ERRATUM TO "CAUCHY PROBLEM AND EXPONENTIAL STABILITY FOR THE INHOMOGENEOUS LANDAU EQUATION"

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We correct a mistake in "Cauchy problem and exponential stability for the inhomogeneous Landau equation", *Arch. Rational Mech. Anal. 221*, 1 (2016), 363–418.

In the study of the linearized equation in Section 2, estimate (2.27) in Lemma 2.8 is not correct, and this error is then straithgforwardly propagated to (2.5) in Theorem 2.3 and to the second estimate in Corollary 3.1. This last estimate is then used to treat the nonlinear equation in the proof of Proposition 3.7.

In this erratum we first show another (weaker) regularity estimate in the place of (2.27), which is then propagated to Theorem 2.3 and Corollary 3.1. Finally we show how the last estimate is used in the proof of Proposition 3.7.

The results of the original paper remain unchanged, and the new regularity estimate we shall prove here is a direct consequence of the techniques already presented in the paper. The only modification to perform is in the condition (H0)-(i) that need to be changed into $k > 3\gamma/2 + 7 + 3/2$.

Remark. Estimates (2.20) and (2.21) are not correct either, and they could also be replaced. However we do not deal with them here since they are not used to treat the nonlinear equation.

Using the notations of Lemma 2.13 and following its proof, we claim that, for any $n \in \mathbb{N}$, the following regulary in velocity estimate holds:

$$\forall\, t \in (0,1], \quad \|\mathcal{S}_{\mathcal{B}}(t)\|_{\mathscr{B}(H^n_x(H^{-1}_{v,*}(\langle v\rangle^{(\gamma+2)/2} m_1)), H^n_x L^2_v(m_0))} \le \frac{C}{\sqrt{t}}, \tag{1}$$

which together with (2.19) in Lemma 2.7 immediately imply that, for any $\lambda < \lambda_{m_0,2}$,

$$\forall\, t > 0, \quad \|\mathcal{S}_{\mathcal{B}}(t)\|_{\mathscr{B}(H^n_x(H^{-1}_{v,*}(\langle v\rangle^{(\gamma+2)/2} m_1)), H^n_x L^2_v(m_0))} \le \frac{C\, e^{-\lambda t}}{\min(\sqrt{t},1)}, \tag{2}$$

and this shall replace (2.27).

Let us now prove (1) for $n = 0$, the general case being the same since ∇_x commutes with $\mathcal{B}$. In Step 1 of Lemma 2.13, we easily observe that we also have the following lower bound for the functional $\mathcal{F}(t,f)$:

$$\forall t \in [0,1], \quad 2\mathcal{F}(t,f) \ge \|f\|^2_{L^2(m_1)} + \alpha_1 t \|\nabla_v f\|^2_{L^2(m_0)} \ge \alpha_1 t \|f\|^2_{L^2_x(H^1_v(m_0))},$$

from which we obtain (by following the proof)

$$\forall t \in (0,1], \quad \|\mathcal{S}_{\mathcal{B}}(t) f\|_{L^2_x(H^1_v(m_0))} \le C\, t^{-1/2}\, \|f\|_{L^2_x L^2_v(m_1)}.$$

In an analogous way, we also get a similiar estimate for the adjoint operator $\mathcal{B}^*_m$,

$$\forall t \in (0,1], \quad \|\mathcal{S}_{\mathcal{B}^*_{m_1}}(t)\phi\|_{L^2_x(H^1_v)} \le C\, t^{-1/2}\, \|\phi\|_{L^2_x L^2_v(\frac{m_1}{m_0})},$$

which implies (1) arguing by duality and using the fact that $H^{-1}_{v,*}(\langle v\rangle^{(\gamma+2)/2} m) \subset H^{-1}_v(m)$.

Following the proof of Theorem 2.3, we then get the new regularity estimate (2) for the semigroup $\mathcal{S}_\Lambda$, more precisely:

- in Theorem 2.3 estimate (2.5) is changed to

$$\forall\, n \in \mathbb{N},\, \forall\, t > 0, \quad \|\mathcal{S}_\Lambda(t)(I - \Pi_0)\|_{\mathscr{B}(H^n_x(H^{-1}_{v,*}(\widetilde{m})), H^n_x L^2_v(m))} \leq \frac{C\, e^{-\lambda_1 t}}{\min(\sqrt{t}, 1)}; \tag{3}$$

- in Corollary 3.1 the second estimate is changed to

$$\forall\, t > 0, \quad \|\mathcal{S}_\Lambda(t)(I - \Pi_0)\|_{\mathscr{B}(\mathcal{H}^3_x(H^{-1}_{v,*}(\widetilde{m})), \mathcal{H}^3_x L^2_v(m))} \leq \frac{C\, e^{-\lambda_1 t}}{\min(\sqrt{t}, 1)}; \tag{4}$$

where $\widetilde{m} = \langle v \rangle^{(\gamma+2)/2} m$ if $\gamma \in [0,1]$ and $\widetilde{m} = \langle v \rangle m$ if $\gamma \in [-2, 0)$.

We now conclude by showing how (4) is used in the proof of Proposition 3.7. The only term we need to treat is

$$I_4 = \int_0^\infty \langle \mathcal{S}_\Lambda(\tau) e^{\lambda_2 \tau} f, \mathcal{S}_\Lambda(\tau) e^{\lambda_2 \tau} Q(f,f) \rangle_{X_0} \, d\tau.$$

Using the first estimate of Corollary 3.1 together with (4), we obtain, denoting $\widetilde{Y}_0' = \mathcal{H}^3_x(H^{-1}_{v,*}(\widetilde{m}_0))$ with $\widetilde{m}_0$ given in (4),

$$\begin{aligned}
&\int_0^\infty \langle \mathcal{S}_\Lambda(\tau) e^{\lambda_2 \tau} f, \mathcal{S}_\Lambda(\tau) e^{\lambda_2 \tau} Q(f,f) \rangle_{X_0} \, d\tau \\
&\quad\leq \int_0^\infty \|\mathcal{S}_\Lambda(\tau) e^{\lambda_2 \tau} f\|_{X_0} \, \|\mathcal{S}_\Lambda(\tau) e^{\lambda_2 \tau} Q(f,f)\|_{X_0} \, d\tau \\
&\quad\leq C \|f\|_{X_0} \, \|Q(f,f)\|_{\widetilde{Y}_0'} \int_0^\infty e^{-(\lambda_1 - \lambda_2)\tau} \, \frac{e^{-(\lambda_1 - \lambda_2)\tau}}{\min(\sqrt{\tau}, 1)} \, d\tau \\
&\quad\leq C \|f\|_{X_0} \, \|Q(f,f)\|_{\widetilde{Y}_0'},
\end{aligned}$$

and then the proof of Proposition 3.7 can be completed.

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