle v\rangle$. It is worth emphasizing that

$$1 \le \omega_A \le \omega \quad \text{and} \quad c_A^{-1}\omega \le \tfrac12 \omega_A \le \widetilde\omega \le \tfrac32 \omega_A, \tag{2.9}$$

with $c_A \in (0,\infty)$. We write

$$\frac{d}{dt}\int_{\mathcal O} f\,\widetilde\omega = \int_{\mathcal O} f\,\mathscr L^*\widetilde\omega - \int_\Sigma \gamma f\,\widetilde\omega\, n_x\cdot v. \tag{2.10}$$

We first compute separately each contribution of the boundary term

$$B := -\int_\Sigma \gamma f\,\widetilde\omega\, n_x\cdot v = B_1 + B_2,$$

with

$$\begin{aligned} B_1 &:= -\int_{\Sigma_+} \gamma_+ f \omega_A |n_x\cdot v| + \int_{\Sigma_-} \{(1-\iota)\mathscr S\gamma_+ f + \iota \mathscr D\gamma_+ f\}\,\omega_A |n_x\cdot v| \\ B_2 &:= -\frac12 \int_\Sigma \gamma f\,(n_x\cdot v)^2. \end{aligned}$$

Making the change of variables $v\mapsto \mathcal V_x v$ in the last integral involved in B_1, we get

$$B_1 = -\int_{\Sigma_+} \iota\gamma_+ f\,\omega_A\,|n_x\cdot v| + \int_{\Sigma_+} \iota\mathscr D\gamma_+ f\,\omega_A\,|n_x\cdot v|.$$

We then define

$$K_1(\omega_A) := \int_{\mathbb R^d} \mathscr M\,\omega_A\,(n_x\cdot v)_+\,dv, \tag{2.11}$$

which is finite by the assumption on ω, so that

$$\int_{\Sigma_+} \iota\mathscr D\gamma_+ f\,\omega_A\,|n_x\cdot v| = \int_{\partial\Omega} \iota K_1(\omega_A)\widetilde{\gamma_+ f}.$$

Since $\omega_A \ge 1$, we then obtain

$$B_1 \le \int_{\partial\Omega} \iota(K_1(\omega_A)-1)\widetilde{\gamma_+ f}.$$

On the other hand, denoting

$$K_0 := \int_{\mathbb R^d} \mathscr M (n_x\cdot\hat v)_+^2\,dv \in (0,\infty), \tag{2.12}$$

which we observe is independent of x, we have

$$-\int_\Sigma \gamma f (n_x\cdot\hat v)^2 \quad\le\quad -\int_{\Sigma_+} \iota\mathscr D\gamma_+ f (n_x\cdot\hat v)^2 = -K_0\int_{\partial\Omega} \iota\widetilde{\gamma_+ f}.$$

Recalling (2.9) and observing that $\omega_A \to 1$ a.e. when $A\to\infty$, we get $K_1(\omega_A)\to K_1(1) = 1$ as $A\to\infty$ thanks to the dominated convergence Theorem of Lebesgue and the normalization condition on $\mathscr M$. We may thus fix $A \ge 1$ large enough in such a way that

$$K_1(\omega_A) - 1 - \frac12 K_0 \le 0,$$

and the contribution of the boundary is nonpositive.

Step 2. For the contribution of the volume integral, we write

$$\mathscr L^*\widetilde\omega = \mathscr C^*\omega + \mathscr C^*[\chi_A(1-\omega)] + \mathscr C^*[n_x\cdot\widetilde v] + v\cdot\nabla_x(n_x\cdot\widetilde v),$$

where we recall that the adjoint Fokker-Planck operator $\mathscr C^*$ is defined in (2.8). Because $\omega\in\mathfrak W$, we have

$$\mathscr C^*\omega \le \varpi_{\omega,1}\,\omega \le \kappa_1\omega,$$

for some $\kappa_1\in\mathbb R$. On the other hand, because χ_A has compact support and because of the regularity assumption of Ω, we have

$$\mathscr C^*[\chi_A(1-\omega)] + \mathscr C^*[n_x\cdot\widetilde v] + v\cdot\nabla_x(n_x\cdot\widetilde v) \le \kappa_2,$$

for some $\kappa_2\in\mathbb R_+$. Coming back to (2.10), we deduce that

$$\frac{d}{dt}\int_{\mathcal O} f\,\widetilde\omega \le \kappa\int_{\mathcal O} f\,\widetilde\omega,$$

with $\kappa := 2\kappa_1 + c_A\kappa_2$. We immediately conclude thanks to Grönwall's lemma and the comparison (2.9) between ω and $\widetilde\omega$. □

We establish now a similar exponential growth estimate in a general weighted L^1 framework for the dual backward problem associated to (1.1)–(1.2)–(1.3), namely

$$\begin{cases} -\partial_t g = v \cdot \nabla_x g + \mathscr{C}^* g & \text{in} \quad (0,T) \times \mathcal{O}, \\ \gamma_+ g = \mathscr{R}^* \gamma_- g & \text{on} \quad (0,T) \times \Sigma_+, \\ g(T) = g_T & \text{in} \quad \mathcal{O}, \end{cases} \tag{2.13}$$

for any $T \in (0,\infty)$ and any final datum g_T. The adjoint Fokker-Planck operator $\mathscr{C}^*$ is defined in (2.8), and the adjoint reflection operator $\mathscr{R}^*$ is defined by

$$\mathscr{R}^* g(x,v) = (1-\iota)\mathscr{S} g(x,v) + \iota \mathscr{D}^* g(x),$$

with

$$\mathscr{D}^* g(x) = \widetilde{\mathscr{M} g}(x) := \int_{\mathbb{R}^d} g(x,w) \mathscr{M}(w) (n_x \cdot w)_- \, dw.$$

Again, we do not discuss the very classical issue about well-posedness in Lebesgue spaces for these problems nor the possibility to approximate the solutions by *smooth enough solutions*, which is useful in the following argument. Consider f a solution to the forward Cauchy problem (1.1)–(1.2)–(1.3) and g a solution to the above dual problem (2.13). We compute (at least formally)

$$\begin{aligned} \int_{\mathcal{O}} f(T) g_T &= \int_{\mathcal{O}} f_0 g(0) + \int_0^T \int_{\mathcal{O}} (\partial_t f g + f \partial_t g) \, ds \\ &= \int_{\mathcal{O}} f_0 g(0) - \int_0^T \int_{\mathcal{O}} (v \cdot \nabla_x f g + f v \cdot \nabla_x g) \, ds \\ &= \int_{\mathcal{O}} f_0 g(0) - \int_0^T \int_{\Sigma} (v \cdot n) \gamma f \gamma g \, ds \\ &= \int_{\mathcal{O}} f_0 g(0) - \int_0^T \int_{\Sigma_+} (v \cdot n)(\gamma_+ f)(\mathscr{R}^* \gamma_- g) \, ds \\ &\quad + \int_0^T \int_{\Sigma_-} |v \cdot n| (\mathscr{R} \gamma_+ f)(\gamma_- g) \, ds, \end{aligned}$$

by using the Green-Ostrogradski formula and the reflection conditions at the boundary. From the very definition of $\mathscr{R}$ and $\mathscr{R}^*$, we then deduce the usual identity

$$\int_{\mathcal{O}} f(T) g_T = \int_{\mathcal{O}} f_0 g(0), \tag{2.14}$$

or equivalently that $g(t) = S^*_{\mathscr{L}}(T-t) g_T$. We observe now that for a weight function ω, we have

$$\mathscr{C} \omega^{-1} = \omega^{-1} \varpi_{\omega,\infty}. \tag{2.15}$$

We then define $\mathfrak{N}$ the class of weight functions $m : \mathbb{R}^d \to (0,\infty)$ such that $\omega = m^{-1} \in \mathfrak{W}$. In particular, because of (2.15) and the definition of $\mathfrak{W}$, there exists $\kappa' \in \mathbb{R}$ such that

$$\mathscr{C} m \le \kappa' m. \tag{2.16}$$

We also define

$$\mathfrak{N}_0 := \{ m \in \mathfrak{N}; \ \mathscr{M} \lesssim m, \ m v \in L^1(\mathbb{R}^d) \}. \tag{2.17}$$

Lemma 2.4. *For any weight function $m \in \mathfrak{N}_0$, there exists $\kappa \in \mathbb{R}$ such that*

$$S^*_{\mathscr{L}}(t) : L^1_m \to L^1_m, \quad \mathcal{O}(e^{\kappa t}).$$

More precisely, there exists $C \ge 1$ such that for any $T > 0$ and any $g_T \in L^1_m$, the associated solution g to the backward dual problem (2.13) satisfies

$$\|g(0)\|_{L^1_m} \le C e^{\kappa T} \|g_T\|_{L^1_m}. \tag{2.18}$$

Proof of Lemma 2.4. Without loss of generality we may suppose that $m \ge \mathscr{M}$. For $T \in (0,\infty)$ and $0 \le g_T \in L^1_m$, we denote by $g = g(t,x,v)$ the solution to the backward dual Cauchy problem (2.13). We introduce the weight functions

$$m_A := \chi_A \mathscr{M} + (1-\chi_A) m, \quad \widetilde{m} := m_A - \frac{1}{2}(n_x \cdot \widetilde{v}) \mathscr{M}, \tag{2.19}$$

with the notations of Lemma 2.3. It is worth emphasizing that

$$\mathscr{M} \le m_A \le m \quad \text{and} \quad c_A^{-1} m \le \tfrac{1}{2} m_A \le \widetilde{m} \le \tfrac{3}{2} m_A, \tag{2.20}$$

with $c_A \in (0,\infty)$. Similarly as in the proof of Lemma 2.3, we compute

$$\begin{aligned} -\frac{d}{dt}\int_{\mathcal O} g\, m_A &= \int_{\mathcal O} g\,(\mathscr{C} m_A) + \int_\Sigma \gamma g\, m_A\, n_x\cdot v \\ &= \int_{\mathcal O} g\,(\mathscr{C} m_A) + \int_{\Sigma_+} [(1-\iota)\mathscr{S}\gamma_- g + \iota \widetilde{\gamma_- g \mathscr{M}}]\, m_A\, |n_x\cdot v| - \int_{\Sigma_-} \gamma_- g\, m_A\, |n_x\cdot v|, \end{aligned}$$

where we have used again the Green-Ostrogradski formula in the first line and the reflection condition at the boundary in the second line. We deduce

$$-\frac{d}{dt}\int_{\mathcal O} g\, m_A = \int_{\mathcal O} g\,(\mathscr{C} m_A) - \int_{\Sigma_-} \iota\gamma_- g\, m_A\, |n_x\cdot v| + \left(\int_{\mathbb{R}^d} m_A (n_x\cdot v)_+\, dv\right) \int_{\Sigma_-} \iota \mathscr{M}\gamma_- g\, |n_x\cdot v|,$$

by making the change of variables $v\mapsto \mathcal{V}_x v$ on the *outgoing part* Σ_+ of the boundary (which is in fact the *incoming part* of the boundary for the backward dual problem). Since $m_A \ge \mathscr{M}$, we have established a first estimate

$$-\frac{d}{dt}\int_{\mathcal O} g\, m_A \le \int_{\mathcal O} g\,(\mathscr{C} m_A) + \int_{\Sigma_-} \iota (K_1(m_A) - 1)\mathscr{M}\gamma_- g\, |n_x\cdot v|,$$

with now

$$K_1(m_A) := \int_{\mathbb{R}^d} m_A\,(n_x\cdot v)_+\, dv \to 1, \quad \text{as} \quad A\to\infty.$$

On the other hand, with the same notations as in the proof of Lemma 2.3, we have

$$\frac{d}{dt}\int_{\mathcal O} g\mathscr{M}\,(n_x\cdot \widetilde v) = \int_{\mathcal O} g\mathscr{C}(\mathscr{M}\,(n_x\cdot\widetilde v)) - \int_{\mathcal O} g\,\mathscr{M}\,\hat v\cdot D_x n_x \hat v - \int_\Sigma \gamma g\,\mathscr{M}\,(n_x\cdot\hat v)^2.$$

For the last term, there holds

$$\begin{aligned} \int_\Sigma \gamma g\mathscr{M}\,(n_x\cdot\hat v)^2 &\ge \int_{\Sigma_+} \iota(\mathscr{D}^*\gamma_- g)(n_x\cdot\hat v)^2\,\mathscr{M} \\ &\ge \left(\int_{\mathbb{R}^d} \mathscr{M}(n_x\cdot\hat v)_+^2\right)\int_{\Sigma_-} \iota\mathscr{M}\gamma_- g|n_x\cdot v|, \end{aligned}$$

which implies a second estimate

$$\frac{d}{dt}\int_{\mathcal O} g\mathscr{M}\,(n_x\cdot\widetilde v) \le \int_{\mathcal O} g\mathscr{C}(\mathscr{M}\,(n_x\cdot\widetilde v)) - \int_{\mathcal O} g\,\mathscr{M}\,\hat v\cdot D_x n_x\hat v - K_0\int_{\Sigma_-} \iota\mathscr{M}\gamma_- g|n_x\cdot v|,$$

with now

$$K_0 := \int_{\mathbb{R}^d} \mathscr{M}\,(n_x\cdot\hat v)_+^2\, dv \in (0,\infty). \tag{2.21}$$

Choosing $A>0$ large enough such that $K_1(m_A) - 1 - \frac12 K_0 \le 0$, the contribution of the boundary is nonpositive and we obtain

$$\begin{aligned} -\frac{d}{dt}\int_{\mathcal O} g\,\widetilde m &\le \int_{\mathcal O} g\,[\mathscr{C} m + \mathscr{C}[\chi_A(\mathscr{M} - m)] + \mathscr{C}[n_x\cdot\widetilde v\mathscr{M}] - v\cdot\nabla_x(n_x\cdot\widetilde v\mathscr{M})] \\ &\le \kappa\int_{\mathcal O} g\widetilde m, \end{aligned}$$

for some $\kappa\in\mathbb{R}$, by arguing similarly as during the proof of Lemma 2.3 and in particular using (2.16). By the Grönwall's lemma, we then deduce

$$\|g(0)\|_{L^1(\widetilde m)} \le e^{\kappa T}\|g_T\|_{L^1(\widetilde m)}, \tag{2.22}$$

from which we immediately conclude to (2.18). □

We may now come to the proof of the main result of this section.

Proof of Proposition 2.1. For $f_0\in L^\infty_\omega$, let us define $f(t) := S_{\mathscr{L}}(t) f_0$ the associated flow. Because of the duality identity (2.14), for any $g_t \in L^1_{\omega^{-1}}$, we have

$$\int_{\mathcal O} f(t) g_t = \int_{\mathcal O} f_0 g(0) \le \|f_0\|_{L^\infty_\omega}\|g(0)\|_{L^1_{\omega^{-1}}}.$$

Together with (2.18), we deduce

$$\int_{\mathcal O} f(t) g_t \le \|f_0\|_{L^\infty_\omega} C e^{\kappa t}\|g_t\|_{L^1_{\omega^{-1}}}.$$

Taking the supremum on g_t over the unit ball of $L^1(\omega^{-1})$, we thus conclude that

$$\|f(t)\|_{L^\infty_\omega} \le C e^{\kappa t}\|f_0\|_{L^\infty_\omega},$$

for any f_0, which is the desired estimate (2.7) when $p=\infty$. The estimate (2.7) for $p=1$ has been established in Lemma 2.4. We then conclude to the estimate (2.7) for any $p\in[1,\infty]$ by using a standard interpolation argument. □

Remark 2.5. The conditions on the weight function ω in the statement of Proposition 2.1 are not optimal but they are more than enough for our purpose. As a matter of fact, we may observe that
- Lemma 2.2 gives an estimate on $S_{\mathcal L}$ in L^1 and in $L^\infty_{\mathscr M^{-1}}$;
- Lemma 2.3 gives an estimate on $S_{\mathcal L}$ in L^1_ω from $\omega=1$ and up to $\omega=\mathscr M^{-1}\langle v\rangle^{-d-1-\varepsilon}$, $\varepsilon>0$;
- Lemma 2.4 gives an estimate on $S^*_{\mathcal L}$ in L^1_m from $m=\langle v\rangle^{-d-1-\varepsilon}$, $\varepsilon>0$, and up to $m=\mathscr M$, and thus an estimate on $S_{\mathcal L}$ in L^∞_ω from $\omega=\langle v\rangle^{d+1+\varepsilon}$, $\varepsilon>0$, and up to $\omega=\mathscr M^{-1}$.

3. Ultracontractivity: Proof of Theorem 1.1

3.1. **An improved weighted L^2 estimate at the boundary.** The DeGiorgi-Nash-Moser theory tells us that for parabolic equations some gain of integrability estimates can be obtained by elementary manipulations when evaluating the evolution of functions f^q for $q\neq 1$. That kind of regularity effect is also called ultracontractivity. More recently, a similar theory has been developed for the Kolomogorov equation in the whole space, see in particular [47, 23]. Our purpose is to generalise these techniques to a bounded domain framework. In the present framework and in order to be able to deduce next (by interpolation) the same kind of regularity effect in the border L^1_ω space, we first consider $q<1$. Let us observe that for $q\neq 0$ and f a positive solution to the KFP equation (1.1), we may compute

$$\partial_t f^q + v\cdot\nabla_x f^q - v\cdot\nabla_v f^q - qdf^q - \Delta_v f^q - 4\frac{(1-q)}{q}|\nabla_v f^{q/2}|^2 = 0.$$

Multiplying the equation by $\Phi^q := \varphi^q m^q$ with $q\in(0,1)$, $\varphi\in\mathcal D((0,T))$, and integrating in all the variables, we obtain

$$\frac1q\int_\Gamma(\gamma f)^q\Phi^q n_x\cdot v + \frac1q\int_{\mathcal U} f^q\mathcal T^*\Phi^q = 4\frac{(1-q)}{q^2}\int_{\mathcal U}|\nabla_v(f\Phi)^{q/2}|^2 + \int_{\mathcal U} f^q\Phi^q\varpi, \tag{3.1}$$

with $\mathcal U := (0,T)\times\mathcal O$, $\Gamma := (0,T)\times\Sigma$, $T\in(0,\infty)$,

$$\mathcal T^*\Psi := -\partial_t\Psi - v\cdot\nabla_x\Psi \tag{3.2}$$

and $\varpi := \varpi_{m,q}$ is defined in (2.3). Alternatively, defining

$$\mathcal T := \partial_t + v\cdot\nabla_x \tag{3.3}$$

and recalling that $\mathscr C$ has been defined in (2.1), we may write

$$\mathcal T\frac{f^q}{q} = f^{q-1}\mathcal T f = f^{q-1}\mathscr C f,$$

so that

$$\frac1q\int_\Gamma(\gamma f)^q\Phi^q n_x\cdot v + \frac1q\int_{\mathcal U} f^q\mathcal T^*\Phi^q = \int_{\mathcal U} f^{q-1}(\mathscr C f)\Phi^q,$$

from what we deduce (3.1) with the help of (2.2)-(2.3).

We now establish a key new moment estimate on the KFP equation (1.1)–(1.2)–(1.3) which makes possible to control a solution near the boundary. The proof is based on the introduction of an appropriate weight function which combines the twisting term used in the previous section and the twisting term used in [21, Section 11], that last one being in the spirit of moment arguments used in [36, 42].

Proposition 3.1. *Let $q\in(0,1)$ and $m:\mathbb R^d\to(0,\infty)$ be a radially symmetric decreasing weight function such that $m^{\frac{q}{1-q}}|v|\in L^1(\mathbb R^d)$. There exists $C=C(q,m,\Omega)>0$ such that for any nonnegative solution f to the KFP equation (1.1)–(1.2)–(1.3) and any test function $0\le\varphi\in\mathcal D((0,T))$, there holds*

$$\int_{\mathcal U} f^q\widetilde m^q\frac{(n_x\cdot\hat v)^2}{\delta^{1/2}}\varphi^q + \int_{\mathcal U}|\nabla_v(f^{q/2}\widetilde m^{q/2})|^2\varphi^q \le C\int_{\mathcal U} f^q m^q[|\partial_t\varphi^q| + \langle\varpi_-\rangle\varphi^q],$$

where $\widetilde m$ is a modified weight function such that $m\lesssim\widetilde m\lesssim m$ and $\varpi := \varpi_{\widetilde m,q}$ is defined in (2.3).

Proof of Proposition 3.1. We fix $q\in(0,1)$ and we introduce the modified weight functions

$$m_A^q := \chi_A\mathscr M^{1-q} + (1-\chi_A)m^q, \tag{3.4}$$

for $A \geq 1$ and with the notations of Lemma 2.3. We next introduce the function

$$\Phi^q := \varphi^q \, \widetilde m^q, \quad \widetilde m^q := m_A^q - \frac{m_A^q}{4} n_x \cdot \widetilde v + \frac{m_A^q}{4D^{1/2}} \delta(x)^{1/2} n_x \cdot \widetilde v,$$

where $D = \sup \delta$ is half the diameter of Ω, so that in particular an estimate similar to (2.20) holds. From (3.1), we have

$$\begin{aligned} &4 \frac{(1-q)}{q} \int_{\mathcal U} |\nabla_v (f\Phi)^{q/2}|^2 - \int_\Gamma (\gamma f)^q \Phi^q n_x \cdot v - \int_{\mathcal U} f^q \mathcal T_2^* \Psi_3 \\ &= \int_{\mathcal U} f^q \mathcal T_2^* \Psi_{12} - q \int_{\mathcal U} f^q \Phi^q \varpi + \int_{\mathcal U} f^q \mathcal T_1^* \Phi^q, \end{aligned} \tag{3.5}$$

where $\mathcal T_1^* = -\partial_t$, $\mathcal T_2^* = -v \cdot \nabla_x$, $\varpi = \varpi_{\tilde m, q}$ and

$$\Psi_{12} := \varphi^q m_A^q \left(1 - \frac14 n_x \cdot \widetilde v \right), \quad \Psi_3 := \varphi^q \, \frac{m_A^q}{4D^{1/2}} \delta(x)^{1/2} n_x \cdot \widetilde v.$$

We now compute each term separately.

Step 1. For the second term at the left-hand side of (3.5), we observe that

$$-\int_\Sigma (\gamma f)^q m_A^q n_x \cdot v = -\int_{\Sigma_+} (\gamma_+ f)^q m_A^q |n_x \cdot v| + \int_{\Sigma_-} (\gamma_- f)^q m_A^q |n_x \cdot v|$$

and, using the boundary condition together with the fact that the map $s \mapsto s^q$ is concave, we get

$$\begin{aligned} &\int_{\Sigma_-} \left\{ (1-\iota) \mathscr S \gamma_+ f + \iota \mathscr D \gamma_+ f \right\}^q m_A^q |n_x \cdot v| \\ &\qquad \geq \int_{\Sigma_-} (1-\iota) (\mathscr S \gamma_+ f)^q m_A^q |n_x \cdot v| + \int_{\Sigma_-} \iota (\widetilde{\gamma_+ f})^q \mathscr M^q m_A^q |n_x \cdot v|. \end{aligned}$$

Removing the contribution of the specular reflection thanks to the change of variables $v \mapsto \mathcal V_x v$ as in the proof of Lemmas 2.3 and 2.4 and using the Hölder inequality in order to manage the term involving K_2, we therefore obtain

$$\begin{aligned} -\int_\Sigma (\gamma f)^q m_A^q n_x \cdot v \ &\geq \ \int_{\Sigma_-} \iota (\widetilde{\gamma_+ f})^q \mathscr M^q m_A^q (n_x \cdot v)_- - \int_{\Sigma_+} \iota (\gamma_+ f)^q m_A^q (n_x \cdot v)_+ \\ &\geq \ (K_1(m_A) - K_2(m_A)^{1-q}) \int_{\partial\Omega} \iota (\widetilde{\gamma_+ f})^q, \end{aligned}$$

with

$$K_1(m_A) := \int_{\mathbb R^d} \mathscr M^q m_A^q (n_x \cdot v)_- dv < +\infty, \quad K_2(m_A) := \int_{\mathbb R^d} m_A^{\frac{q}{1-q}} (n_x \cdot v)_+ dv < +\infty.$$

On the other hand, we have

$$\int_\Sigma (\gamma f)^q m_A^q \frac{(n_x \cdot \hat v)^2}{4} \ \geq \ K_0(m_A) \int_{\partial\Omega} \iota (\widetilde{\gamma_+ f})^q$$

with

$$K_0(m_A) := \frac14 \int_{\mathbb R^d} \mathscr M^q m_A^q (n_x \cdot \hat v)_-^2 \, dv.$$

Both together, we obtain

$$-\int_\Sigma (\gamma f)^q \widetilde m^q n_x \cdot v \geq \left[K_0(m_A) + K_1(m_A) - K_2(m_A)^{1-q} \right] \int_{\partial\Omega} \iota (\widetilde{\gamma_+ f})^q.$$

Observing that $m_A \to \mathscr M^{\frac1q - 1}$ when $A \to \infty$, we deduce that $K_1(m_A) \to K_1(\mathscr M^{\frac1q - 1}) = 1$, $K_2(m_A) \to K_2(\mathscr M^{\frac1q - 1}) = 1$ and $K_0(m_A) \to K_0(\mathscr M^{\frac1q - 1}) > 0$ as $A \to \infty$, thanks to the integrability condition made on m and the dominated convergence theorem of Lebesgue. We may thus choose $A > 0$ large enough in such a way that

$$K_0(m_A) + K_1(m_A) - K_2(m_A)^{1-q} \geq 0. \tag{3.6}$$

Step 2. In order to deal with the third term at the left-hand side of (3.5), we define $\psi := \delta(x)^{1/2} n_x \cdot \widetilde v$. Observing that $\langle v \rangle \psi \in L^\infty(\mathcal O)$, $\nabla_v \psi \in L^\infty(\mathcal O)$ and

$$v \cdot \nabla_x \psi = \frac12 \frac{1}{\delta(x)^{1/2}} (\hat v \cdot n_x)^2 + \delta(x)^{1/2} \hat v \cdot D_x n_x \hat v,$$

we compute

$$-\int_{\mathcal{U}} f^q \mathcal{T}_2^* \Psi_3 = \frac{1}{4D^{1/2}} \int_{\mathcal{U}} f^q \varphi^q m_A^q \left\{ \frac{1}{2} \frac{1}{\delta(x)^{1/2}} (\hat{v} \cdot n_x)^2 + \delta(x)^{1/2} \hat{v} \cdot D_x n_x \hat{v} \right\}.$$

We may now conclude. Because of (3.6), we may get rid of the boundary term, and together with the last inequality, we get

$$\begin{aligned} &4\frac{1-q}{q} \int_{\mathcal{U}} |\nabla_v (f\widetilde{m})^{q/2}|^2 \varphi^q + \frac{1}{8D^{1/2}} \int_{\mathcal{U}} f^q \varphi^q m_A^q \, \frac{1}{\delta(x)^{1/2}} (n_x \cdot \hat{v})^2 \\ &\le \frac{1}{4} \int_{\mathcal{U}} f^q m_A^q \varphi^q \, v \cdot \nabla_x (n_x \cdot \widetilde{v}) - q \int_{\mathcal{U}} f^q \varphi^q \widetilde{m}^q \varpi - \int_{\mathcal{U}} f^q \widetilde{m}^q \partial_t \varphi^q \\ &\quad - \frac{1}{4D^{1/2}} \int_{\mathcal{U}} f^q \varphi^q m_A^q \delta(x)^{1/2} \hat{v} \cdot D_x n_x \hat{v} \\ &\le C_{\Omega,A} \int_{\mathcal{U}} f^q m^q \langle \varpi_- \rangle \varphi^q + C_A \int_{\mathcal{U}} f^q m^q |\partial_t \varphi^q|, \end{aligned}$$

where we have used that $\delta \in W^{2,\infty}(\Omega)$ and Ω is bounded. $\square$

Using an interpolation argument, we may write our previous weighted L^q estimate in a more convient way where the penalization of a neighborhood of the boundary is made clearer. In order to do this, we use the following interpolation estimate.

Lemma 3.2. *We set $\beta := (2(d+1))^{-1}$. For any function $g : \mathcal{O} \to \mathbb{R}$, there holds*

$$\int_{\mathcal{O}} \frac{g^2}{\delta^\beta} \lesssim \int_{\mathcal{O}} (g\langle v \rangle)^2 \frac{(n_x \cdot \hat{v})^2}{\delta^{1/2}} + \int_{\mathcal{O}} |\nabla_v (g\langle v \rangle)|^2. \tag{3.7}$$

Proof of Lemma 3.2. For $\eta, \zeta > 0$, we start by writing

$$\int_{\mathcal{O}} g^2 \delta^{-2\eta} \;\; = \;\; \int_{\mathcal{O}} \frac{g^2}{\delta^{2\eta}} \mathbf{1}_{(n_x \cdot v)^2 > \delta^{2\zeta}} + \int_{\mathcal{O}} \frac{g^2}{\delta^{2\eta}} \mathbf{1}_{|n_x \cdot v| \le \delta^\zeta} =: T_1 + T_2.$$

For the first term, we have

$$T_1 \le \int_{\mathcal{O}} g^2 \mathbf{1}_{(n_x \cdot v)^2 > \delta^{2\zeta}} \frac{(n_x \cdot v)^2}{\delta^{2\zeta + 2\eta}} \le \int_{\mathcal{O}} g^2 \frac{(n_x \cdot v)^2}{\delta^{1/2}},$$

by choosing $2\zeta + 2\eta = 1/2$. For the second term, we define $2^* := 2d/(d-2)$ the Sobolev exponent in dimension $d \ge 3$, and we compute

$$\begin{aligned} T_2 \;\; &\le \;\; \int_{\Omega} \delta^{-2\eta} \Big(\int_{\mathbb{R}^d} (\langle v \rangle g)^{2^*} \Big)^{2/2^*} \Big(\int_{\mathbb{R}^d} \langle v \rangle^{-d} \mathbf{1}_{|n_x \cdot v| \le \delta^\zeta} \Big)^{2/d} \\ &\lesssim \;\; \int_{\Omega} \delta^{-2\eta + 2\zeta/d} \int_{\mathbb{R}^d} |\nabla_v (\langle v \rangle g)|^2, \end{aligned}$$

where we have used the Hölder inequality in the first line and the Sobolev inequality in the second line together with the observation that $\langle v \rangle^{-d} \in L^\infty(\mathbb{R}; L^1(\mathbb{R}^{d-1}))$. Choosing $2\zeta/d = 2\eta$, we get $\eta = (4(d+1))^{-1}$ and we conclude to (3.7). $\square$

Gathering the estimates of Proposition 3.1 and Lemma 3.2, we immediately obtain the following result.

Proposition 3.3. *Let $q \in (0,1)$ and $m : \mathbb{R}^d \to (0,\infty)$ be a radially symmetric decreasing weight function such that $m^{\frac{q}{1-q}} |v| \in L^1(\mathbb{R}^d)$. There exists $C = C(q, m, \Omega) > 0$ such that for any nonnegative solution f to the KFP equation (1.1)–(1.2)–(1.3) and any test function $0 \le \varphi \in \mathcal{D}((0,T))$, there holds*

$$\int_{\mathcal{U}} \frac{f^q}{\delta^\beta} \frac{m^q}{\langle v \rangle^2} \varphi^q \le C \int_{\mathcal{U}} f^q m^q [|\partial_t \varphi^q| + \langle \varpi_- \rangle \varphi^q],$$

where $\beta := (2(d+1))^{-1}$ and $\varpi := \varpi_{m^{-1},q}$ is defined in (2.3).

By particularizing the choice of m, we obtain a first boundary penalizing weighted $L^1 - L^q$ estimate which will be convenient for our purpose in the next steps.

Proposition 3.4. *For any $q \in ((d+1)/(d+2), 1)$, for any nonnegative solution f to the KFP equation (1.1)–(1.2)–(1.3) and any test function $0 \le \varphi \in \mathcal{D}((0,T))$, there holds*

$$\int_{\mathcal{U}} \frac{f^q}{\delta^\beta} \frac{\varphi^q}{\langle v \rangle^{2 + (d+2)q(1-q)}} \le C T^{1-q} \|\varphi^q\|_{W^{1,\infty}(0,T)} \|f\|^q_{L^1(\mathcal{U})},$$

with $C = C(q, d, \Omega) > 0$ and $\beta = (2(d+1))^{-1}$ defined just above.

Proof of Proposition 3.4. We choose $m := \langle v\rangle^{-(d+2)(1-q)}$ and we observe that $m^{\frac{q}{1-q}}\langle v\rangle \in L^1$ and $\varpi_{m^{-1},q} \in L^\infty$. From Proposition 3.3, we thus get

$$\int_{\mathcal{U}} \frac{f^q}{\delta^\beta} \frac{\varphi^q}{\langle v\rangle^{2+(d+2)q(1-q)}} \le C\|\varphi^q\|_{W^{1,\infty}(0,T)} \int_{\mathcal{U}} f^q \langle v\rangle^{-(d+2)(1-q)q}.$$

On the other hand, using the Hölder inequality, we have

$$\int_{\mathcal{U}} f^q \langle v\rangle^{-(d+2)(1-q)q} \le \left(\int_{\mathcal{U}} f\right)^q (T|\Omega|)^{1-q} \left(\int_{\mathbb{R}^d} \langle v\rangle^{-(d+2)q}\right)^{1-q},$$

and the last integral if finite because $(d+2)q > d$. We conclude by just gathering the two estimates. $\square$

3.2. **A weak weighted $L^1 - L^p$ estimate.**

Taking advantage of a known $L^1 - L^p$ estimate available for the KFP equation set in the whole space and thus in the interior of the domain, we deduce a downgrade weighted $L^1 - L^p$ estimate. We define

$$\mathfrak{W}_3 := \left\{\omega : \mathbb{R}^d \to (0,\infty)\,;\, \omega_0 := \omega/\langle v\rangle \in \mathfrak{W},\ |\nabla\omega_0|\omega_0^{-1}\langle v\rangle^{-1} \in L^\infty(\mathbb{R}^d)\right\}. \tag{3.8}$$

Proposition 3.5. *Assume that $p \in (1, 1+1/(2d))$, $\alpha > p$ and $\omega \in \mathfrak{W}_3$. There exists some constant $C = C(\Omega, p, \alpha, \omega) \in (0,\infty)$ such that any solution f to the KFP equation (1.1)–(1.2)–(1.3) satisfies*

$$\left\| f\varphi \frac{\omega}{\langle v\rangle}\delta^{\alpha/p}\right\|_{L^p(\mathcal{U})} \le C T^{1/p+2d(1-1/p)} \|\varphi\|_{W^{1,\infty}(0,T)} \|f\omega\|_{L^1(\mathcal{U})}, \tag{3.9}$$

for any $0 \le \varphi \in \mathcal{D}((0,T))$ and any $T > 0$.

Proof of Proposition 3.5. For $\chi \in \mathcal{D}(\Omega)$ such that $0 \le \chi \le 1$, we define $0 \le \bar f := f\varphi\chi\omega_0$, which is a solution to the equation

$$\partial_t \bar f + v\cdot\nabla_x \bar f - \Delta_v \bar f - (v + 2\frac{\nabla_v\omega_0}{\omega_0})\cdot\nabla_v \bar f = F$$

set on $(0,T)\times\mathbb{R}^d\times\mathbb{R}^d$, with

$$F := f\omega_0(\varphi'\chi + \varphi v\cdot\nabla_x\chi) + \bar f\Big(d - v\cdot\frac{\nabla_v\omega_0}{\omega_0} + 2\frac{|\nabla_v\omega_0|^2}{\omega_0^2} - \frac{\Delta_v\omega_0}{\omega_0}\Big).$$

Because $\omega_0 \in \mathfrak{W}$, we have

$$F_+ \le f\omega_0\langle v\rangle(|\varphi'|\chi + \varphi|\nabla_x\chi|) + f\varphi\chi\omega_0\kappa_{\omega_0}.$$

From [4, Theorem 1.5] for instance and because $|\nabla\omega_0|\omega_0^{-1} \lesssim \langle v\rangle$, we know that

$$\bar f \le \int_0^t K_{t-s} \star F_{+s}\, ds,$$

where $\star = \star_{x,v}$ stands for a convenient convolution operation and K_τ is the Kolmogorov kernel defined by

$$K_\tau(x,v) := \frac{C_1}{\tau^{2d}} \exp\left(-\frac{3C_2}{\tau^3}|x - \frac{\tau}{2}v|^2 - \frac{C_2}{4\tau}|v|^2\right), \quad C_i > 0.$$

We next compute

$$\begin{aligned} \|\bar f\|^p_{L^p([0,T]\times\mathbb{R}^{2d})} &\le \int_0^T \|\int_0^t K_{t-s}\star F_{+s}\|^p_{L^p(\mathbb{R}^{2d})} dt \\ &\le \|K\|^p_{L^p([0,T]\times\mathbb{R}^{2d})} \|F_+\|^p_{L^1([0,T]\times\mathbb{R}^{2d})}, \end{aligned}$$

and because $1 \le p < 1 + 1/(2d)$, we find

$$\|K\|^p_{L^p([0,T]\times\mathbb{R}^{2d})} = C_{K,p} T^{1-2d(p-1)}.$$

As a consequence, we have

$$\|f\varphi\omega_0\chi\|_{L^p(\mathcal{U})} \lesssim C_T \|\varphi\|_{W^{1,\infty}} \|\chi\|_{W^{1,\infty}} \|f\omega\|_{L^1(\mathcal{U})}, \tag{3.10}$$

with $C_T := T^{1/p+2d(1-1/p)}$.

Step 2. We define $\Omega_k := \{x \in \Omega \mid \delta(x) > 2^{-k}\}$ and we choose $\chi_k \in \mathcal{D}(\Omega)$ such that $\mathbf{1}_{\Omega_{k+1}} \le \chi_k \le \mathbf{1}_{\Omega_k}$ and $2^{-k}\|\chi_k\|_{W^{1,\infty}} \lesssim 1$ uniformly in $k \ge 1$. We also denote $\mathcal{U}_k := (0,T)\times\Omega_k\times\mathbb{R}^d$. We deduce from (3.10) that

$$\|f\varphi\omega_0\|_{L^p(\mathcal{U}_{k+1})} \lesssim 2^k C_T \|\varphi\|_{W^{1,\infty}(0,T)} \|f\omega\|_{L^1(\mathcal{U})}, \quad \forall\, k \ge 1.$$

Summing up, we obtain

$$\begin{aligned}\int_{\mathcal U}\delta^\alpha(\varphi f\omega_0)^p &= \sum_k\int_{\mathcal U_{k+1}\backslash\mathcal U_k}\delta^\alpha(\varphi f\omega_0)^p\\ &\lesssim \sum_k 2^{-k\alpha}\int_{\mathcal U_{k+1}}(\varphi f\omega_0)^p\\ &\lesssim \sum_k 2^{k(p-\alpha)}C_T^p\|\varphi\|^p_{W^{1,\infty}}\|f\omega\|^p_{L^1(\mathcal U)}\\ &\lesssim C_T^p\|\varphi\|^p_{W^{1,\infty}(0,T)}\|f\omega\|^p_{L^1(\mathcal U)},\end{aligned}$$

because $\alpha>p$, what is nothing but (3.9). □

3.3. **The L^1-L^r estimate up to the boundary.** We start with a classical interpolation result.

Lemma 3.6. *For any exponent $0<r_0<r_1<\infty$, $\alpha,\beta>0$, $0<\theta<1$ and any weight functions $\sigma_i:\mathcal U\to(0,\infty)$, there holds*

$$\|g\|_{L^r_\sigma}\le\|g\|^{1-\theta}_{L^{r_0}_{\sigma_0}}\|g\|^\theta_{L^{r_1}_{\sigma_1}},$$

with $1/r:=(1-\theta)/r_0+\theta/r_1$ and $\sigma:=\sigma_0^{1-\theta}\sigma_1^\theta$.

We include the very classical proof because the statement is usually written assuming rather $1\le r_0<r_1<\infty$, but that last restriction is not needed.

Proof of Lemma 3.6. We write

$$\begin{aligned}\Big(\int f^r\sigma^r\Big)^{1/r} &= \Big(\int(f\sigma_0)^{r(1-\theta)}(f\sigma_1)^{r\theta}\Big)^{1/r}\\ &\le \Big(\int(f\sigma_0)^{ar(1-\theta)}\Big)^{1/ar}\Big(\int(f\sigma_1)^{a'r\theta}\Big)^{1/a'r}\end{aligned}$$

thanks to the Hölder inequality with $a:=\frac{p}{\theta r}=1+\frac{1-\theta}{\theta}\frac{p}{q}>1$. We conclude by observing that $ar(1-\theta)=r_0$ and $a'r\theta=r_1$. □

We are now in position of stating our weighted L^1-L^r estimate up to the boundary which is the well-known cornerstone step in the proof of DeGiorgi-Nash-Moser gain of integrability estimate.

Proposition 3.7. *There exist an exponent $r>1$ and some constants $\eta>0$, $\theta,q\in(0,1)$ such that any solution f to the KFP equation (1.1)-(1.2) satisfies*

$$\|\varphi f\omega^\sharp\|_{L^r(\mathcal U)}\le CT^\eta\|\varphi^q\|^{1/q}_{W^{1,\infty}(0,T)}\|f\omega\|_{L^1(\mathcal U)},\tag{3.11}$$

for any weight function $\omega\in\mathfrak W_3$ and any test function $0\le\varphi\in\mathcal D((0,T))$, with $\omega^\sharp:=\omega^\theta\langle v\rangle^{-4}$ and $C=C(d,\Omega,\omega)$.

Proof of Proposition 3.7. From Proposition 3.4, we have

$$\Big\|f\varphi\frac{1}{\delta^{\beta/q}}\frac{1}{\langle v\rangle^{2/q+(d+2)(1-q)}}\Big\|_{L^q(\mathcal U)}\le CT^{1/q-1}\|\varphi^q\|^{1/q}_{W^{1,\infty}(0,T)}\|f\omega\|_{L^1(\mathcal U)},$$

for some exponent $q\in((d+1)/(d+2),1)$ and with $\beta:=(2(d+1))^{-1}$. Together with Proposition 3.5 and Lemma 3.6, we deduce that

$$\|f\varphi\sigma\|_{L^r}\le CT^\eta\|\varphi^q\|^{1/q}_{W^{1,\infty}}\|f\omega\|_{L^1(\mathcal U)},$$

for any $\theta\in(0,1)$ with

$$\frac1r=\frac{1-\theta}{q}+\frac{\theta}{p},\quad \sigma:=\frac{\delta^{\alpha\theta/p}}{\delta^{(1-\theta)\beta/q}}\frac{\omega^\theta}{\langle v\rangle^{\theta+(2/q+(d+2)(1-q))(1-\theta)}},$$

and

$$\eta:=(1-\theta)(1/q-1)+\theta(1/p+2d(1-1/p)),$$

where we recall here that $p\in(1,1+1/(2d))$ and $\alpha>1$. We first choose

$$\theta=\theta_q:=\frac{\beta/q}{\beta/q+\alpha/p}$$

in such a way that $\delta^{\alpha\theta/p-(1-\theta)\beta/q}\equiv1$. Because $\theta_q\to\theta_1\in(0,1)$ as $q\to1$ and $r=r_q\to r_*$ as $q\to1$ with

$$\frac{1}{r_*}=1-\theta_1+\frac{\theta_1}{p}<1,$$

we may choose $q \in ((d+1)/(d+2), 1)$ large enough in such a way that $r > 1$. We finally observe that $2/q + (d+2)(1-q) \le 4$ so that $\sigma \gtrsim \omega^\sharp$. □

3.4. **The $L^1 - L^p$ estimate on the dual problem.** We consider the dual backward problem (2.13) for which we establish the same kind of estimate as for the forward KFP problem (1.1)-(1.2). We define

$$\mathfrak{N}_1 := \Big\{ m : \mathbb{R}^d \to (0,\infty);\ m \in L^1(\mathbb{R}^d),\ m\mathscr{M}^{-1} \in L^d(\mathbb{R}^d), \quad m_\ell := m\langle v\rangle^{-2-2\ell} \text{ satisfies } \frac{|\nabla m_\ell|^2}{m_\ell^2} + \frac{|\Delta m_\ell|}{m_\ell} \lesssim \langle v\rangle^2 \text{ for } \ell = 0,1 \Big\}. \tag{3.12}$$

Proposition 3.8. *There exist some exponent $r_1 > 1$ and some constants $\eta_1 > 0$, $q \in (0,1)$ such that any solution g to the dual backward problem (2.13) satisfies*

$$\|\varphi g \frac{m}{\langle v\rangle^8}\|_{L^{r_1}(\mathcal{U})} \lesssim T^{\eta_1} \|\varphi^q\|^{1/q}_{W^{1,\infty}(0,T)} \|gm\|_{L^1(\mathcal{U})}, \tag{3.13}$$

for any weight function $m \in \mathfrak{N}_1$ and any test function $0 \le \varphi \in \mathcal{D}((0,T))$, with $C = C(d,\Omega,m)$.

Proof of Proposition 3.8. The proof follows the same steps as for the proof of Proposition 3.7 and we thus repeat it without too much details.

Step 1. Boundary penalizing $L^1 - L^q$ estimate, $q < 1$. From [21, Lemma 7.7] or a direct computation, we have

$$\int (\mathscr{C}^* g)\, g^{q-1} m^q = -\frac{4(q-1)}{q^2} \int |\nabla_v (gm)^{q/2}|^2 + \int g^q m^q \wp, \tag{3.14}$$

with $\mathscr{C}^*$ defined in (2.8) and

$$\wp := 2\left(1 - \frac{1}{q}\right) \frac{|\nabla_v m|^2}{m^2} + \left(\frac{2}{q} - 1\right) \frac{\Delta_v m}{m} + \frac{d}{q} + v \cdot \frac{\nabla_v m}{m}. \tag{3.15}$$

Considering a solution g to the dual backward problem (2.13) and $q \ne 1$, we may write

$$\mathcal{T}^* \frac{g^q}{q} = g^{q-1} \mathcal{T}^* g = g^{q-1} \mathscr{C}^* g, \tag{3.16}$$

with $\mathcal{T}^*$ defined in (3.2). Let us fix an exponent $q \in (0,1)$. For a weight function $m \in \mathfrak{N}$ satisfying

$$m^q |v| \in L^1(\mathbb{R}^d),\ (m\mathscr{M}^{-1})^{\frac{q}{1-q}} |v| \in L^1(\mathbb{R}^d), \tag{3.17}$$

we define the modified weight function $\widetilde{m}$ by

$$\widetilde{m}^q := m_A^q \left(1 + \frac{1}{4} n_x \cdot \widetilde{v} - \frac{1}{4D^{1/2}} \delta(x)^{1/2} n_x \cdot \widetilde{v}\right),$$

where similarly as in (2.19) and with the same notations, we have defined

$$m_A^q := \chi_A \mathscr{M} + (1-\chi_A) m^q, \quad A \ge 1.$$

Multiplying the equation (3.16) by $\Phi^q := \varphi^q \widetilde{m}^q$ with $\varphi \in \mathcal{D}(0,T)$, and integrating in all the variables, we obtain

$$-\frac{1}{q} \int_\Gamma (\gamma g)^q \Phi^q n_x \cdot v + \frac{1}{q} \int_{\mathcal{U}} g^q \mathcal{T} \Phi^q = \int_{\mathcal{U}} g^{q-1} (\mathscr{C}^* g) \Phi^q,$$

with $\mathcal{T}$ defined in (3.3). Together with (3.14), we thus deduce

$$4\frac{1-q}{q^2} \int_{\mathcal{U}} |\nabla_v (g\Phi)^{q/2}|^2 + \frac{1}{q} \int_\Gamma (\gamma g)^q \Phi^q n_x \cdot v = \frac{1}{q} \int_{\mathcal{U}} g^q \mathcal{T} \Phi^q - \int g^q \Phi^q \wp, \tag{3.18}$$

with $\wp = \wp_{\widetilde{m},q}$. In order to deal with the second boundary term at the LHS, we first set

$$K_1(m_A^q) := \int_{\mathbb{R}^d} m_A^q n_x \cdot v < \infty, \quad K_2(m_A^q) := \int_{\mathbb{R}^d} (\mathscr{M}^{-q} m_A^q)^{\frac{1}{1-q}} (n_x \cdot v)_- < \infty.$$

We next observe that

$$\begin{aligned} \int_\Sigma (\gamma g)^q m_A^q n_x \cdot v &\ge \int_{\Sigma_+} \iota (\widetilde{\gamma_- g \mathscr{M}})^q m_A^q (n_x \cdot v)_+ - \int_{\Sigma_-} \iota (\gamma_- g)^q m_A^q (n_x \cdot v)_- \\ &\ge \int_{\partial\Omega} \iota \left(K_1(m_A^q) - K_2(m_A^q)^{1-q}\right) (\widetilde{\gamma_- g \mathscr{M}})^q, \end{aligned}$$

where we have used the concavity of the function $G \mapsto G^q$, we have removed the contribution of the specular reflection in the first line, and we have used Hölder's inequality

$$\int_{\Sigma^x_-} \gamma_- g^q m_A^q (n_x \cdot v)_- \le (\widetilde{\gamma_- g \mathscr{M}})^q \Big(\int_{\mathbb{R}^d} (\mathscr{M}^{-q} m_A^q)^{\frac{1}{1-q}} (n_x \cdot v)_- \Big)^{1-q}$$

in the second line. On the other hand, we have

$$\begin{aligned} \int_\Sigma (\gamma g)^q m_A^q \frac{(n_x \cdot \hat v)^2}{4} &\ge \int_{\Sigma_+} (\mathscr{R}^* \gamma_- g)^q m_A^q \frac{(n_x \cdot \hat v)^2}{4} \\ &\ge \int_{\partial\Omega} \iota K_0(m_A^q) (\widetilde{\gamma_- g \mathscr{M}})^q, \end{aligned}$$

with

$$K_0(m_A^q) := \frac14 \int_{\mathbb{R}^d} m_A^q (n_x \cdot \hat v)_-^2 \, dv \in (0,\infty).$$

Both estimates together, the contribution of the boundary is bounded by below in the following way

$$\int_\Sigma (\gamma g)^q \widetilde m^q n_x \cdot v \ge \int_{\partial\Omega} \iota \left(K_0(m_A^q) + K_1(m_A^q) - (K_2(m_A^q))^{1-q} \right) (\widetilde{\gamma_- g \mathscr{M}})^q.$$

Because

$$\lim_{A\to\infty} K_0(m_A^q) + K_1(m_A^q) - (K_2(m_A^q))^{1-q} = K_0(\mathscr{M}) > 0,$$

we may choose $A > 0$ large enough in such a way that $K_0(m_A^q) + K_1(m_A^q) - (K_2(m_A^q))^{1-q} \ge 0$. We may then proceed exactly as in the proof of Proposition 3.1, and we obtain

$$\int g^q \widetilde m^q \frac{(n_x \cdot \hat v)^2}{\delta^{1/2}} \varphi^q + \int |\nabla_v (g^{q/2} \widetilde m^{q/2})|^2 \varphi^q \le \frac{C_\Omega}{1-q} \int g^q m^q [|\partial_t \varphi^q| + \varphi^q \langle \wp_- \rangle].$$

As in Proposition 3.3 and with the help of the interpolation Lemma 3.2, we deduce

$$\int \frac{g^q}{\delta^\beta} \frac{\widetilde m^q}{\langle v\rangle^2} \varphi^q \le \frac{C_\Omega}{1-q} \int g^q m^q [|\partial_t \varphi^q| + \varphi^q \langle \wp_- \rangle], \tag{3.19}$$

for the same $\beta := (2(d+1))^{-1}$. Finally, arguing similarly as in the proof of Proposition 3.4 with the help of a last Holder inequality for handling the left-hand side term, we get

$$\int \frac{g^q}{\delta^\beta} \frac{m^q}{\langle v\rangle^2} \varphi^q \le C T^{1-q} \|\varphi^q\|_{W^{1,\infty}} \| g m \langle \wp_- \rangle^{1/q} \langle v \rangle^{(1-q)(d+1)/q} \|^q_{L^1(\mathcal{U})}, \tag{3.20}$$

for some constant $C = C(q, \Omega) > 0$.

Step 2. Weak weighted $L^1 - L^p$ estimate, $p > 1$. Consider again a solution g of the dual problem (2.13), $0 \le \varphi \in \mathcal{D}((0,T))$, $0 \le \chi \in \mathcal{D}(\Omega)$ and a weight function $m : \mathbb{R}^d \to (0,\infty)$ such that $m_0 := m\langle v\rangle^{-2}$ satisfies

$$\frac{|\nabla m_0|^2}{m_0^2} + \frac{|\Delta m_0|}{m_0} \lesssim \langle v \rangle^2.$$

We set $\bar g := g \varphi \chi m_0$ and we easily compute

$$-\partial_t \bar g - v \cdot \nabla_x \bar g - \Delta_v \bar g + (2 \frac{\nabla_v m_0}{m_0} - v) \cdot \nabla_v \bar g = G,$$

with

$$G := \bar g \Big[2 \frac{|\nabla_v m_0|^2}{m_0^2} - \frac{\Delta_v m_0}{m_0} + v \cdot \frac{\nabla_v m_0}{m_0} \Big] - g m_0 (\partial_t + v \cdot \nabla_x)(\varphi \chi).$$

Proceeding as in the proof of Proposition 3.5, we get first

$$\|\bar g\|_{L^p(\mathbb{R}^{2d+1})} \le C T^{\eta_2} \|\varphi\|_{W^{1,\infty}} \|\chi\|_{W^{1,\infty}} \|gm\|_{L^1(\mathcal{U})},$$

for any $p \in (1, 1 + 1/(2d))$ and with $\eta_2 := 1/p + 2d(1 - 1/p)$. By interpolation, we then conclude

$$\|g \varphi \frac{m}{\langle v\rangle^2} \delta^{\alpha/p} \|_{L^p(\mathcal{U})} \le C T^{\eta_2} \|\varphi\|_{W^{1,\infty}} \|gm\|_{L^1(\mathcal{U})}, \tag{3.21}$$

for any $\alpha > 1$ and some constant $C = C(\alpha, \Omega, m) > 0$.

Step 3. Weighted $L^1 - L^r$ estimate, $r > 1$. We consider a weight function $m \in \mathfrak{N}_1$ so that $m_1 := m/\langle v\rangle^4$ satisfies the condition (3.17) with $q \in (d/(d+1), 1)$ as well as

$$\frac{|\nabla m_1|^2}{m_1^2} + \frac{|\Delta m_1|}{m_1} \lesssim \langle v \rangle^2.$$

From Step 1 applied to m_1, we find

$$\Big\| \frac{g}{\delta^{\beta/q}} \frac{m}{\langle v\rangle^{4+2/q}} \varphi \Big\|_{L^q(\mathcal{U})} \le C T^{\eta_1} \|\varphi^q\|^{1/q}_{W^{1,\infty}} \|gm\|_{L^1(\mathcal{U})},$$

with $\eta_1 := 1/q - 1$. We observe that m satisfies the requirement of Step 2 because $m \in \mathfrak{N}_1$. We may thus use the above estimate together with (3.21) and the interpolation Lemma 3.6, in order to get

$$\|g\varphi\sigma\|_{L^r(\mathcal{U})} \le C T^{\eta} \|\varphi^q\|^{1/q}_{W^{1,\infty}} \|gm\|_{L^1(\mathcal{U})},$$

with

$$\frac{1}{r} = \frac{1-\theta}{q} + \frac{\theta}{p}, \quad \sigma := \frac{m}{\langle v\rangle^{2\theta+(4+2/q)(1-\theta)}},$$

and

$$\eta := (1-\theta)(1/q-1) + \theta(1/p + 2d(1-1/p)),$$

where we have fixed

$$p \in (1, 1+1/(2d)), \quad \alpha > 1, \quad \theta := \frac{\beta/q}{\beta/q + \alpha/p}.$$

For $q \in (d/(d+1), 1)$ large enough, we find $\sigma \gtrsim m/\langle v\rangle^8$ and $r > 1$. □

We finally deduce a slightly modified weighted $L^1 - L^r$ estimate which will be more convenient for our purpose in the last step that we present in the next section. We define

$$\mathfrak{N}_2 := \Big\{ m \in \mathfrak{N}_0;\ m^2 \in \mathfrak{N}_1,\ m^{3/2}\langle v\rangle^{-4} \in \mathfrak{N}_0 \Big\}. \tag{3.22}$$

Proposition 3.9. *There exist some exponent $r > 1$ and some constants $\eta > 0$, $q \in (0,1)$ such that any solution g to the dual backward problem (2.13) satisfies*

$$\|\varphi g \frac{m^{3/2}}{\langle v\rangle^4}\|_{L^r(\mathcal{U})} \lesssim T^{\eta} \|\varphi^q\|^{1/q}_{W^{1,\infty}(0,T)} \|gm\|_{L^1(\mathcal{U})}, \tag{3.23}$$

for any weight function $m \in \mathfrak{N}_2$ and any test function $0 \le \varphi \in \mathcal{D}((0,T))$, with $C = C(d,\Omega,m)$.

Proof of Proposition 3.9. From the interpolation Lemma 3.6 (which is nothing but the Cauchy-Schwarz inequality in that case), we have

$$\|\varphi g \frac{m^{3/2}}{\langle v\rangle^4}\|_{L^r(\mathcal{U})} \le \|\varphi g \frac{m^2}{\langle v\rangle^8}\|^{1/2}_{L^{r_1}(\mathcal{U})} \|\varphi g m\|^{1/2}_{L^1(\mathcal{U})},$$

with $r > 1$ defined by $1/r = 1/2 + 1/(2r_1)$ and $r_1 > 1$ defined in Proposition 3.8. We conclude with $\eta = \eta_1/2$ by using (3.13) with the weight function m^2 and by observing that $m^2 \lesssim m$ since m is a radially symmetric decreasing function. □

3.5. **Conclusion of the proof.** We now conclude the proof of Theorem 1.1 in several elementary and classical (after Nash's work) steps.

Proof of Theorem 1.1. We split the proff into four steps.

Step 1. Take $\omega_1 := \omega \in \mathfrak{W}_0 \cap \mathfrak{W}_3$ such that $\omega_r := \omega^\sharp = \omega^\theta \langle v\rangle^{-4} \in \mathfrak{W}_0$, where $r > 1$ and $\theta \in (0,1)$ are defined in the statement of Proposition 3.7. We first claim that there exists $\nu_1 > 0$ and $\kappa_1 \in \mathbb{R}$ such that

$$T^{\nu_1} \|S_{\mathcal{L}}(T) f_0\|_{L^r_{\omega_r}(\mathcal{O})} \lesssim e^{\kappa_1 T} \|f_0\|_{L^1_{\omega_1}(\mathcal{O})}, \quad \forall T > 0,\ \forall f_0 \in L^1_{\omega_1}(\mathcal{O}). \tag{3.24}$$

We set $f_t := S_{\mathcal{L}}(t) f_0$. On the one hand, from Proposition 2.1 with $p = r$, we have

$$\begin{aligned} \frac{T}{2} \|f_T\|^r_{L^r_{\omega_r}} &\lesssim \int_{T/2}^T e^{r\kappa(T-t)} \|f_t\|^r_{L^r_{\omega_r}} dt \\ &\lesssim e^{r\kappa T} \int_0^T \|f_t \varphi_0(t/T)\|^r_{L^r_{\omega_r}} dt, \end{aligned}$$

with $\varphi_0 \in C^1_c((0,2))$, $\mathbf{1}_{[1/2,1]} \le \varphi \le 1$, $\varphi^q \in W^{1,\infty}$ for any $q \in ((d+1)/(d+2), 1)$. On the other hand, thanks to Proposition 3.7 applied with $\varphi(t) := \varphi_0(t/T)$ and next to Proposition 2.1 with $p = 1$, we deduce

$$\begin{aligned} \frac{T}{2} \|f_T\|^r_{L^r_{\omega_r}} &\lesssim e^{r\kappa T} T^{r\eta} \big(1 + \frac{1}{T}\big)^{r/q} \Big(\int_0^T \|f_t\|_{L^1_{\omega_1}} dt \Big)^r \\ &\lesssim e^{r\kappa T} T^{r\eta} \big(1 + \frac{1}{T}\big)^{r/q} \Big(\int_0^T e^{\kappa t} dt \Big)^r \|f_0\|^r_{L^1_{\omega_1}}, \end{aligned}$$

from what (3.24) follows with $\nu_1 := 1/r - \eta - 1/q$ and any $\kappa_1 \ge 3\kappa$.

Step 2. Take $m_1 := m \in \mathfrak{N}_2$ so that $m_r := m^{3/2}\langle v\rangle^{-4} \in \mathfrak{N}_0$, where $r > 1$ is defined in the statement of Proposition 3.9. We now claim that there exist $\nu_2 > 0$ and $\kappa_2 \in \mathbb{R}$ such that

$$T^{\nu_2}\|S^*_{\mathcal{L}}(T)g_0\|_{L^r_{m_r}(\mathcal{O})} \lesssim e^{\kappa_2 T}\|g_0\|_{L^1_{m_1}(\mathcal{O})}, \quad \forall T>0,\ \forall g_0 \in L^1_{m_1}(\mathcal{O}). \tag{3.25}$$

We repeat the argument presented in Step 1. We set $g_t := S^*_{\mathcal{L}}(t)g_0$. On the one hand, from the dual counterpart of Proposition 2.1 with $p = r'$ and next from Proposition 3.9, we have

$$\begin{aligned}\frac{T}{2}\|g_T\|^r_{L^r_{m_r}} &\lesssim e^{r\kappa T}\int_0^T \|g_t\varphi_0(t/T)\|^r_{L^r_{m_r}}\,dt,\\ &\le e^{r\kappa T}T^{r\eta}\big(1+\frac{1}{T}\big)^{r/q}\Big(\int_0^T \|g_t\|_{L^1_{m_1}}\,dt\Big)^r,\end{aligned}$$

where φ_0 is the same function as above. We conclude to (3.25) thanks to the dual counterpart of Proposition 2.1 with $p = \infty$ (which is nothing but Lemma 2.4).

Step 3. Take ω a weight function such that $m_1 := \omega^{-1} \in \mathfrak{N}_2$. The dual counterpart of (3.25) writes

$$T^{\nu_2}\|S_{\mathcal{L}}(T)f_0\|_{L^\infty_{\omega_\infty}(\mathcal{O})} \lesssim e^{\kappa_2 T}\|f_0\|_{L^s_{\omega_s}(\mathcal{O})}, \quad \forall T>0,\ \forall f_0 \in L^s_{\omega_s}(\mathcal{O}), \tag{3.26}$$

with $\omega_\infty := m_1^{-1}$, $s = r' \in (1,\infty)$ and $\omega_s := m_r^{-1}$. Interpolating (3.24) and (3.26), for any $1 \le p < q \le \infty$, we obtain

$$\|S_{\mathcal{L}}(T)f_0\|_{L^q_{\omega_q}} \le C_1\frac{e^{C_2T}}{t^{\nu(1/p-1/q)}}\|f_0\|_{L^p_{\omega_p}}, \quad \forall T>0,\ \forall f_0 \in L^p_{\omega_p}(\mathcal{O}),$$

with $\nu := \max(\nu_1,\nu_2)(1-1/r)^{-1}$, $C_2 > \max(\kappa_1,\kappa_2)$, $C_2 > 0$, and the interpolated weight functions ω_p and ω_q.

Step 4. In this final step, we exhibit some weight function ω such that estimate (1.9) indeed holds, so that $\mathfrak{W}_1$ is not empty! We define

$$\omega := e^{\frac{4}{9}|v|^2}.$$

We clearly have $\omega \in \mathfrak{W}_0 \cap \mathfrak{W}_3$ and $\omega^\theta\langle v\rangle^{-4} \in \mathfrak{W}_0$ with $\theta \in (0,1)$ defined in the statement of Proposition 3.7 from the very definitions (2.5) and (3.8), so that (1.9) holds with $p = 1$, $\omega_1 = \omega$, $q = r$ and $\omega_r := \omega^{\theta_1}$ whatever is $\theta_1 \in (0,\theta)$.

We now define

$$\widetilde{\omega} := e^{\frac{1}{4}|v|^2+|v|}.$$

We also clearly have $m := \widetilde{\omega}^{-1} \in \mathfrak{N}_0$, $m^{3/2}\langle v\rangle^{-4} \in \mathfrak{N}_0$ from the very definition (2.17) and $m^2 \in \mathfrak{N}_1$ from the very definition (3.12), so that $\widetilde{\omega}^{-1} \in \mathfrak{N}_2$ from the very definition (3.22). We may thus write (3.25) and next (3.26) with $m = \widetilde{\omega}^{-1}$. Observing that the associated weight functions $\widetilde{\omega}_\infty$ and $\widetilde{\omega}_s$ satisfy $\omega^{9/16} \le \widetilde{\omega} = \widetilde{\omega}_\infty$ and $\widetilde{\omega}_s = \widetilde{\omega}^{3/2}\langle v\rangle^4 \le \omega$, we have established that (1.9) holds with $p = s$, $\omega_s = \omega$, $q = \infty$ and $\omega_\infty := \omega^{9/16}$. We conclude that (1.9) holds with the choice ω and $\theta := \min(\theta_1, 9/16)$ whatever is $1 \le p \le q \le \infty$ by the same interpolation argument as in Step 3. ☐

4. Hypocoercivity: Proof of Theorem 1.2

We adapt the proof of [7, Theorem 1.1]. We start introducing some notations and recalling some classical results about the Poisson equation. For any convenient function or distribution $\xi : \Omega \to \mathbb{R}$, we define $u := (-\Delta_x)^{-1}\xi : \Omega \to \mathbb{R}$ as the associated solution to the Poisson equation with Neumann condition. More precisely, for any $\eta_i \in L^2(\Omega)$, $\langle \eta_1\rangle = 0$, we define $u \in H$, with $H := \{u \in H^1(\Omega),\ \langle u\rangle = 0\}$, as the solution of the variational problem

$$\int_\Omega \nabla_x u\cdot\nabla_x w = \int_\Omega \{w\eta_1 - \nabla_x w\cdot\eta_2\}, \quad \forall w \in H, \tag{4.1}$$

which is indeed a variational solution to the Poisson equation with Neumann condition

$$-\Delta_x u = \eta_1 + \operatorname{div}_x \eta_2 \ \text{ in } \ \Omega, \quad n_x\cdot(\nabla_x u - \eta_2) = 0 \ \text{ on } \ \partial\Omega. \tag{4.2}$$

It is well-known that the above variational problem has a unique solution thanks to the Poincaré-Wirtinger inequality and the Lax-Milgram Theorem, that

$$\|u\|_{H^1(\Omega)} \lesssim \sum_{i=1}^2 \|\eta_i\|_{L^2(\Omega)}, \tag{4.3}$$

holds true and that the additional regularity estimates

$$\|u\|_{H^1(\partial\Omega)} \lesssim \|u\|_{H^2(\Omega)} \lesssim \|\eta_1\|_{L^2(\Omega)} \tag{4.4}$$

holds when $\eta_2 = 0$. We define

$$\mathcal{H} := L^2(\mu^{-1} dv dx), \quad \mathcal{H}_0 := \{f \in \mathcal{H};\ \langle\!\langle f \rangle\!\rangle = 0\},$$

where μ is defined in (1.7) and $\langle\!\langle \cdot \rangle\!\rangle$ in (1.8). We next define the new (twisted) scalar product $(\!(\cdot,\cdot)\!)$ on $\mathcal{H}_0$ by

$$(\!(f,g)\!) := (f,g)_{\mathcal{H}} + \varepsilon(\nabla_x(-\Delta_x)^{-1}\varrho_f, j_g)_{L^2} + \varepsilon(\nabla_x(-\Delta_x)^{-1}\varrho_g, j_f)_{L^2},$$

with $\varepsilon > 0$ small enough to be fixed later, $L^2 := L^2_x(\Omega)$ and where the mass ϱ_f and the momentum j_f are defined respectively by

$$\varrho_h(x) = \varrho[h](x) := \langle h \rangle, \quad j_h(x) = j[h](x) := \langle hv \rangle, \quad \langle H \rangle := \int_{\mathbb{R}^d} H(x,v)\, dv.$$

For any $f \in \mathcal{H}_0$, we next decompose

$$f = \pi f + f^\perp, \tag{4.5}$$

with the macroscopic part πf given by

$$\pi f(x,v) = \varrho_f(x)\mu(v),$$

and we remark that

$$\|f\|^2_{\mathcal{H}} = \|f^\perp\|^2_{\mathcal{H}} + \|\pi f\|^2_{\mathcal{H}}, \quad \|\pi f\|^2_{\mathcal{H}} = \|\varrho_f\|^2_{L^2}. \tag{4.6}$$

as well as

$$\|\varrho_f\|_{L^2} \le \|f\|_{\mathcal{H}}, \quad \|j_f\|_{L^2} \lesssim \|f^\perp\|_{\mathcal{H}} \le \|f\|_{\mathcal{H}}. \tag{4.7}$$

It is worth emphasizing that

$$\begin{aligned} |(\nabla_x(-\Delta_x)^{-1}\varrho_f, j_f)_{L^2}| &\le \|\nabla_x(-\Delta_x)^{-1}\varrho_f\|_{L^2}\|j_f\|_{L^2} \\ &\lesssim \|\varrho_f\|_{L^2}\|f^\perp\|_{\mathcal{H}} \lesssim \|f\|^2_{\mathcal{H}}, \end{aligned}$$

from the Cauchy-Schwarz inequality, (4.4) and (4.7). Denoting by $|\!|\!| \cdot |\!|\!|$ the norm associated to the scalar product $(\!(\cdot,\cdot)\!)$, we in particular deduce that

$$\|f\|_{\mathcal{H}} \lesssim |\!|\!| f |\!|\!| \lesssim \|f\|_{\mathcal{H}}, \quad \forall f \in \mathcal{H}_0. \tag{4.8}$$

We finally define the Dirichlet form associated to the operator $\mathscr{L}$ defined in (1.10) for the twisted scalar product

$$D[f] := (\!(-\mathscr{L} f, f)\!), \quad f \in \mathcal{H}_0.$$

More explicitely, we have

$$D[f] = D_1[f] + D_2[f] + D_3[f],$$

with

$$D_1[f] := (-\mathscr{L} f, f)_{\mathcal{H}}, \quad D_2[f] := \varepsilon(\nabla_x \Delta_x^{-1}\varrho_f, j[\mathscr{L} f])_{L^2}, \quad D_3[f] := \varepsilon(\nabla_x\Delta_x^{-1}\varrho[\mathscr{L} f], j_f)_{L^2},$$

and we estimate each term separately. For simplicity we introduce the notations $\mathscr{D}^\perp := \mathrm{Id} - \mathscr{D}$, where we recall that $\mathscr{D}$ is given by (1.5) and $\partial\mathcal{H}_+ := L^2(\Sigma_+; \mu^{-1}(v) n_x \cdot v dv d\sigma_x)$. It is worth emphasizing that because $f \in \mathrm{Dom}(\mathscr{L})$, the trace functions $\gamma_\pm f$ are well defined. We refer the interested reader to [38, 21, 12] and the references therein for a suitable definition of the trace function for solutions to the KFP equation.

We estimate the first term involved in the Dirichlet form D.

Lemma 4.1. *For any $f \in \mathcal{H}$, there holds*

$$(-\mathscr{L} f, f)_{\mathcal{H}} \ge \|f^\perp\|^2_{\mathcal{H}} + \frac{1}{2}\|\sqrt{\iota(2-\iota)}\mathscr{D}^\perp\gamma_+ f\|^2_{\partial\mathcal{H}_+}.$$

Proof of Lemma 4.1. Recalling (1.10) and (2.1), we write

$$(-\mathscr{L} f, f)_{\mathcal{H}} = (-\mathscr{C} f, f)_{\mathcal{H}} + (v \cdot \nabla_x f, f)_{\mathcal{H}}.$$

On the one hand, we recall the classical Poincaré inequality

$$\|h - \langle h\mu \rangle\|_{L^2(\mu)} \le \|\nabla_v h\|_{L^2(\mu)}, \quad \forall h \in L^2(\mu dv dx),$$

from what we classically deduce

$$\begin{aligned}(-\mathscr{C}f,f)_{\mathcal{H}} &= -\int_{\mathcal{O}} \mathrm{div}_v(\mu\nabla_v(f/\mu))f/\mu\,dvdx \\ &= \int_{\mathcal{O}} |\nabla_v(f/\mu)|^2\mu\,dv\,dx \\ &\geq \int_{\mathcal{O}} |f/\mu-\langle f\rangle|^2\mu\,dv\,dx = \|f^\perp\|^2_{\mathcal{H}}.\end{aligned}$$

The second part of the estimate has been proved during the proof of [7, Lemma 3.1]. □

We recall the identity established in [7, Lemma 3.2].

Lemma 4.2. *Let $\phi:\mathbb{R}^d\to\mathbb{R}$. For any $x\in\partial\Omega$, there holds*

$$\begin{aligned}\int_{\mathbb{R}^d}\phi(v)\gamma f(x,v)\,n_x\cdot v\,dv = &\int_{\Sigma^x_+}\phi(v)\iota(x)\mathscr{D}^\perp\gamma_+ f\,n_x\cdot v\,dv \\ &+\int_{\Sigma^x_+}\{\phi(v)-\phi(\mathcal{V}_x v)\}\,(1-\iota(x))\mathscr{D}^\perp\gamma_+ f\,n_x\cdot v\,dv \\ &+\int_{\Sigma^x_+}\{\phi(v)-\phi(\mathcal{V}_x v)\}\,\mathscr{D}\gamma_+ f\,n_x\cdot v\,dv.\end{aligned}$$

We estimate the second term involved in the Dirichlet form D.

Lemma 4.3. *There is a constant $C_2>0$, such that*

$$(\nabla_x\Delta_x^{-1}\varrho_f, j[\mathscr{L}f])_{L^2}\geq \frac{1}{2}\|\varrho_f\|^2_{L^2}-C_2\|f^\perp\|^2_{\mathcal{H}}-C_2\|\iota\mathscr{D}^\perp\gamma_+ f\|^2_{\partial\mathcal{H}_+},\quad \forall\, f\in\mathcal{H}.$$

Proof of Lemma 4.3. We repeat the proof of [7, Lemma 3.8]. Writing

$$j[\mathscr{L}f]=j[-v\cdot\nabla_x f]-j[f^\perp]$$

where we have observed that $\mathscr{C}\pi f=0$ and $j[\mathscr{C}g]=j[g]$, and denoting $u:=(-\Delta_x)^{-1}\varrho_f$, we have

$$(-\nabla_x u, j[\mathscr{L}f])_{L^2}=\big(\partial_{x_i}u,\partial_{x_j}\int_{\mathbb{R}^d}v_iv_j f\,dv\big)_{L^2}+\big(\nabla_x u, j[f^\perp]\big)_{L^2}.$$

On the one hand, using the Green formula, we may write

$$\big(\partial_{x_i}u,\partial_{x_j}\int_{\mathbb{R}^d}v_iv_j f\,dv\big)_{L^2}=A+B,$$

with

$$A:=-\big(\partial_{x_j}\partial_{x_i}u,\int_{\mathbb{R}^d}v_iv_j f\,dv\big)_{L^2},\quad B:=\int_{\partial\Omega}\partial_{x_i}u\,n_j(x)\left(\int_{\mathbb{R}^d}v_iv_j\,\gamma f\,dv\right)d\sigma_x.$$

Thanks to the decomposition (4.5), we get

$$\int_{\mathbb{R}^d}v_iv_j f\,dv=\delta_{ij}\varrho_f+\int_{\mathbb{R}^d}v_iv_j f^\perp\,dv,$$

and hence

$$\begin{aligned}A&=(-\Delta_x u,\varrho_f)_{L^2}-(\partial_{x_j}\partial_{x_i}u,\int_{\mathbb{R}^d}v_iv_j f^\perp\,dv)_{L^2}\\ &=\|\varrho\|^2_{L^2}-(\partial_{x_j}\partial_{x_i}u,\int_{\mathbb{R}^d}v_iv_j f^\perp\,dv)_{L^2},\end{aligned}$$

since $-\Delta_x u=\varrho$ by definition of u. Using (4.4), we have

$$\begin{aligned}\left|(\partial_{x_j}\partial_{x_i}u,\int_{\mathbb{R}^d}v_iv_j f^\perp\,dv)_{L^2}\right| &\lesssim \|D^2_x u\|_{L^2}\|f^\perp\|_{\mathcal{H}}\\ &\lesssim \|\varrho_f\|_{L^2}\|f^\perp\|_{\mathcal{H}},\end{aligned}$$

from what it follows, thanks to Young's inequality,

$$A\geq\frac{3}{4}\|\varrho_f\|^2_{L^2}-C\|f^\perp\|^2_{\mathcal{H}}.$$

We now investigate the boundary term B. Thanks to Lemma 4.2, we have

$$\begin{aligned} B &= \int_{\Sigma} \nabla_x u \cdot v\, \gamma f\, n_x \cdot v \, dv \, d\sigma_x \\ &= \int_{\Sigma_+} \nabla_x u \cdot v \iota(x) \mathscr{D}^\perp \gamma_+ f\, n_x \cdot v \, dv \, d\sigma_x \\ &\quad + \int_{\Sigma_+} \nabla_x u \cdot [v - \mathcal{V}_x v](1-\iota(x)) \mathscr{D}^\perp \gamma_+ f\, n_x \cdot v \, dv \, d\sigma_x \\ &\quad + \int_{\Sigma_+} \nabla_x u \cdot [v - \mathcal{V}_x v] D \gamma_+ f\, n_x \cdot v \, dv \, d\sigma_x \\ &=: B_1 + B_2 + B_3, \end{aligned}$$

and we remark that

$$v - \mathcal{V}_x v = 2 n_x (n_x \cdot v),$$

so that

$$\nabla_x u \cdot [v - \mathcal{V}_x v] = 2 \nabla_x u \cdot n_x \, (n_x \cdot v).$$

We therefore obtain $B_2 = B_3 = 0$ thanks to the boundary condition satisfied by u in (4.2). On the other hand, the Cauchy-Schwarz inequality and (4.4) yield

$$\begin{aligned} |B_1| &\le \|\nabla_x u\|_{L^2_x(\partial\Omega)} \|v\mu\|_{L^1}^{1/2} \|\iota \mathscr{D}^\perp \gamma_+ f\|_{\partial\mathcal{H}_+} \\ &\lesssim \|\varrho_f\|_{L^2} \|\iota \mathscr{D}^\perp \gamma_+ f\|_{\partial\mathcal{H}_+}. \end{aligned}$$

Similarly as for the term A, we last have

$$\left| \left(\nabla_x u, j[f^\perp] \right)_{L^2} \right| \le \|\nabla_x u\|_{L^2} \|j[f^\perp]\|_{L^2} \lesssim \|\varrho_f\|_{L^2} \|f^\perp\|_{\mathcal{H}},$$

where we have used the estimate (4.4) and twice the Cauchy-Schwarz inequality. The proof is then complete by gathering all the previous estimates and by using Young's inequality. □

We finally estimate the third term involved in the Dirichlet form D.

Lemma 4.4. *There is a constant $C_3 > 0$ such that*

$$(\nabla_x \Delta_x^{-1} \varrho[\mathscr{L} f], j_f)_{L^2} \ge -C_3 \|f^\perp\|^2_{\mathcal{H}}$$

Proof of Lemma 4.4. From (1.10), (2.1) and $\varrho[\mathscr{C} f] = 0$, one has

$$\varrho[\mathscr{L} f] = \varrho[-v \cdot \nabla_x f] = -\operatorname{div}_x \int_{\mathbb{R}^d} v f \, dv = -\operatorname{div}_x j_f.$$

On the other hand, we also classically observe

$$\begin{aligned} j_f \cdot n_x &= \int_{\mathbb{R}^d} \gamma f \, v \cdot n_x dv \\ &= \iota \Big\{ \int_{\Sigma^x_+} \gamma_+ f \, v \cdot n_x dv - \int_{\Sigma^x_-} \mathscr{M}(v) \widetilde{\gamma_+ f} \, |v \cdot n_x| dv \Big\} \\ &\quad + (1-\iota) \Big\{ \int_{\Sigma^x_+} \gamma_+ f \, v \cdot n_x dv - \int_{\Sigma^x_-} \gamma_+ f \circ \mathcal{V}_x \, |v \cdot n_x| dv \Big\}, \end{aligned}$$

and using the very definition of $\widetilde{\gamma_+ f}$ and $\mathscr{M}$ in (1.5) and (1.6) in the second integral and the change of variables $v \mapsto \mathcal{V}_x v$ in the last integral, we see that both contributions vanish and we thus obtain the zero flux condition

$$j_f \cdot n_x = 0. \tag{4.9}$$

Now let us define

$$u := (-\Delta_x)^{-1} \varrho[\mathscr{L} f] = (-\Delta_x)^{-1} (-\operatorname{div}_x j_f)$$

the unique variational solution to (4.2) with Neumann boundary condition associated to the source term $\xi = \varrho[\mathscr{L} f] = \operatorname{div} \eta_2$, $\eta_2 := -j_f$. From the variational formulation (4.1), we have

$$\begin{aligned} \|\nabla_x u\|^2_{L^2} &= -\int_{\Omega} (\nabla_x \cdot j_f) \, u \, dx \\ &= \int_{\Omega} j_f \cdot \nabla_x u \, dx - \int_{\partial\Omega} j_f \cdot n_x \, u \, d\sigma_x = \int_{\Omega} j_f \cdot \nabla_x u \, dx, \end{aligned}$$

where we have used the Green formula and finally (4.9) in order to obtain the last equality. We deduce

$$\|\nabla_x u\|_{L^2} \lesssim \|j_f\|_{L^2}.$$

thanks to the Cauchy-Schwarz inequality, and thus

$$|(-\nabla_x u, j_f)_{L^2}| \lesssim \|\nabla_x u\|_{L^2}\|j_f\|_{L^2} \lesssim \|j_f\|^2_{L^2}.$$

We conclude thanks to (4.7). □

We are now able to conclude the proof of Theorem 1.2.

Proof of Theorem 1.2. Let f satisfy the assumptions of Theorem 1.2. Observing that $\sqrt{\iota(2-\iota)} \geq \iota$ since ι takes values in $[0,1]$, and gathering Lemmas 4.1, 4.3 and 4.4, one has

$$\begin{aligned}((-\mathscr{L}f, f)) \geq \|f^\perp\|^2_{\mathcal{H}} + \frac{1}{2}\|\sqrt{\iota(2-\iota)}\mathscr{D}^\perp\gamma_+ f\|^2_{\partial\mathcal{H}_+} \\ + \varepsilon\Big(\frac{1}{2}\|\varrho_f\|^2_{L^2} - (C_2+C_3)\|f^\perp\|^2_{\mathcal{H}} - C_2\|\sqrt{\iota(2-\iota)}\mathscr{D}^\perp\gamma_+ f\|^2_{\partial\mathcal{H}_+}\Big).\end{aligned}$$

Choosing $0<\varepsilon<1$ small enough, we get

$$((-\mathscr{L}f, f)) \geq \kappa\left(\|f^\perp\|^2_{\mathcal{H}} + \|\varrho_f\|^2_{L^2}\right) + \kappa'\|\sqrt{\iota(2-\iota)}\mathscr{D}^\perp\gamma_+ f\|^2_{\partial\mathcal{H}_+}$$

for some constants $\kappa, \kappa' > 0$. We thus obtain (1.11) by using the identity (4.6) and the equivalence (4.8) of the norms $\|\cdot\|_{\mathcal{H}}$ and $|||\cdot|||$. □

5. Asymptotic behavior: Proof of Theorem 1.3

We repeat the proof of [24, Theorem 3.1] and [39, Theorem 1.4], so that we just sketch the arguments.

Proof of Theorem 1.3. We introduce the splitting

$$\mathcal{A}f := M\chi_R(v)f, \quad \mathcal{B} := \mathscr{L} - \mathcal{A},$$

with $\chi_R(v) := \chi(v/R)$, $\chi \in \mathcal{D}(\mathbb{R}^d)$, $\mathbf{1}_{B_1} \leq \chi \leq \mathbf{1}_{B_2}$, and some constants $M, R > 0$ to be fixed below. We denote by $S_\mathcal{B}$ the semigroup associated to the modified KFP equation associated to the partial differential operator $\mathcal{B}$ and the same reflection condition (1.2). We define

$$\mathfrak{W}_2 := \left\{\omega \in \mathfrak{W}_0\,;\ \sup_{p\in[1,\infty]}\ \limsup_{|v|\to\infty} \varpi_{\omega,p} =: \kappa^* < -1\right\}, \tag{5.1}$$

where we recall that $\varpi_{\omega,p}$ is defined in (2.3). In particular, $\omega := \langle v\rangle^k e^{\zeta|v|^s} \in \mathfrak{W}_2$ if $s=2$ and $\zeta \in (0,1/2)$, or if $s \in [0,2)$, or if $s=0$ and $k > d+1$. By repeating the proof of Proposition 2.1, for any $\kappa > \kappa^*$, we may find $M, R > 0$ large enough such that for any $\omega \in \mathfrak{W}_2$, we have

$$\sup_{p\in[1,\infty]}\ \sup_{v\in\mathbb{R}^d} (\varpi_{\omega,p}(v) - M\chi_R(v)) \leq (\kappa^*+\kappa)/2,$$

and thus there exists a constant $C = C(\omega) > 0$ such that

$$\|S_\mathcal{B}(t)f_0\|_{L^p_\omega} \leq Ce^{\kappa t}\|f_0\|_{L^p_\omega}, \quad \forall t \geq 0, \tag{5.2}$$

for any $f_0 \in L^p_\omega$, $1 \leq p \leq \infty$. By repeating the proof of Theorem 1.1, we also have

$$\|S_\mathcal{B}(t)f_0\|_{L^\infty_{\omega^\theta}} \leq C\frac{e^{\kappa t}}{t^\nu}\|f_0\|_{L^p_\omega}, \quad \forall t > 0. \tag{5.3}$$

Recalling the definition of total mass $\langle\langle\cdot\rangle\rangle$ in (1.8), we define

$$\Pi g := g - \langle\langle g\rangle\rangle\mu$$

and

$$\bar{S}_\mathscr{L} := \Pi S_\mathscr{L} = S_\mathscr{L}\Pi = \Pi S_\mathscr{L}\Pi.$$

Iterating the Duhamel formulas

$$\begin{aligned} S_\mathscr{L} &= S_\mathcal{B} + S_\mathcal{B}\mathcal{A} * S_\mathscr{L} \\ S_\mathscr{L} &= S_\mathcal{B} + S_\mathscr{L} * \mathcal{A}S_\mathcal{B}, \end{aligned}$$

where $*$ stands the time convolution between operator defined on $\mathbb{R}$ with support on $\mathbb{R}_+$, we deduce that

$$\bar{S}_\mathscr{L} = V_1\Pi + W_1 * \bar{S}_\mathcal{L}, \tag{5.4}$$

and

$$\bar{S}_{\mathscr{L}} = \Pi V_1 + \bar{S}_{\mathscr{L}} * W_2, \tag{5.5}$$

with

$$V_1 := \sum_{j=0}^{n-1} (S_{\mathcal{B}}\mathcal{A})^{*j} * S_{\mathcal{B}}, \quad W_1 := (S_{\mathcal{B}}\mathcal{A})^{*n}, \quad W_2 := (\mathcal{A}S_{\mathcal{B}})^{*n},$$

where we use the shorthand $U^{*0} := Id$, $U^{*(j+1)} := U * U^{*j}$. Both estimates (5.4) and (5.5) together, we obtain

$$\bar{S}_{\mathscr{L}} = V_2 + W_1 * \bar{S}_{\mathscr{L}} * W_2, \tag{5.6}$$

with

$$V_2 := V_1 \Pi + W_1 * \Pi V_1.$$

For any $\kappa > \kappa^*$ and $n \in \mathbb{N}$, we deduce from (5.2) that

$$\|V_2(t) f_0\|_{L^p_\omega} \le C e^{\kappa t} \|f_0\|_{L^p_\omega}, \quad \forall t \ge 0, \tag{5.7}$$

For any $\kappa > \kappa^*$, we deduce from (5.2) and (5.3) (see [24, 39] as well as [40, Proposition 2.5]) that we may find $n \in \mathbb{N}^*$ such that

$$\|W_1(t) f_0\|_{L^p_\omega} \le C e^{\kappa t} \|f_0\|_{L^2(\mu)}, \quad \forall t \ge 0, \tag{5.8}$$

$$\|W_2(t) f_0\|_{L^2_{\mu^{-1/2}}} \le C e^{\kappa t} \|f_0\|_{L^p_\omega}, \quad \forall t \ge 0. \tag{5.9}$$

We also recall that from Theorem 1.2, we have

$$\|\bar{S}_{\mathscr{L}} f_0\|_{L^2_{\mu^{-1/2}}} \le C e^{-\lambda t} \|f_0\|_{L^2_{\mu^{-1/2}}}, \quad \forall t \ge 0. \tag{5.10}$$

We conclude to (1.12) by just writing the representation formula (5.6) and using the estimates (5.7), (5.8), (5.9) and (5.10). □

Acknowledgments. The authors warmly thank Clément Mouhot for his PhD course *De Giorgi-Nash-Moser theory for kinetic equations* teached at Université Paris Dauphine-PSL during Spring 2023, which has been a source of inspiration for the present work and also the enlightening discussions in many occasions about the same subject. K.C. was partially supported by the Project CONVIVIALITY ANR-23-CE40-0003 of the French National Research Agency.

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