# On the trend to equilibrium for the homogeneous Landau equation with hard potentials

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**Resumen**

En esta nota expositiva se presentan algunos resultados recientes obtenidos en [2] al respecto a la convergencia al equilibrio de soluciones de la ecuación de Landau espacialmente homogénea con potenciales duros.

**Abstract**

In this expository note we present some recents results obtained in [2] concerning the trend to equilibrium for solutions to the spatially homogeneous Landau equation with hard potentials.

## 1. Introduction

This expository note presents recent results obtained in [2] concerning the trend to equilibrium for solutions to the spatially homogeneous Landau equation with hard potentials. It is well know that these solutions converge towards the Maxwellian equilibrium (Gaussian measure) when time goes to infinity and we are interested in quantitative rates of convergence.

Let us present the problem in a precise manner before going further on known results and on the main contribution of [2]. In kinetic theory, the Landau equation is a model in plasma physics that describes the evolution of the density in the phase space of all positions and velocities of particles. Assuming that the density function does not depend on the position, we obtain the spatially homogeneous Landau equation in the form

$$\left\{ \begin{array}{rcl} \partial_t f & = & Q(f,f) \\ f_{|t=0} & = & f_0 \end{array} \right. \tag{1}$$

where $f = f(t,v) \geq 0$ is the density of particles with velocity $v$ at time $t$, $v \in \mathbb{R}^3$ and $t \in \mathbb{R}^+$. The Landau operator $Q$ is a bilinear operator given by

$$Q(g,f) = \partial_i \int_{\mathbb{R}^3} a_{ij}(v - v_*) \left[ g_* \partial_j f - f \partial_{*j} g_* \right] \mathrm{d}v_*, \tag{2}$$

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where here and below we shall use the convention of implicit summation over repeated indices and we use the shorthand $g_* = g(v_*)$, $\partial_{*j} g_* = \partial_{v_{*j}} g(v_*)$, $f = f(v)$ and $\partial_j f = \partial_{v_j} f(v)$.

The matrix $a$ is nonnegative, symmetric and depends on the interaction between particles. If two particles interact with a potential proportional to $1/r^s$, where $r$ denotes their distance, $a$ is given by (see for instance [13])

$$a_{ij}(v) = |v|^{\gamma+2}\left(\delta_{ij} - \frac{v_i v_j}{|v|^2}\right), \tag{3}$$

with $\gamma = (s-4)/s$. We call hard potentials if $\gamma \in (0,1]$, Maxwellian molecules if $\gamma = 0$, soft potentials if $\gamma \in (-3,0)$ and Coulombian potential if $\gamma = -3$. Our results concerns the case of hard potentials $\gamma \in (0,1]$.

It is wellknow that, at least formally, the Landau equation conserves mass, momentum and energy (see e.g. [12]), more precisely

$$\int Q(f,f)\varphi(v) = 0 \qquad \text{for} \qquad \varphi(v) = 1, v, |v|^2. \tag{4}$$

Moreover, the entropy defined by $H(f) := \int f \log f$ is nonincreasing, indeed, at least formally, since $a_{ij}$ is nonnegative we have the following inequality for $D(F)$ the entropy dissipation,

$$\begin{aligned} D(f) &:= -\frac{d}{dt} H(f) \\ &= \frac{1}{2}\int_{\mathbb{R}^3\times\mathbb{R}^3} f f_* \, a_{ij}(v-v_*)\left(\frac{\partial_i f}{f} - \frac{\partial_{i*} f_*}{f_*}\right)\left(\frac{\partial_j f}{f} - \frac{\partial_{j*} f_*}{f_*}\right) \mathrm{d}v_* \, \mathrm{d}v \geq 0. \end{aligned} \tag{5}$$

It follows that any equilibrium is a Maxwellian distribution

$$\mu_{\rho,u,T}(v) := \frac{\rho}{(2\pi T)^{3/2}} e^{-\frac{|v-u|^2}{2T}},$$

for some $\rho > 0$, $u \in \mathbb{R}^3$ and $T > 0$. This is the Landau version of the famous Boltzmann's $H$-theorem (for more details we refer to [5, 11]), from which the solution $f(t,\cdot)$ of the Landau equation is expected to converge towards the Maxwellian $\mu_{\rho_f,u_f,T_f}$ when $t \to +\infty$, where $\rho_f$ is the density of the gas, $u_f$ the mean velocity and $T_f$ the temperature, defined by

$$\rho_f = \int f(v), \quad u_f = \frac{1}{\rho}\int v f(v), \quad T_f = \frac{1}{3\rho}\int |v-u|^2 f(v),$$

and these quantities are defined by the initial datum $f_0$ thanks to the conservation properties (4).

We may only consider the case of initial datum $f_0$ satisfying

$$\int_{\mathbb{R}^3} f_0(v)\, dv = 1, \quad \int_{\mathbb{R}^3} v f_0(v)\, dv = 0, \quad \int_{\mathbb{R}^3} |v|^2 f_0(v)\, dv = 3, \tag{6}$$

the general case being reduced to (6) by a simple change of coordinates (see [5]). Then, we shall denote $\mu(v) = (2\pi)^{-3/2} e^{-|v|^2/2}$ the standard Gaussian distribution in $\mathbb{R}^3$, which corresponds to the Maxwellian with $\rho = 1$, $u = 0$ and $T = 1$, i.e. the Maxwellian with same mass, momentum and energy of $F_0$ (6).

## 1.1. Known results

The Landau equation (1) with hard potentials was studied in great details by Desvillettes and Villani [4, 5]. In particular, concerning the trend to equilibrium problem, they proved a polynomial in time convergence of solutions to the equilibrium by a entropy method. More precisely, they prove the following inequality

$$D(f) \geq \min\left\{\delta_1 H(f|\mu), \delta_2 H(f|\mu)^{1+\gamma/2}\right\}, \tag{7}$$

for all $f$ satisfying (6) and some constants $\delta_1, \delta_2 > 0$, where $D(f)$ is the entropy dissipation (5) and $H(f|\mu)$ is the relative entropy of $f$ with respect to $\mu$ given by

$$H(f|\mu) = \int_{\mathbb{R}^3} \frac{f}{\mu} \log \frac{f}{\mu}\, d\mu.$$

The inequality (7) implies that solutions $f_t$ converges to $\mu$ in relative entropy

$$\forall t > 0 \qquad H(f(t)|\mu) \leq C(f_0)(1+t)^{-2/\gamma},$$

for some constant $C(f_0) > 0$ depending on the initial data $f_0$, which implies, by the Csiszár-Kullback-Pinsker inequality, a polynomial in time convergence in $L^1$-norm

$$\forall t > 0 \qquad \|f(t) - \mu\|_{L^1} \leq C'(f_0)(1+t)^{-1/\gamma}. \tag{8}$$

Another approach to the convergence issue consists to study the linearized equation. We can linearize the Landau equation around the equilibrium $\mu$, with the perturbation $f = \mu + h$, hence the equation satisfied by $h = h(t,v)$ takes the form

$$\partial_t h = \mathcal{L}h + Q(h,h),$$

with initial datum $h_0$ defined by $h_0 = f_0 - \mu$, and where the linearized Landau operator $\mathcal{L}$ is given by

$$\mathcal{L}h = Q(\mu, h) + Q(h, \mu). \tag{9}$$

Furthermore, from the conservations properties (4), we observe that the null space of $\mathcal{L}$ has dimension 5 and is given by (see e.g. [3, 7, 1, 8, 10])

$$\mathcal{N}(\mathcal{L}) = \mathrm{Span}\{\mu, v_1\mu, v_2\mu, v_3\mu, |v|^2\mu\}. \tag{10}$$

We can then study the long-time behavoiur of the linearized equation

$$\begin{cases} \partial_t h = \mathcal{L}h \\ h_{|t=0} = h_0. \end{cases} \tag{11}$$

by studying spectral properties of the operator $\mathcal{L}$.

Consider the weighted space $L^2(\mu^{-1/2})$ associated to the scalar product and norm

$$\langle h,g\rangle_{L^2(\mu^{-1/2})} := \int_{\mathbb{R}^3} hg\,\mu^{-1}\,\mathrm{d}v \qquad \text{and} \qquad \|h\|^2_{L^2(\mu^{-1/2})} := \int_{\mathbb{R}^3} |h|^2\,\mu^{-1}\,\mathrm{d}v.$$

In this space, the operator $\mathcal{L}$ is self-adjoint, moreover we have

$$\langle \mathcal{L}h,h\rangle_{L^2(\mu^{-1/2})} = -\frac{1}{2}\iint_{\mathbb{R}^3\times\mathbb{R}^3} a_{ij}(v-v_*)\left\{\partial_j(\mu^{-1}h)-\partial_{*j}(\mu_*^{-1}h_*)\right\}\left\{\partial_j(\mu^{-1}h)-\partial_{*j}(\mu_*^{-1}h_*)\right\}\mu_*\mu\,\mathrm{d}v_*\,\mathrm{d}v \le 0,$$

since $a$ is positive, which implies that the spectrum of $\mathcal{L}$ is included in $\mathbb{R}_-$.

From Degond-Lemou [3], Guo [7], Baranger-Mouhot [1], Mouhot [8], Mouhot-Strain [10], there exists $\lambda_0 > 0$ such that

$$\langle -\mathcal{L}h,h\rangle_{L^2(\mu^{-1/2})} \ge \lambda_0\|h\|^2_{L^2(\mu^{-1/2})} \qquad \forall\, h \in \mathcal{N}(\mathcal{L})^\perp.$$

This spectral gap estimate implies, for the linearized equation 11,

$$\|h(t)-\Pi h_0\|_{L^2(\mu^{-1/2})} \le e^{-\lambda_0 t}\|h_0-\Pi h_0\|_{L^2(\mu^{-1/2})} \qquad \forall\, h_0 \in L^2(\mu^{-1/2}), \tag{12}$$

where $\Pi$ denotes the projection onto $\mathcal{N}(\mathcal{L})$.

## 2. Exponential convergence

As we can see above, the result (8) tell us that any solution to the Landau equation converges to the equilibrium in polynomial time. Moreover, (12) gives us an exponential convergence to equilibrium, but only if the solution lies in some suitable neighborhood of the equilibrium, when the linear term dominates the nonlinear one. One could then expect to combine these two results : for small times one uses (8), then for large times, when the solution enters in some appropriated neighborhood of the equilibrium in $L^2(\mu^{-1/2})$-norm, one uses (12). However, the spectral gap for the linearized operator holds in $L^2(\mu^{-1/2})$ and the Cauchy theory [4] for the nonlinear Landau equation is constructed in $L^1$-spaces with polynomial weight, which means that to be able to use this strategy, starting from some initial datum in weighted $L^1$-space, one would need the creation of the $L^2(\mu^{-1/2})$-norm, and this is not known to be true.

The main result of [2] is an exponential in time convergence of solutions to the Landau equation towards the equilibrium, given by the following theorem.

**Theorem 1** (Exponential convergence to equilibrium)**.** *Let $\gamma \in (0,1]$ and a nonnegative $f_0 \in L^1(\langle v\rangle^{2+\delta})$ for some $\delta > 0$, satisfying* (6)*. Then, for any weak solution $(F_t)_{t\ge 0}$ to the spatially homogeneous Landau equation* (1) *with initial datum $F_0$, there exists a constant $C > 0$ such that*

$$\forall t \ge 0, \qquad \|f(t)-\mu\|_{L^1} \le Ce^{-\lambda_0 t},$$

*where $\lambda_0$ is the spectral gap of the linearized operator $\mathcal{L}$ on $L^2(\mu^{-1/2})$.*

The strategy to prove this theorem is the following:

(1) New spectral gap estimates for the linearized Landau operator $\mathcal{L}$ in weighted (polynomial and exponential) $L^p$-spaces. Theses estimates are based on the method of *enlargement of the functional space of the semigroup decay* developed by Gualdani, Mischler and Mouhot [6]. More precisely, we xetend the known spectral gap on the *small space* $L^2(\mu^{-1/2})$ to *bigger spaces* of the $L^p$-type with weight.

(2) The Cauchy theory for the (nonlinear) homogeneous Landau equation developed by Desvillettes and Villani [5].

(3) The coupling between the linear and nonlinear theories: for small times we use the polynomial convergence from (8); then, for large times, when the solution enters in some appropriated neighborhood of the equilibrium, we use the exponential decay given by the new estimates for the linearized theory from (1).

This strategy was introduced by Mouhot [9] in order to prove the exponential convergence for the homogeneous Boltzmann equation with hard potentials. Recently, Gualdani, Mischler and Mouhot [6] used the same approach to prove the exponential convergence for the inhomogeneous Boltzmann equation with hard spheres.

Point (1) above is given by the following result from [2].

**Theorem 2.** *Consider the linearized Landau operator* $\mathcal{L}$ (9) *with hard potentials* $\gamma \in (0,1]$ *and the equation* (11). *Let* $p \in [1,2]$ *and a polynomial or exponential weigth m. Then, there exists* $C > 0$ *such that*

$$\forall t > 0,\ \forall h_0 \in L^p(m), \quad \|h(t) - \Pi h\|_{L^p(m)} \leq C e^{-\lambda_0 t} \|h - \Pi h\|_{L^p(m)},$$

*where* $\lambda_0$ *is the spectral gapr of* $\mathcal{L}$ *on* $L^2(\mu^{-1/2})$ *and* $\Pi$ *is the projection onto* $\mathcal{N}(\mathcal{L})$.

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