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par

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Analyse asymptotique de quelques équations cinétiques, fluides et d'agrégation-diffusion

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Contents

Avant-propos

Ce mémoire a pour objectif de présenter une synthèse de mes travaux de recherche depuis ma soutenance de thèse.

Le manuscrit est composé de cinq chapitres qui peuvent être lus indépendamment, et concernent l'analyse de certaines équations aux dérivées partielles dans le domaine des équations cinétiques, des équations fluides et des équations d'agrégation-diffusion. Outre le problème habituel du caractère bien posé de telles équations, nous nous intéresserons également à certains problèmes asymptotiques pour ces équations, lesquels problèmes constitueront le fil conducteur de ce mémoire.

Chaque chapitre est consacré à l'analyse asymptotique d'une équation ou d'une classe d'équations. La plupart du temps, nous nous intéresserons au comportement asymptotique en temps long des équations d'évolution, comme cela sera discuté dans les Chapitres 1, 2, 3 et 5. Dans le Chapitre 4 cependant, nous nous intéresserons à une équation stationnaire et nous étudierons le comportement asymptotique des solutions dans une limite d'échelle appropriée.

La première partie du manuscrit traite des équations de la théorie cinétique et est composée des Chapitres 1 et 2. La deuxième partie est consacrée aux équations de la mécanique des fluides et correspond aux Chapitres 3 et 4. Enfin, la troisième partie de ce mémoire est constituée par le Chapitre 5 et concerne les équations d'agrégation-diffusion.

Équations cinétiques

La première partie de ce manuscrit est consacrée à l'étude de certaines équations cinétiques. Le but de la théorie cinétique est de décrire l'évolution d'un système formé d'un très grand nombre de composants indiscernables, tels que des gaz, des plasmas ou des galaxies. En raison de la présence d'un grand nombre de composants, l'analyse du comportement individuel de chacun d'entre eux n'est pas un problème traitable. Cependant, nous ne sommes pas intéressés par la description complète et détaillée de tels systèmes, mais plutôt par une description statistique de ceux-ci.

On s'intéresse donc à la description de l'évolution de la densité $f = f(t,x,v) \geqslant 0$ de particules qui au temps $t \in \mathbf{R}_+$ et position $x \in \Omega \subseteq \mathbf{R}^d$ sont animées de la vitesse $v \in \mathbf{R}^d$. La forme générale d'une équation cinétique est donnée par

$$\partial_t f + v \cdot \nabla_x f + (\mathscr{F}[f] + \mathscr{F}_{\text{ext}}) \cdot \nabla_v f = \mathscr{C}(f).$$

L'opérateur de collision $\mathscr{C}(f)$ modélise l'interaction des particules du système et est supposé agir seulement en la variable de vitesse $v \in \mathbf{R}^d$. Les exemples fondamentaux sont l'opérateur de Boltzmann, introduit par Maxwell [151] et Boltzmann [27], et l'opérateur de Landau, introduit par Landau [143].

La force dite de champ-moyen $\mathscr{F}[f] = \mathscr{F}[f](t,x)$ représente la force auto-induite créée par la distribution f à travers ses quantités macroscopiques, c'est-à-dire la densité de masse, la quantité d'impulsion et l'énergie, ce qui correspond à des moyennes de f par rapport à la variable v. Finalement le terme $\mathscr{F}_{\text{ext}}$ représente un champ de force extérieur.

Chapitre 1 : Équation de Landau

Dans le Chapitre 1 on considère l'équation de Landau qui s'écrit sous la forme

$$\partial_t f + v \cdot \nabla_x f = Q(f, f),$$

où l'opérateur de collision de Landau Q est un opérateur intégro-différentiel bilinéaire qui sera décrit en détails dans la Section 1.1.

Les équations de Boltzmann et de Landau partagent deux propriétés fondamentales qui sont supposées dicter le comportement en temps long des solutions. D'une part, ces équations satisfont aux lois de conservation physique de la masse, de la quantité de mouvement et de l'énergie. D'autre part, elles vérifient le celèbre théorème-H de Boltzmann, introduit par Boltzmann [27], qui nous donne l'existence d'une fonctionnelle de Lyapunov le long des solutions et qui caractérise également les minima de telle fonctionnelle. Plus précisément, le théorème-H stipule que la fonctionnelle d'entropie

$$\mathscr{H}(f) = \int f \log f \, \mathrm{d}v \, \mathrm{d}x$$

est décroissante au cours du temps et le long des solutions des équations de Boltzmann et de Landau, et que les distributions minimisant l'entropie sont données par les maxwelliennes, c'est-à-dire des gaussiennes en la variable de vitesse v.

On conjecture ainsi que les solutions de l'équation de Landau convergent en temps longs vers l'unique équilibre maxwellien associé à la donnée initiale. Le Chapitre 1 est consacré à ce problème pour l'équation de Landau, et nous présenterons des résultats qui démontrent cette convergence dans différentes situations et qui donnent également des taux de convergence.

Chapitre 2 : Équations cinétiques linéaires avec potentiel de confinement

Dans le Chapitre 2, nous nous intéressons aux équations cinétiques collisionnelles linéaires dans tout l'espace avec potentiel de confinement, qui s'écrit

$$\partial_t f + v \cdot \nabla_x f - \nabla_x \phi \cdot \nabla_v f = \mathscr{C}(f).$$

L'opérateur $\mathscr{C}(f)$ est un opérateur de collision linéaire qui satisfait les lois de conservation physique de la masse, de la quantité de mouvement et de l'énergie, ainsi qu'une version linéarisée du théorème-H.

De plus, on considère que les particules sont confinées dans tout l'espace $x \in \mathbf{R}^d$ via un potentiel extérieur $\phi : \mathbf{R}^d \to \mathbf{R}$ de confinement, ce qui signifie que $\mathrm{e}^{-\phi}$ est une mesure de probabilité. En prenant en compte les propriétés géométriques du potentiel ϕ ainsi que l'harmonicité de ϕ, nous étudierons ensuite le comportement en temps grand des solutions à cette équation et présenterons des résultats concernant la convergence vers les états d'équilibre et les solutions stationnaires.

Équations fluides

La deuxième partie de ce mémoire concerne les équations aux dérivées partielles de la mécanique des fluides.

Chapitre 3 : Fluides isothermes

Le premier problème sur les équations des fluides que nous aborderons concerne l'évolution de certains fluides compressibles isentropiques.

Ces modèles fluides décrivent l'évolution de la densité $\varrho = \varrho(t, x)$ et du champ de vitesse $u = u(t, x)$ du fluide, où $t \in \mathbf{R}_+$ représente la variable de temps et $x \in \mathbf{R}^d$ la variable spatiale.

Les équations satisfaites par le couple (ϱ, u) décrivent les deux lois de conservation : la première équation correspond à l'équation de continuité

$$\partial_t \varrho + \operatorname{div}(\varrho u) = 0;$$

la seconde équation représente la conservation de la quantité de mouvement

$$\partial_t(\varrho u) + \operatorname{div}(\varrho u \otimes u) + \nabla P(\varrho) = \operatorname{div} \mathbb{T},$$

où P est la pression du fluide et $\mathbb{T}$ correspond au tenseur de contraintes dépendant du modèle fluide considéré. La pression P ne dépend que de la densité ϱ et on suppose qu'elle vérifie une équation d'état. Nous étudierons le cas des fluides isothermes, ce qui correspond à $P(\varrho) = \varrho$, par opposition au modèle plus courant des fluides polytropiques $P(\varrho) = \varrho^\gamma$ avec $\gamma > 1$. En outre, nous nous intéresserons aux équations fluides suivantes :

— l'équation d'Euler isotherme, ce qui correspond à $\operatorname{div} \mathbb{T} = 0$;

— l'équation d'Euler–Korteweg isotherme, ce qui correspond à

$$\operatorname{div} \mathbb{T} = \frac{\varepsilon^2}{2} \varrho \nabla \left(\frac{\Delta\sqrt{\varrho}}{\sqrt{\varrho}} \right)$$

avec $\varepsilon > 0$;

— l'équation de Navier–Stokes isotherme avec viscosité dégénérée dependant de la densité, ce qui correspond à

$$\operatorname{div} \mathbb{T} = \nu \operatorname{div} \left(\varrho \, \frac{(\nabla u + \nabla u^\top)}{2} \right)$$

avec $\nu > 0$;

— l'équation de Navier–Stokes–Korteweg isotherme avec viscosité dégénérée dependant de la densité, ce qui correspond à

$$\operatorname{div} \mathbb{T} = \frac{\varepsilon^2}{2} \varrho \nabla \left(\frac{\Delta\sqrt{\varrho}}{\sqrt{\varrho}} \right) + \nu \operatorname{div} \left(\varrho \, \frac{(\nabla u + \nabla u^\top)}{2} \right)$$

avec $\varepsilon, \nu > 0$.

Le Chapitre 3 est consacré à l'existence de solutions aux équations fluides ci-dessus ainsi qu'à leur comportement asymptotique en temps long. Comme pour les Chapitres 1 et 2, un ingrédient essentiel dans l'étude du comportement asymptotique consiste en l'existence d'une fonctionnelle de type entropie.

Chapitre 4 : Équation de Stokes dans un domaine perforé

La deuxième question concernant les équations fluides qui nous intéresse correspond à l'étude d'un problème stationnaire.

Nous considérerons l'équation de Stokes

$$\begin{cases} -\Delta u + \nabla P = 0 \\ \operatorname{div} u = 0 \end{cases}$$

dans un domaine perforé correspondant à l'espace tout entier $\mathbf{R}^3$ privé de N obstacles sphériques identiques, où u représente encore le champ de vitesse du fluide et P sa pression.

Le but d'un tel problème est de décrire le comportement d'un nuage donné de N particules immergées dans un fluide visqueux. En considérant une mise à l'échelle appropriée, du type *champ-moyen*, nous nous intéressons à la description du comportement asymptotique des solutions dans la limite $N \to \infty$, ce qui correspond à un problème d'homogénéisation pour l'équation de Stokes, et ainsi à la dérivation de l'équation effective associée.

Nous présenterons dans le Chapitre 4 quelques résultats sur ce problème lorsque les obstacles sont choisis aléatoirement.

Équations d'agrégation-diffusion

La dernière partie du manuscrit est consacrée à l'étude des équations d'évolution du type agrégation-diffusion. Ces équations modélisent l'évolution d'une densité $f = f(t,x)$, où $t \in \mathbf{R}_+$ représente la variable temporelle et $x \in \mathbf{R}^d$ la variable spatiale, qui est soumis à deux mécanismes en concurrence: l'agrégation qui a tendance à concentrer la densité; et la diffusion qui a tendance à l'étaler.

Nous nous intéressons dans le Chapitre 5 à un modèle classique en chimiotaxie décrivant le mouvement collectif des cellules qui sont attirées par une substance chimique qu'elles-mêmes émettent. Ce système correspond à l'équation de Keller–Segel parabolique-parabolique dans $\mathbf{R}^2$, qui s'écrit

$$\begin{cases} \partial_t f = \Delta f - \operatorname{div}(f \nabla u) \\ \varepsilon \partial_t u = \Delta u + f \end{cases}.$$

La fonction $f = f(t,x)$ représente la densité de cellules, tandis que $u = u(t,x)$ représente la concentration de l'attractant chimique, et $\varepsilon > 0$ est un paramètre strictement positif. Une propriété importante de l'équation ci-dessus est que les deux mécanismes en compétition d'agrégation et de diffusion sont presque du même ordre, ce qui rend son analyse difficile et intéressante.

Dans le Chapitre 5, nous présenterons des résultats concernant l'unicité et la régularité des solutions ainsi que leur comportement asymptotique en temps grand.

Foreword

The aim of this memoir is to present the research conducted by the author since the end of his PhD thesis.

The manuscript is composed by five chapters that can be read independently, and regards the analysis of some partial differential equations in the realm of kinetic equations, fluid equations, and aggregation-diffusion equations. Besides the usual well-posedness issue related to the above partial differential equations, we shall also be interested in some asymptotic problems for these equations, which constitute the common thread of this memoir.

Each chapter will be devoted to the asymptotic analysis of a (class of) equation. Most of the time we will be interested in the asymptotic behavior for large times of evolution equations, as will be discussed in Chapters 1, 2, 3, and 5. In Chapter 4 however, we will be interested in a stationary equation and will investigate the asymptotic behavior of solutions in a suitable scaling limit.

The first part of this manuscript concerns equations in kinetic theory and are composed by Chapters 1 and 2. The second part is devoted to equations of fluid mechanics and corresponds to Chapters 3 and 4. Finally, the third part of this memoir is composed by Chapter 5 and regards aggregation-diffusion equations.

Kinetic equations

The first part of this manuscript is devoted to the study of some kinetic equations. The purpose of kinetic theory is to describe the evolution of system composed by a large number of indistinguishable components, such as gases, plasmas or galaxies. Because of the presence of a large number of components, the analysis of the individual behavior of each component is not a tractable problem. However, we are not interested in the complete and detailed description of such systems, but rather in a statistical description of them.

We are therefore interested in describing the evolution of the density $f = f(t, x, v) \geqslant 0$ of particles that at time $t \in \mathbf{R}_+$ and position $x \in \Omega \subseteq \mathbf{R}^d$ have the velocity $v \in \mathbf{R}^d$. The general form of a kinetic equation is

$$\partial_t f + v \cdot \nabla_x f + (\mathscr{F}[f] + \mathscr{F}_{\text{ext}}) \cdot \nabla_v f = \mathscr{C}(f).$$

The collision operator $\mathscr{C}(f)$ models the collision between particles in the system and is supposed to act only on the velocity variable $v \in \mathbf{R}^d$. The fundamental examples are the Boltzmann operator, introduced by Maxwell [151] and Boltzmann [27], and the Landau operator, introduced by Landau [143].

The mean-field force $\mathscr{F}[f] = \mathscr{F}[f](t, x)$ represents the self-generated force created by the distribution f through its macroscopic quantities, i.e. density of mass, momentum and energy, which corresponds to averages of f with respect to v. Finally $\mathscr{F}_{\text{ext}}$ represents an exterior force field.

Chapter 1: Landau equation

In Chapter 1 we shall consider the Landau equation that takes the form

$$\partial_t f + v \cdot \nabla_x f = Q(f, f),$$

where the Landau collision operator Q is a bilinear integro-differential operator that will be described in Section 1.1.

The Boltzmann and Landau equations share two fundamental properties that are expected to dictate the large-time behavior of solutions. On the one hand these equations satisfy the physical conservation laws of mass, momentum and energy. On the other hand, they satisfy the H-theorem of Boltzmann, introduced in Boltzmann [27], which describes a Lyapunov functional along solutions to these equations and also characterizes the distribution states reaching the minimum of such functional. More precisely, the H-Theorem states that the entropy functional

$$\mathscr{H}(f) = \int f \log f \, \mathrm{d}v \, \mathrm{d}x$$

is non-increasing along solutions to the Boltzmann and Landau equations and that the distributions that minimize the entropy are the Maxwellian states, that is, Gaussian functions in the velocity variable v.

We therefore conjecture that solutions to the Landau equation converge to the associated unique Maxwellian equilibrium in large time. Chapter 1 is devoted to the analysis of this problem for the Landau equation, and we shall present results that prove this convergence in different situations and also give rates of convergence.

Chapter 2: Linear kinetic equation with a confining potential

In Chapter 2 we shall be interested in linear collisional kinetic equations in the whole space with confining potential, which reads

$$\partial_t f + v \cdot \nabla_x f - \nabla_x \phi \cdot \nabla_v f = \mathscr{C}(f).$$

Here $\mathscr{C}(f)$ is a linear collisional operator that satisfies the physical conservation laws of mass, momentum and energy, as well as a linearized version of the H-Theorem.

Furthermore, we consider here that particles are confined in the whole space $x \in \mathbf{R}^d$ via an exterior potential $\phi : \mathbf{R}^d \to \mathbf{R}$, meaning that $\mathrm{e}^{-\phi}$ is a probability measure. By taking into account geometric properties of the potential ϕ, we will then investigate the large-time behavior of solutions to this equation and shall present results concerning the convergence to equilibria states and stationary solutions.

Fluid equations

The second part of this memoir concerns partial differential equations from fluid mechanics.

Chapter 3: Isothermal fluids

The first problem concerning fluid equations that we shall address regards the evolution in time of some isentropic compressible fluids.

These fluid models describe the evolution of the density $\varrho = \varrho(t, x)$ and the velocity-field $u = u(t, x)$ of the fluid, where $t \in \mathbf{R}_+$ stands for the time variable and $x \in \mathbf{R}^d$ for the spatial variable.

The equations satisfied by the couple (ϱ, u) describe the two basic conservation laws: The first equation corresponds to the continuity equation that reads

$$\partial_t \varrho + \operatorname{div}(\varrho u) = 0.$$

The second equation stands for the conservation of momentum and reads

$$\partial_t(\varrho u) + \operatorname{div}(\varrho u \otimes u) + \nabla P(\varrho) = \operatorname{div} \mathbb{T}$$

where P stands for the pressure of the fluid and $\mathbb{T}$ corresponds to the stress tensor depending on the fluid model we consider. The pressure P depends only on the density ϱ and it is assumed to verify an equation of state. We shall investigate the case of isothermal fluids which corresponds to the case $P(\varrho) = \varrho$, as opposed to the more common model of polytropic fluids $P(\varrho) = \varrho^\gamma$ with $\gamma > 1$. Furthermore, we shall be interested in the following fluid equations:

— isothermal Euler equation, corresponding to $\operatorname{div} \mathbb{T} = 0$;

— isothermal Euler–Korteweg equation, corresponding to

$$\operatorname{div} \mathbb{T} = \frac{\varepsilon^2}{2}\, \varrho \nabla \left(\frac{\Delta \sqrt{\varrho}}{\sqrt{\varrho}} \right)$$

with $\varepsilon > 0$;

— isothermal Navier–Stokes equation with degenerate density dependent viscosities, corresponding to

$$\operatorname{div} \mathbb{T} = \nu \operatorname{div} \left(\varrho\, \frac{(\nabla u + \nabla u^\top)}{2} \right)$$

with $\nu > 0$;

— isothermal Navier–Stokes–Korteweg equation with degenerate density dependent viscosities, corresponding to

$$\operatorname{div} \mathbb{T} = \frac{\varepsilon^2}{2}\, \varrho \nabla \left(\frac{\Delta \sqrt{\varrho}}{\sqrt{\varrho}} \right) + \nu \operatorname{div} \left(\varrho\, \frac{(\nabla u + \nabla u^\top)}{2} \right)$$

with $\varepsilon, \nu > 0$.

Chapter 3 is devoted to the existence of solutions to the above equations as well as their large-time behavior. As for Chapters 1 and 2, an important ingredient of the analysis of the large-time issue relies on the existence of an entropy-type functional.

Chapter 4: Stokes equation in a perforated domain

The second issue regarding fluid equations we shall be interested, corresponds to the study of a stationary problem.

We shall consider the Stokes equation

$$\begin{cases} -\Delta u + \nabla P = 0 \\ \operatorname{div} u = 0 \end{cases}$$

in a perforated domain, corresponding to the whole space $\mathbf{R}^3$ deprived of N-spherical indistinguishable obstacles, where u still represents the velocity-field of the fluid and P its pressure.

The purpose of such problem is to describe the behavior of a given cloud of N-particles immersed in a viscous fluid. By considering a suitable scaling, we will be interested in describing the asymptotic behavior of solutions in the limit $N \to \infty$, which corresponds to a homogenization problem for the Stokes equation, and therefore in deriving the associated effective equation.

We shall present in Chapter 4 some results on the problem of the Stokes equation around a random array of spheres.

Aggregation-diffusion equations

The last part of the manuscript is devoted to the study of aggregation-diffusion evolution equations. These type of equations model the evolution of a density $f = f(t, x)$, where $t \in \mathbf{R}_+$ stands for the time variable and $x \in \mathbf{R}^d$ for the spatial variable, that is subjected to two mechanisms in competition: an aggregation mechanism which has tendency to concentrate the density; and a diffusion mechanism which spreads out the density.

We shall be interested in Chapter 5 in a classical model in chemotaxis that describes the collective motion of cells which are attracted by a chemical substance that they emit. This system is the parabolic-parabolic Keller–Segel equation in the plane $\mathbf{R}^2$ that reads

$$\begin{cases} \partial_t f = \Delta f - \operatorname{div}(f \nabla u) \\ \varepsilon \partial_t u = \Delta u + f \end{cases}.$$

Here $f = f(t, x)$ represents the density of cells, whereas $u = u(t, x)$ stands for the concentration of the chemo-attractant, and $\varepsilon > 0$ is a positive parameter. An important property of the above equation is that the two competing mechanisms of aggregation and diffusion are almost of the same order, resulting therefore in a difficult and interesting analysis.

In Chapter 5 we shall present results concerning the uniqueness and regularity of solutions as well as their large-time behavior.

List of publications

Below we may find the list of publications of the author. They are presented in anti-chronological order of publication, and the reference numbers are the same as in the general bibliography presented at the end of the manuscript.

Research articles presented in this memoir

[52] K. Carrapatoso, J. Dolbeault, F. Hérau, S. Mischler, C. Mouhot, and C. Schmeiser. Linear stability of a confined system of charged particles. In preparation.

[51] K. Carrapatoso, J. Dolbeault, F. Hérau, S. Mischler, and C. Mouhot. Weighted Korn inequalities in the whole space. In preparation.

[43] R. Carles, K. Carrapatoso, and M. Hillairet. Global weak solutions for quantum isothermal fluids. Preprint arXiv:1905.00732.

[54] K. Carrapatoso and M. Hillairet. On the derivation of a Stokes-Brinkman problem from Stokes equations around a random array of moving spheres. *Comm. Math. Phys.*, 373(1):265–325, 2020.

[44] R. Carles, K. Carrapatoso, and M. Hillairet. Rigidity results in generalized isothermal fluids. *Ann. H. Lebesgue*, 1:47–85, 2018.

[55] K. Carrapatoso and S. Mischler. Landau equation for very soft and Coulomb potentials near Maxwellians. *Ann. PDE*, 3(1):Art. 1, 65 pp., 2017.

[50] K. Carrapatoso, L. Desvillettes, and L. He. Estimates for the large time behavior of the Landau equation in the Coulomb case. *Arch. Ration. Mech. Anal.*, 224(2):381–420, 2017.

[56] K. Carrapatoso and S. Mischler. Uniqueness and long time asymptotics for the parabolic-parabolic Keller-Segel equation. *Comm. Partial Differential Equations*, 42(2):291–345, 2017.

[57] K. Carrapatoso, I. Tristani, and K.-C. Wu. Cauchy problem and exponential stability for the inhomogeneous Landau equation. *Arch. Ration. Mech. Anal.*, 221(1):363–418, 2016.

— Erratum to: Cauchy problem and exponential stability for the inhomogeneous Landau equation. *Arch. Ration. Mech. Anal.*, 223(2):1035–1037, 2017.

[47] K. Carrapatoso. On the rate of convergence to equilibrium for the homogeneous Landau equation with soft potentials. *J. Math. Pures Appl. (9)*, 104(2):276–310, 2015.

Research articles not presented in this memoir

The following papers are part of the author's PhD thesis and are not presented in this memoir.

[49] K. Carrapatoso. Propagation of chaos for the spatially homogeneous Landau equation for Maxwellian molecules. *Kinet. Relat. Models*, 9(1):1–49, 2016.

[46] K. Carrapatoso. Exponential convergence to equilibrium for the homogeneous Landau equation with hard potentials. *Bull. Sci. Math.*, 139(7):777–805, 2015.

[48] K. Carrapatoso. Quantitative and qualitative Kac's chaos on the Boltzmann's sphere. *Ann. Inst. Henri Poincaré Probab. Stat.*, 51(3):993–1039, 2015.

[53] K. Carrapatoso and A. Einav. Chaos and entropic chaos in Kac's model without high moments. *Electron. J. Probab.*, 18:no. 78, 38, 2013.

Chapter 1

The Landau equation

In this chapter we present the works [47]; [57] in collaboration with I. Tristani and K.-C. Wu; [50] jointly with L. Desvillettes and L. He; and [55] with S. Mischler.

1.1 Introduction

The Landau equation is a fundamental model in kinetic theory that describes the evolution of a weakly interacting dense plasma, taking into account collisions between the charged particles. The unknown is the distribution $f = f(t,x,v) \geqslant 0$ of particles that at time $t \in \mathbf{R}^+$ and position $x \in \Omega_x \subseteq \mathbf{R}^3$ possess velocity $v \in \mathbf{R}^3$. The evolution of f is described by the *spatially inhomogeneous Landau equation*

$$\partial_t f + v \cdot \nabla_x f = Q(f,f), \tag{1.1}$$

which is complemented with a non-negative initial datum $f_{|t=0} = f_{\rm in}$. One typically assumes that particle are in the whole space $\Omega_x = \mathbf{R}^3$; or that they satisfy periodic boundary conditions, in which case $\Omega_x = \mathbf{T}^3$; or that Ω_x is a bounded domain of $\mathbf{R}^3$, in which case equation (1.1) shall be complemented with boundary conditions. We shall consider hereafter that the particles are confined in the torus so that $\Omega_x = \mathbf{T}^3$.

In the case of a spatially homogeneous initial datum, that is when $f_{\rm in} = f_{\rm in}(v)$ depends only on the velocity variable v, the solution $f = f(t,v)$ also depends only on v, then equation (1.1) simplifies into the *spatially homogeneous Landau equation*

$$\partial_t f = Q(f,f). \tag{1.2}$$

The Landau collision operator Q, introduced by Landau [143], is a non-local integro-differential bilinear operator that acts only on the velocity variable v and is given by

$$Q(g,f)(v) = \nabla_v \cdot \int_{\mathbf{R}^3} a(v-v_*) \left\{ g(v_*) \nabla_v f(v) - f(v) \nabla_{v_*} g(v_*) \right\} \mathrm{d}v_*, \tag{1.3}$$

where a is a matrix-valued function that is symmetric, nonnegative and takes the form

$$a(z) = |z|^{\gamma+2} \Pi^\perp(z) = |z|^{\gamma+2} \left(\mathrm{I}_d - \frac{z}{|z|} \otimes \frac{z}{|z|} \right), \quad -3 \leqslant \gamma \leqslant 1, \tag{1.4}$$

where $\Pi^\perp(z)$ denotes the orthogonal projection onto $z^\perp$. One usually classifies the different cases as follows:

— Hard potentials if $0 < \gamma \leqslant 1$;

— Maxwellian molecules if $\gamma = 0$;

— Moderately soft potentials if $-2 \leqslant \gamma < 0$;

— Very soft potentials if $-3 < \gamma < -2$;

— Coulomb potential if $\gamma = -3$.

It is worth mentioning that the Coulomb potential is the most physically interesting case, and it is also the most difficult one to study because of the singularity in (1.4).

1.1.1 Boltzmann and Landau collision operators

Let us briefly present the Boltzmann operator, which is a fundamental kinetic collision operator and shares the same fundamental properties as the Landau operator, namely the conservation laws and the H-theorem that will be presented in Section 1.2. For a detailed account we refer the reader to the books of Cercignani [59] and Villani [187].

The Boltzmann equation describes the evolution of a rarefied gaz out of equilibrium, taking into account the binary collisions between particles. It was introduced by Maxwell [151] in 1867 and Boltzmann [27] in 1872, and it reads

$$\partial_t f + v \cdot \nabla_x f = Q_B(f, f)$$

where as above $f = f(t, x, v)$ represents the density of particles. The collision operator Q_B is bilinear and acts only on the velocity variable $v \in \mathbf{R}^3$, which represents the fact that collisions are supposed to be localized in space, and it reads

$$Q_B(g, f) = \iint_{\mathbf{R}^3 \times \mathbf{S}^2} B(v - v_*, \sigma)\Big(g(v'_*)f(v') - g(v_*)f(v)\Big)\, \mathrm{d}\sigma\, \mathrm{d}v_*.$$

The pre- and post-collision velocities (v', v'_*) and (v, v_*) are given by

$$v' = \frac{v + v_*}{2} + \frac{|v - v_*|}{2}\sigma \quad \text{and} \quad v'_* = \frac{v + v_*}{2} - \frac{|v - v_*|}{2}\sigma$$

which is one possible parametrization of the conservation of momentum and energy in an elastic collision

$$v' + v'_* = v + v_* \quad \text{and} \quad |v'|^2 + |v'_*|^2 = |v|^2 + |v_*|^2.$$

The collision kernel $B(v - v_*, \sigma)$ encodes the physics of the interaction between particles. It is assumed to be nonnegative and to depend only on the relative velocity $|v - v_*|$ and the angle $\cos\theta = \sigma \cdot \frac{(v - v_*)}{|v - v_*|}$.

Maxwell [151] has computed the collision kernels in terms of the interaction potential. On the one hand, in the case of hard-spheres, that is when particles collide to each other as billiard ball, the collision kernel is given by

$$B(v - v_*, \sigma) = C|v - v_*|$$

for some constant $C > 0$. On the other hand, when particles interact via an inverse power law interaction potential, which falls in the class of long-range interactions,

$$U(r) = \frac{C}{r^{s-1}} \quad \text{with} \quad s > 2,$$

where r represents here the distance between particles, then one has (in the three-dimensional case)

$$B(v - v_*, \sigma) = |v - v_*|^\gamma b(\cos\theta) \quad \text{with} \quad \gamma = \frac{s - 5}{s - 1}$$

and the angular kernel b is an implicit function that is locally smooth and has a non-integrable singularity at $\theta = 0$ as

$$\sin\theta\, b(\cos\theta) \underset{\theta\sim 0}{\sim} C\,\theta^{-1-\nu} \quad \text{with} \quad \nu = \frac{2}{s-1}$$

for some constant $C > 0$. One usually classifies the different cases into: hard potentials if $0 < \gamma \leqslant 1$; Maxwellian molecules if $\gamma = 0$; and soft potentials if $-3 < \gamma < 0$.

A possible simplification consists in removing the singularity of the angular kernel by supposing that b is locally integrable, in which case these kernels are said to be *with Grad's cutoff*.

It worth mentioning that the case of Coulomb potential, that is $s = 2$ and thus $\gamma = -3$, the Boltzmann operator does not make sense, see for instance Villani [187]. On the other hand, we remark that the singularity appearing in the angular kernel is consequence of grazing collisions, that is collisions for which the resulting deviation is very small.

Landau [143] in 1936 has introduced the Landau collision operator (1.3) by considering the Boltzmann operator with cutoff Coulomb interaction and then performing a grazing collisions limit. We mention that the original operator derived by Landau [143] corresponds to the case $\gamma = -3$ in (1.4), which is the only physical model. All the other cases $-3 < \gamma \leqslant 1$ are not realistic in the physical point of view, but are of course interesting from a mathematical viewpoint and important since they can be seen as an approximation of the Boltzmann operator.

1.2 Fundamental properties

In this section we gather some fundamental properties of the Landau equation. Let us define the following quantities

$$b(z) = \nabla \cdot a(z) = -2z|z|^{\gamma},$$

and

$$c(z) = \nabla \cdot b(z) = \begin{cases} -2(\gamma+3)|z|^{\gamma} & \text{if } -3 < \gamma \leqslant 1, \\ -8\pi\,\delta_0(z) & \text{if } \gamma = -3. \end{cases}$$

The Landau operator can therefore be rewritten into two other forms:

$$Q(g,f) = \nabla_v \cdot \{(a \star g)\nabla_v f - (b \star g) f\} \tag{1.5}$$

and also

$$Q(g,f) = (a \star g) : \nabla_v^2 f - (c \star g) f, \tag{1.6}$$

where $\star$ denotes the convolution with respect to the velocity variable $v \in \mathbf{R}^3$, $\nabla_v^2 f$ stands for teh Hessian of f, and for two matrices A and B we denote $A : B = \sum_{ij} A_{ij} B_{ij}$.

At the formal level, we can write a weak formulation of the Landau operator Q, thanks to (1.3) and the symmetry of a, in the following way: for any smooth test function $\varphi = \varphi(v)$ there holds

$$\begin{aligned} &\int_{\mathbf{R}^3} Q(f,f)(v)\varphi(v)\,\mathrm{d}v \\ &= -\frac{1}{2}\iint_{\mathbf{R}^3\times\mathbf{R}^3} a(v-v_*)\left\{\frac{\nabla_v f}{f}(v) - \frac{\nabla_{v_*} f}{f}(v_*)\right\} \cdot \{\nabla_v\varphi(v) - \nabla_{v_*}\varphi(v_*)\}\, f(v_*) f(v)\,\mathrm{d}v_*\mathrm{d}v. \end{aligned} \tag{1.7}$$

Furthermore, based on the equations (1.5) or (1.6), another weak formulation also holds at the formal level, namely

$$\begin{aligned} \int_{\mathbf{R}^3} Q(f,f)(v)\varphi(v)\,\mathrm{d}v = \frac{1}{2}&\iint_{\mathbf{R}^3\times\mathbf{R}^3} a(v-v_*) : \{\nabla_v^2\varphi(v) + \nabla_{v_*}^2\varphi(v_*)\}\, f(v_*) f(v)\,\mathrm{d}v_*\mathrm{d}v \\ &+ \iint_{\mathbf{R}^3\times\mathbf{R}^3} b(v-v_*) \cdot \{\nabla_v\varphi(v) - \nabla_{v_*}\varphi(v_*)\}\, f(v_*) f(v)\,\mathrm{d}v_*\mathrm{d}v. \end{aligned} \tag{1.8}$$

From these weak formulations we are now able to deduce two fundamental properties of the Landau collision operator Q, which hold at least formally: the conservation of mass, momentum and energy; and the (Landau's version of) Boltzmann's H-theorem, which we present in more details below.

1.2.1 Conservation laws

Taking $\varphi = 1$ or $\varphi(v) = v_\alpha$, for $\alpha \in \{1,2,3\}$, we easily observe that $\partial_{v_j}\varphi(v) - \partial_{v_{*,j}}\varphi(v_*) = 0$ so that (1.7) vanishes. Moreover, for $\varphi(v) = |v|^2$ we get $\partial_{v_j}\varphi(v) - \partial_{v_{*,j}}\varphi(v_*) = 2(v_j - v_{*j})$ and $a_{ij}(v-v_*)(v_j - v_{*,j}) = 0$ thanks to (1.4). We hence deduce that the collision operator Q (1.3) has the following *collision invariants*

$$\int_{\mathbf{R}^3} Q(f,f)(v)\varphi(v)\,\mathrm{d}v = 0 \quad \text{for} \quad \varphi(v) = 1, v, |v|^2.$$

From this last estimate we obtain that solutions to the spatially inhomogeneous Landau equation (1.1), or the spatially homogeneous Landau equation (1.2), satisfy

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbf{T}^3\times\mathbf{R}^3} f(t,x,v)\varphi(v)\,\mathrm{d}x\,\mathrm{d}v = 0 \quad \text{for} \quad \varphi(v) = 1, v, |v|^2,$$

and thus one obtain the conservation laws:

— *conservation of mass*: for all $t \geqslant 0$ there holds

$$\int_{\mathbf{T}^3\times\mathbf{R}^3} f(t,x,v)\,\mathrm{d}x\,\mathrm{d}v = \int_{\mathbf{T}^3\times\mathbf{R}^3} f_{\mathrm{in}}(x,v)\,\mathrm{d}x\,\mathrm{d}v;$$

— *conservation of momentum*: for all $t \geqslant 0$ there holds

$$\int_{\mathbf{T}^3\times\mathbf{R}^3} v f(t,x,v)\,\mathrm{d}x\,\mathrm{d}v = \int_{\mathbf{T}^3\times\mathbf{R}^3} v f_{\mathrm{in}}(x,v)\,\mathrm{d}x\,\mathrm{d}v;$$

— *conservation of energy*: for all $t \geqslant 0$ there holds

$$\int_{\mathbf{T}^3\times\mathbf{R}^3} |v|^2 f(t,x,v)\,\mathrm{d}x\,\mathrm{d}v = \int_{\mathbf{T}^3\times\mathbf{R}^3} |v|^2 f_{\mathrm{in}}(x,v)\,\mathrm{d}x\,\mathrm{d}v.$$

1.2.2 Boltzmann's H-Theorem

Still from the weak formulation (1.7) and at a formal level, choosing now the test function $\varphi(v) = \log f(v)$ we deduce the Landau's version of the celebrated Boltzmann's H-theorem.

The first part of the H-Theorem states that the entropy of a solution to the Landau equation is non-increasing along time. More precisely, one defines the entropy functional

$$\mathscr{H}(f) = \int_{\mathbf{T}^3\times\mathbf{R}^3} f\log f\,\mathrm{d}x\,\mathrm{d}v \tag{1.9}$$

as well as the entropy-dissipation functional

$$\begin{aligned}\mathscr{D}(f) &= -\int_{\mathbf{R}^3} Q(f,f)\log f\,\mathrm{d}v \\ &= \frac{1}{2}\iint_{\mathbf{R}^3\times\mathbf{R}^3} a(v-v_*)\left\{\frac{\nabla_v f}{f}(v) - \frac{\nabla_{v_*} f}{f}(v_*)\right\}\left\{\frac{\nabla_v f}{f}(v) - \frac{\nabla_{v_*} f}{f}(v_*)\right\} f(v_*)f(v)\,\mathrm{d}v_*\,\mathrm{d}v,\end{aligned} \tag{1.10}$$

which verifies $\mathscr{D}(f) \geqslant 0$ since the matrix a is nonnegative. Thanks to (1.7) one hence obtains that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{H}(f) = -\int_{\mathbf{T}^3} \mathscr{D}(f)\,\mathrm{d}x \leqslant 0, \tag{1.11}$$

and therefore the entropy $\mathscr{H}(f)$ is non-increasing along solutions to (1.1), that is, for all $t \geqslant 0$ one has

$$\mathscr{H}(f(t)) \leqslant \mathscr{H}(f_{\rm in}).$$

The second part of the H-Theorem characterizes the functions that minimize the entropy. More precisely, one can show that $\mathscr{D}(f) = 0$ if and only if one has

$$f = \mu_{\varrho,u,\theta}(t,x,v) = \frac{\varrho(t,x)}{(2\pi\theta(t,x))^{3/2}} \exp\left(-\frac{|v-u(t,x)|^2}{2\theta(t,x)}\right)$$

where $\varrho \geqslant 0$ denotes the density, $u \in \mathbf{R}^3$ the bulk velocity and $\theta > 0$ the temperature. The function $\mu_{\varrho,u,\theta}$ is called a *local Maxwellian*. As a consequence, the global equilibrium of (1.1) is given by a *global Maxwellian*, that is a function $\mu = \mu(v)$ that does not depend on the space variable x nor the time variable t.

1.3 Trend to equilibrium

A very important physical feature predicted by Boltzmann [27], when studying the equation that now bares his name, is the phenomenon of relaxation to the equilibrium state as predicted by the H-Theorem described above. This can be seen as a mathematical foundation of the second law of thermodynamics.

One is therefore interested in proving this convergence, in some sense to be precised, and then in obtaining quantitative rates of convergence. Let us emphasize that this trend-to-equilibrium issue for Boltzmann and Landau equations has been tackled with different strategies that we resume below:

1.3.1 Compactness methods

After obtaining the global existence of some weak solutions, one can prove the convergence to the equilibrium by a compactness method, by exploiting the regularity-type information one can obtain with the entropy-dissipation estimate (1.11). Of course this approach is not constructive and it does not give any information on convergence rates.

1.3.2 Entropy methods

One can employ entropy methods, which corresponds to the obtention of functional inequalities linking the entropy-dissipation functional (1.10) to the relative entropy (1.15) of a solution with respect the equilibrium. Once this is achieved, one could hope to obtain a closed differential inequality for the relative entropy thanks to the entropy identity (1.16), from which a convergence to equilibrium is obtained. One remarks that the rate of convergence will depend on the functional inequalities obtained, and moreover this type of method can handle nonlinear equations directly, that is, without linearization techniques. For example, if one obtain a linear inequality relating the entropy-functional and the relative entropy, one would obtain an exponential convergence.

1.3.3 Linearization methods

Another possible strategy is to employ linear techniques by considering the linearized equation around the Maxwellian equilibrium. One hence is brought to study the linearized operator and employ arguments from spectral theory and operator semigroups. For instance, when this linearized operator possesses a spectral gap one can hope to obtain an exponential convergence for the linearized equation. However this method can only be relevant for the nonlinear equation in a certain neighborhood of the equilibrium when linear terms are dominant. On the one hand, one could obtain from this a trend-to-equilibrium result in a perturbative framework (for instance this is what we shall obtain for the inhomogeneous Landau equation). On the other hand, if one wants to obtain a convergence result in a non perturbative setting, one has to show, by other methods, that solutions indeed enter in this appropriated neighborhood of the equilibrium, starting from which the linear techniques lead to the convergence to the equilibrium.

We shall now present our main results regarding the trend to equilibrium for the Landau equation. In Section 1.4 we focus our attention in the spatially homogeneous equation and present the results of [47] and [50]. After that, in Section 1.5, we turn to the spatially inhomogeneous equation and present the results of [57] and [55].

1.4 The spatially homogeneous equation

In this section we shall consider the *spatially homogeneous* Landau equation

$$\begin{cases} \partial_t f = Q(f,f) \\ f_{|t=0} = f_{\mathrm{in}}. \end{cases} \tag{1.12}$$

We always suppose that the initial data f_{in} satisfies the natural physical assumptions: $f_{\mathrm{in}} = f_{\mathrm{in}}(v) \geqslant 0$ is nonnegative and has finite mass, energy and entropy, that is

$$\int_{\mathbf{R}^3} (1 + |v|^2 + \log f_{\mathrm{in}})\, f_{\mathrm{in}} \,\mathrm{d}v < +\infty.$$

From this last bound it is standard to obtain that $f_{\mathrm{in}} \in L^1_2 \cap L\log L(\mathbf{R}^3)$, namely that

$$f_{\mathrm{in}} \geqslant 0 \quad \text{and} \quad \int_{\mathbf{R}^3} (1 + |v|^2 + |\log f_{\mathrm{in}}|)\, f_{\mathrm{in}} \,\mathrm{d}v < +\infty, \tag{1.13}$$

where we denote, for $k \in \mathbf{R}$,

$$L^1_k(\mathbf{R}^3) = \left\{ f : \mathbf{R}^3 \to \mathbf{R} \;\Big|\; \int_{\mathbf{R}^3} \langle v \rangle^k |f| \,\mathrm{d}v < \infty \right\}, \quad \langle v \rangle = (1+|v|^2)^{\frac12},$$

and

$$L\log L(\mathbf{R}^3) = \left\{ f \in L^1(\mathbf{R}^3) \;\Big|\; \int_{\mathbf{R}^3} |f \log |f|| \,\mathrm{d}v < \infty \right\}.$$

We also suppose, without loss of generality, that f_{in} satisfies the normalization

$$\int_{\mathbf{R}^3} f_{\mathrm{in}} \,\mathrm{d}v = 1, \quad \int_{\mathbf{R}^3} v f_{\mathrm{in}} \,\mathrm{d}v = 0, \quad \int_{\mathbf{R}^3} |v|^2 f_{\mathrm{in}} \,\mathrm{d}v = 3, \tag{1.14}$$

and denote by

$$\mu(v) = \frac{1}{(2\pi)^{\frac32}}\, \mathrm{e}^{-\frac{|v|^2}{2}}$$

the Maxwellian equilibrium with same mass, momentum and energy than the initial data f_{in}.

The conservation laws and the H-theorem provides us with the following natural physical a priori estimates for a solution f to (1.12), namely

$$\sup_{t \geqslant 0} \int_{\mathbf{R}^3} (1 + |v|^2 + |\log f(t)|)\, f(t) \,\mathrm{d}v \leqslant C(f_{\mathrm{in}})$$

and

$$\int_0^\infty \mathscr{D}(f)(t) \,\mathrm{d}t \leqslant C(f_{\mathrm{in}}),$$

for some constant $C(f_{\mathrm{in}})$ depending on the mass, energy and entropy of f_{in}.

1.4.1 Well-posedness

From the previous a priori estimates, we define the notion of weak solutions as done in Villani [185].

Definition 1.1 (Weak solutions)**.** Let f_{in} satisfies (1.13). One says that f is a global weak solution to the spatially homogeneous Landau equation (1.12) if the following conditions are fulfilled:

(i) $f \geqslant 0$ is a nonnegative function verifying

$$f \in \mathscr{C}(\mathbf{R}_+; \mathscr{D}'(\mathbf{R}^3)) \cap L^\infty(\mathbf{R}_+; L^1_2(\mathbf{R}^3) \cap L \log L(\mathbf{R}^3)) \cap L^1_{\text{loc}}(\mathbf{R}_+; L^1_{2+\gamma}(\mathbf{R}^3));$$

(ii) the conservation of mass, momentum and energy hold;

(iii) the entropy inequality holds: for all $t \geqslant 0$ one has

$$\mathscr{H}(f)(t) + \int_0^t \mathscr{D}(f)(s)\,\mathrm{d}s \leqslant \mathscr{H}(f_{\text{in}});$$

(iv) f satisfies (1.12) in the distributional sense: for any $\varphi \in \mathscr{C}(\mathbf{R}_+; \mathscr{D}(\mathbf{R}^3))$ and all $t \geqslant 0$, there holds

$$\begin{aligned}\int_{\mathbf{R}^3} f(t)\varphi(t)\,\mathrm{d}v - \int_{\mathbf{R}^3} f_{\text{in}}\varphi(0)\,\mathrm{d}v - \int_0^t \int_{\mathbf{R}^3} f(s)\partial_t\varphi(s)\,\mathrm{d}v\,\mathrm{d}s \\ = \int_0^t \int_{\mathbf{R}^3} Q(f(s), f(s))\varphi(s)\,\mathrm{d}v\,\mathrm{d}s\end{aligned}$$

where $\int_{\mathbf{R}^3} Q(f,f)\varphi$ is defined by (1.8).

Villani [185] remarked that for the case $-3 \leqslant \gamma < -2$ the formulation (1.8) does not make sense, due to the appearance of a singular term that cannot be defined with the only assumption $f \in L^1_2(\mathbf{R}^3) \cap L \log L(\mathbf{R}^3)$. To circumvent this problem Villani [185] has defined a new notion of weak solutions, called H-solutions, which takes advantage of the a priori estimate for the entropy-dissipation $\mathscr{D}(f)$ and uses the the weak formulation (1.7), which is indeed well-defined thanks to (i) and the entropy-dissipation estimate $\mathscr{D}(f) \in L^1(\mathbf{R}_+)$.

Definition 1.2 (H-solutions)**.** Let f_{in} satisfies (1.13). One says that f is a global H-solution to the spatially homogeneous Landau equation (1.12) if f satisfies (i)–(iv) of Definition 1.1 with $\int_{\mathbf{R}^3} Q(f,f)\varphi$ being defined by (1.7).

In a recent work, Desvillettes [76] established a new estimate for the entropy-dissipation $\mathscr{D}(f)$ in the case $-3 \leqslant \gamma < 0$. From this estimate one can deduce that a H-solution satisfies $\langle v\rangle^\gamma f \in L^1_{\text{loc}}(\mathbf{R}_+, L^3(\mathbf{R}^3))$, which in turn implies that the weak formulation (1.8) makes sense and hence weak solutions and H-solutions are actually equivalent. Hereafter, we shall simply say that f is a weak solution.

Before presenting our results, let us now briefly describe some well-posedness results for the spatially homogeneous Landau equation (1.12) for all cases $-3 \leqslant \gamma \leqslant 1$. We shall always assume that the initial data f_{in} is a nonnegative function with finite mass, energy and entropy, namely satisfying (1.13).

We start by referring to the work of Villani [185] which proved the existence of global weak solutions (or equivalently H-solutions) for all potentials $-3 \leqslant \gamma \leqslant 1$, assuming in addition, in the case of hard potentials $0 < \gamma \leqslant 1$, that the initial data f_{in} lies in $L^1_{2+\delta}(\mathbf{R}^3)$ for some $\delta > 0$.

In the particular case of Maxwellian molecules $\gamma = 0$, Villani [186] proved the existence and uniqueness of global smooth $\mathscr{C}^\infty$ solutions. For the hard potentials case $0 < \gamma \leqslant 1$, Desvillettes-Villani [79] established existence of global smooth $\mathscr{C}^\infty$ solutions, and uniqueness is proven supposing in addition that $f_{\rm in}$ lies is some weighted L^2 space. More recently, Fournier-Guillin [103] obtained uniqueness assuming the initial data has some stretched exponential moments, that is $e^{|v|^\alpha} f_{\rm in} \in L^1(\mathbf{R}^3)$ for some α sufficiently large. For the above cases of Maxwellian molecules $\gamma = 0$ and hard potentials $0 < \gamma \leqslant 1$, there are some results regarding the smoothness effect beyond $\mathscr{C}^\infty$ regularity, and we refer the reader to Chen-Li-Xu [63, 62, 64], Morimoto-Pravda–Starov-Xu [156], Li-Xu [145] and the references therein.

In the case of moderately soft potentials $-2 < \gamma < 0$, Fournier-Guérin [101] have proved, thanks to a probabilistic approach, global uniqueness assuming further that the initial data lies in $L^1_k(\mathbf{R}^3)$ for some k sufficiently large. Furthermore, Alexandre-Lao-Lin [1] and Wu [191] established some global well-posedness in L^p spaces with $p > 1$. By using the uniqueness criteria of Fournier-Guérin [101] together with new a priori estimates established by the author in [47], we obtain uniqueness without further assumption on the initial data. Finally we mention the works of Silvestre [173] and Gualdani-Guillin [113] for some regularity results.

The case of very soft and Coulomb potentials is less understood. For very soft potentials $-3 < \gamma \leqslant -2$, Fournier-Guérin [101] has established local uniqueness assuming in addition that the initial data belongs to $L^p(\mathbf{R}^3)$ for some exponent p sufficiently large. Moreover, Alexandre-Lao-Lin [1] obtained some local well-posedness results in weighted L^2-spaces.

Finally, in the most important case of Coulomb potential $\gamma = -3$, Arsenev-Peskov [10] obtained local existence of bounded solutions assuming further that the initial data is bounded, and Fournier [100] established a uniqueness criteria which in turn yields local well-posedness of such bounded solutions. We also mention the work of Alexandre-Lao-Lin [1] for other local well-posedness results in weighted L^2-spaces. For some conditional regularity results, we refer to the recent works of Silvestre [173] and Gualdani-Guillen [113]. We finally mention the very recent works of Golse-Gualdani-Imbert-Vasseur [109] in which the authors proved a partial regularity result, and Chern-Gualdani [66] which established a new uniqueness criteria.

It is worth mentioning that uniqueness and regularity of solutions to the spatially homogeneous Landau equation with Coulomb potential $\gamma = -3$ is a major open problem.

We now turn our attention to the issue of convergence to equilibrium as discussed in Section 1.3. We define therefore the relative entropy of f with respect to the Maxwellian equilibrium μ by

$$\mathscr{H}(f|\mu) := \int_{\mathbf{R}^3} f \log\left(\frac{f}{\mu}\right) \mathrm{d}v \tag{1.15}$$

that verifies, thanks to (1.11) and the conservation laws,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{H}(f|\mu) = -\mathscr{D}(f) \leqslant 0. \tag{1.16}$$

One remarks that the relative entropy is a good way to measure the distance from f to the equilibrium μ since by the Csiszár-Kullback-Pinsker inequality one has

$$\|f - \mu\|^2_{L^1(\mathbf{R}^3)} \leqslant 2\mathscr{H}(f|\mu).$$

We shall now present our results concerning the long-time behavior of solutions to the spatially homogeneous Landau equation (1.12). In Section 1.4.3 we shall present the results of [47] for moderately soft potentials $-2 < \gamma < 0$, and Section 1.4.4 is devoted to the results of [50] for very soft and Coulomb potentials $-3 \leqslant \gamma \leqslant -2$. For the sake of completeness, we shall also briefly mention the known results in the case of hard potentials $0 < \gamma \leqslant 1$ and Maxwellian molecules $\gamma = 0$ in Section 1.4.2 below.

1.4.2 Long-time behavior for Maxwellian molecules and hard potentials

In the case of Maxwellian molecules $\gamma = 0$, Villani [186] and Desvillettes-Villani [80] have proved a linear functional inequality relating the relative entropy (1.15) and the entropy-dissipation (1.10) by constructive methods, namely

$$\mathscr{D}(f) \geqslant C_0 \mathscr{H}(f|\mu) \tag{1.17}$$

for some positive constant $C_0 > 0$ depending on the mass, energy and entropy of f. Thanks to this inequality, one deduces from (1.16) an exponential convergence to the equilibrium μ in relative entropy

$$\mathscr{H}(f(t)|\mu) \leqslant \mathrm{e}^{-C_0 t} \mathscr{H}(f_{\mathrm{in}}|\mu).$$

This kind of linear functional inequality relating the relative entropy and the entropy-dissipation is known as Cercignani's Conjecture in Boltzmann and Landau theory. Concerning the spatially homogeneous Boltzmann equation we refer to the works of Carlen-Carvalho [41, 42] and Villani [188] as well as the references therein; and for more details on the Cercignani's Conjecture we refer to the review of Desvillettes-Mouhot-Villani [78].

On the other hand, in the case of hard potentials $0 < \gamma \leqslant 1$, Desvillettes-Villani [80] proved another functional inequality relating the relative entropy and the entropy-dissipation, which now is not linear, by using the above linear inequality (1.17) together with an interpolation argument. They have then obtained

$$\mathscr{D}(f) \geqslant \min\left(C_1 \mathscr{H}(f|\mu), C_2 \mathscr{H}(f|\mu)^{1+\frac{\gamma}{2}} \right)$$

for some constructive constants $C_1, C_2 > 0$ depending on the mass, energy and entropy of f, from which we now obtain an algebraic decay to the equilibrium in relative entropy

$$\mathscr{H}(f(t)|\mu) \leqslant C_{\mathrm{in}} (1+t)^{-\frac{\gamma}{2}}$$

for some constant $C_{\mathrm{in}} > 0$ depending on the mass, energy and entropy of the initial data f_{in}.

By combining the above decay with some new spectral estimates for the linearized collision operator in large spaces and using the strategy described in Section 1.3.3, this result was later improved by the author in [46], where an optimal exponential decay to equilibrium was obtained:

$$\|f(t) - \mu\|_{L^1(\mathbf{R}^3)} \leqslant C_{\mathrm{in}}\, \mathrm{e}^{-\lambda_0 t}$$

for some constant $C_{\mathrm{in}} > 0$ depending on the mass, energy and entropy of the initial data f_{in} and where $\lambda_0 > 0$ is the spectral gap of the linearized collision operator (see Section 1.5.3 and estimate (1.33)). It is worth mentioning that this strategy, of obtaining spectral estimates in large spaces in order to be able to connect the nonlinear theory with the linearized one, was initiated by Mouhot [158] for the spatially homogeneous Boltzmann equation with hard potentials, and later it was fully developed in an abstract manner in Gualdani-Mischler-Mouhot [114].

1.4.3 Long-time behavior for moderately soft potentials

We shall consider weight functions $\omega = \omega(v) : \mathbf{R}^3 \to \mathbf{R}_+$ of the following form: polynomial weight functions

$$\omega(v) = \langle v \rangle^{\ell} \quad \text{with} \quad \ell > \ell(\gamma) := 8 + \frac{15}{2}\,|\gamma|, \tag{1.18a}$$

and (stretched) exponential weight functions

$$\omega(v) = \mathrm{e}^{\kappa \langle v \rangle^s} \quad \text{with} \quad \begin{cases} -\gamma \leqslant s < 2 \text{ and } \kappa > 0, & \text{or} \\ s = 2 \text{ and } 0 < \kappa < \frac{1}{2\mathrm{e}}. \end{cases} \tag{1.18b}$$

For each of these classes of weight function, we associate the decay function $\Theta_\omega : \mathbf{R}_+ \to \mathbf{R}_+$ defined as: if $\omega = \langle v \rangle^{\ell}$ verifies (1.18a) we define

$$\Theta_\omega(t) = (1+t)^{-\frac{1}{|\gamma|}(\ell - \ell(\gamma))}, \tag{1.19a}$$

and if $\omega = e^{\kappa\langle v\rangle^s}$ verifies (1.18b) we define

$$\Theta_\omega(t) = e^{-\lambda_0 t}, \tag{1.19b}$$

where $\lambda_0 > 0$ is the spectral gap of the linearized collision operator (see Section 1.5.3 and estimate (1.33)).

We are now able to state our result concerning the decay to equilibrium in the case of moderately soft potentials $-2 < \gamma < 0$ established in [47].

Theorem 1.A. *Assume $-2 < \gamma < 0$. Consider an initial data $f_{\rm in}$ satisfying (1.13) and (1.14). Then there is a unique global weak solution f to the spatially homogeneous Landau equation (1.2). Furthermore, if $\omega f_{\rm in} \in L^1(\mathbf{R}^3)$ with ω satisfying (1.18), then the following decay estimates hold:*

(i) *If ω satisfy (1.18a) then for all $t \geqslant 0$ one has*

$$\mathscr{H}(f(t)|\mu) \leqslant C\Theta_\omega(t),$$

for some constant $C > 0$.

(ii) *If ω satisfy (1.18b) then for all $t \geqslant 0$ one has*

$$\|f(t) - \mu\|_{L^1(\mathbf{R}^3)} \leqslant C\Theta_\omega(t),$$

for some constant $C > 0$.

The proof of point (i) in Theorem 1.A is based on an entropy method as described in Section 1.3.2 and which we shall present in more details below. Once this is achieved, the optimal exponential convergence in point (ii) is then obtained thanks to the decay estimate of point (i) together with some new spectral estimates for the linearized collision operator in large Banach spaces, following the strategy designed in Section 1.3.3. Since these new spectral estimates will be discussed later on in Section 1.5 in the more difficult framework of the spatially inhomogeneous Landau equation, we only focus here on entropy methods and point (i) of the theorem.

A crucial ingredient in the entropy method presented above, in the case of Maxwellian molecules and hard potentials $0 \leqslant \gamma \leqslant 1$, is that the polynomial moments of the solution f, that is the quantities $\int_{\mathbf{R}^d} \langle v\rangle^k f \, dv$, are uniformly bounded in time. However, this is not the case for soft potentials $\gamma < 0$, in which one has relatively bad control of the distribution tails. Toscani-Villani [177] have then developed a new entropy method that compensates this lack of uniform-in-time bounds by some precise logarithmic Sobolev inequalities. They have considered the case of *mollified soft potentials*, that is replacing the matrix a in (1.4) by a regularized version, more precisely

$$a^\dagger(z) = \Psi(z)\Pi^\perp(z) \qquad \text{with} \qquad c_\Psi(1+|z|^2)^{\frac{\gamma}{2}} \leqslant \frac{\Psi(z)}{|z|^2} \leqslant C_\Psi(1+|z|^2)^{\frac{\gamma}{2}},$$

for some constants $c_\Psi, C_\psi > 0$. Denoting $\mathscr{D}_{a^\dagger}$ the entropy-dissipation functional associated to $a^\dagger$, they proved that for all $k > 0$ there is $C_k > 0$ depending on k and on the mass, energy and entropy of f, such that

$$\mathscr{D}_{a^\dagger}(f) \geqslant C_k\, \mathscr{H}(f|\mu)^{1+\frac{|\gamma|}{k}} \left(\|\langle v\rangle^{k+2} f\|_{L^1} + \|\langle v\rangle^{k+2}\nabla\sqrt{f}\|_{L^2}^2 \right)^{-\frac{|\gamma|}{k}} \tag{1.20}$$

and the same estimate holds for the entropy-dissipation $\mathscr{D}(f)$ associated to the true kernel a. This entropy/entropy-dissipation estimate is a consequence of an interpolation argument using the linear entropy/entropy-dissipation inequality (1.17) in the case of Maxwellian molecules

$\gamma = 0$. As a consequence of this functional inequality and (1.16), if one is able to obtain energy estimates that control the growth of the quantity appearing in (1.20) as

$$\|\langle v\rangle^{k+2} f(t)\|_{L^1} + \|\langle v\rangle^{k+2}\nabla\sqrt{f}(t)\|_{L^2}^2 \leqslant C(1+t)^{\theta} \quad \text{with} \quad \theta < \frac{k}{|\gamma|},$$

then one obtains the following decay to equilibrium in relative entropy

$$\mathscr{H}(f(t)|\mu) \leqslant C(1+t)^{-\frac{k}{|\gamma|}+\theta}.$$

Toscani-Villani [177] indeed proved this in the case of mollified soft potentials $a^\dagger$ with $-3 < \gamma < 0$.

The difference between the true moderately soft potentials with matrix a with respect to its mollified version $a^\dagger$ is that we have removed the singularity at $z = 0$ in $a^\dagger$ (recall that we consider now the case $-2 < \gamma < 0$). Thus the energy estimates established by Toscani-Villani [177] in the mollified case does not apply to the case of moderately soft potentials.

We have hence investigated new a priori estimates for the Landau equation (1.12) with moderately soft potentials $-2 < \gamma < 0$. The first one concerns the evolution of polynomial moments, and we have obtained that, if a polynomial moment is initially finite $\|\langle v\rangle^k f_{\mathrm{in}}\|_{L^1(\mathbf{R}^3)} < \infty$, then one has

$$\|\langle v\rangle^k f(t)\|_{L^1(\mathbf{R}^3)} \leqslant C(1+t),$$

which already gives us with some slowly growing bound for one term appearing in the (1.20). Inspired by some new energy estimates established in Wu [191], concerning the propagation of L^p-norms, we have also proved the the appearance and slowly growing bounds of weighted H^2-norms: for all $t \geqslant 1$ one has

$$\|\langle v\rangle^k f(t)\|_{H^2(\mathbf{R}^3)} \leqslant C(1+t)^{7/2}.$$

The proof of Theorem 1.A-(i) is then a consequence of these new a priori estimates, as explained above.

1.4.4 Long-time behavior for very soft and Coulomb potentials

Consider weight functions $\omega = \omega(v) : \mathbf{R}^3 \to \mathbf{R}_+$ of the following form: polynomial weight functions

$$\omega(v) = \langle v\rangle^{\ell} \quad \text{with} \quad \ell > \ell(\gamma) := \frac{1}{4}\sqrt{25|\gamma|^2 - 24|\gamma| + 16} + \frac{7}{4}|\gamma| + 1, \tag{1.21a}$$

and stretched exponential weight functions

$$\omega(v) = \mathrm{e}^{\kappa\langle v\rangle^s} \quad \text{with} \quad \begin{cases} 0 < s < \frac{\gamma+4}{|\gamma+1|} \text{ and } \kappa > 0, \quad \text{or} \\ s = \frac{\gamma+4}{|\gamma+1|} \text{ and } 0 < \kappa < \frac{1}{\mathrm{e}}\frac{|\gamma+1|}{\gamma+4}. \end{cases} \tag{1.21b}$$

For each of these classes of weight function, we associate a (family of) decay functions $\Theta_\omega : \mathbf{R}_+ \to \mathbf{R}_+$ defined as: if $\omega = \langle v\rangle^{\ell}$ verifies (1.21a) we define

$$\Theta_\omega(t) = (1+t)^{-\beta_\ell} \tag{1.22a}$$

for any

$$0 < \beta_\ell < \frac{2\ell^2 - (7|\gamma|+4)\ell + (10+3|\gamma|)|\gamma|}{3|\gamma|(\ell-2)},$$

and if $\omega = \mathrm{e}^{\kappa\langle v\rangle^s}$ verifies (1.21b) we define

$$\Theta_\omega(t) = \exp\left(-c\frac{(1+t)^{\frac{s}{s+|\gamma|}}}{[\log(1+t)]^{\frac{|\gamma|}{s+|\gamma|}}}\right) \tag{1.22b}$$

for some constant $c > 0$.

We now state our result concerning the decay to equilibrium in the case of very soft and Coulomb potentials $-3 \leqslant \gamma \leqslant -2$ established in [50].

Theorem 1.B. *Assume $-3 \leqslant \gamma \leqslant -2$. Consider an initial data f_{in} satisfying (1.13) and (1.14). Let f be a global weak solution to the spatially homogeneous Landau equation (1.2). If $\omega f_{\text{in}} \in L^1(\mathbf{R}^3)$ with ω satisfying (1.21) then, for all $t \geqslant 0$ one has*

$$\mathscr{H}(f(t)|\mu) \leqslant C\Theta_\omega(t),$$

for some constant $C > 0$ and where Θ_ω is defined by (1.22).

By combining this result with new spectral estimates for the linearized collision operator, described in Section 1.5.2 below, and using the strategy described in Section 1.3.3, one can improve the rate of convergence and we obtain the following result in [55].

Corollary. *Under the framework of Theorem 1.B assume that ω satisfies (1.21b). Then for all $t \geqslant 0$ there holds*

$$\|f(t) - \mu\|_{L^1(\mathbf{R}^3)} \leqslant C\,\mathrm{e}^{-ct^{\frac{s}{|\gamma|}}}$$

for some constants $c, C > 0$.

In the case of very soft and Coulomb potentials $-3 \leqslant \gamma \leqslant -2$, apart from the natural a priori estimates presented above: conservation of mass, momentum and energy, and the entropy estimate from the H-theorem, we do not know any other global a priori estimate. Therefore, one cannot hope to use the same strategy as for the case $-2 < \gamma < 0$ presented above by taking advantage of the entropy/entropy-dissipation estimate (1.20) of Toscani-Villani [177], which is valid for soft potentials with $-3 < \gamma < 0$.

However, as said before, the estimate (1.20) is obtained by an interpolation argument using the linear entropy/entropy-dissipation estimate for the Maxwellian case $\gamma = 0$ established by Desvillettes-Villani [80]. Therefore one could imagine that there is room for improvement in the entropy/entropy-dissipation estimate for soft potentials.

A first important result in that direction is the work of Desvillettes [76] which established a new entropy-dissipation estimate for potentials in the range $-3 \leqslant \gamma < 0$ that reads

$$\|\langle v\rangle^{\frac{\gamma}{2}} \nabla \sqrt{f}\|^2_{L^2(\mathbf{R}^3)} \leqslant \mathscr{D}(f) + C_f \tag{1.23}$$

where the constant $C_f > 0$ depends only on the mass, energy and entropy of f. This inequality was obtained by following a strategy similar in spirit to the proof of the linear entropy/entropy-dissipation estimate (1.17) for $\gamma = 0$ by Desvillettes-Villani [80]. One remarks that this estimate implies better integrability bounds on the solution, namely $\langle v\rangle^\gamma f \in L^1_{\text{loc}}(\mathbf{R}_+; L^3(\mathbf{R}^3))$, which in turn yields that weak solutions and H-solutions are equivalent, as already discussed before. However this estimate cannot be used to tackle the trend-to-equilibrium issue.

Let us now describe the main ideas behind the proof of Theorem 1.B.

1.4.5 A variant of the entropy method

Our first step is to look for new functional inequalities relating $\mathscr{H}(f|\mu)$ and $\mathscr{D}(f)$. Inspired by the new inequality (1.23) of Desvilletes [76], more precisely by its proof, we first obtain a new entropy-dissipation estimate that bounds from below the entropy-dissipation $\mathscr{D}(f)$ by a weighted relative Fisher information of f with respect to the associated Maxwellian distribution μ. More precisely, for any $k \in \mathbf{R}$ we define the weighted relative Fisher information $I_k(f|\mu)$ of f with respect to μ by

$$I_k(f|\mu) := \int_{\mathbf{R}^3} \langle v\rangle^k \left|\nabla \log\left(\frac{f}{\mu}\right)\right|^2 f\,\mathrm{d}v = \int_{\mathbf{R}^3} \langle v\rangle^k \left|\frac{\nabla f}{f} + v\right|^2 f\,\mathrm{d}v.$$

We then obtain:

Theorem 1.1. *Assume $-3 \leqslant \gamma < 0$. There exists a constant $C > 0$, depending on the mass, energy and entropy of f, such that*

$$\mathscr{D}(f) \geqslant C \, \|\langle v \rangle^{2+|\gamma|} f\|_{L^1}^{-1} \, I_\gamma(f|\mu).$$

As a consequence of this last estimate, using the logarithmic Sobolev inequality, we shall prove a variant of the so-called weak Cercignani's conjecture for the Landau equation:

Corollary. *There holds*

$$\mathscr{D}(f) \geqslant C \, \|\langle v \rangle^{2+|\gamma|} f\|_{L^1(\mathbf{R}^3)}^{-1} \int_{\mathbf{R}^3} \left(f \log \left(\frac{Z_1}{Z_2} \frac{f}{\mu} \right) + \frac{Z_2}{Z_2} \mu - f \right) \mathrm{d}v \tag{1.24}$$

where $Z_1 = \int_{\mathbf{R}^3} \langle v \rangle^\gamma \mu \, \mathrm{d}v$ and $Z_2 = \int_{\mathbf{R}^3} \langle v \rangle^\gamma f \, \mathrm{d}v$. As a consequence, for any $R > 0$ there holds

$$\begin{aligned} \mathscr{D}(f) \geqslant C \, \|\langle v \rangle^{2+|\gamma|} f\|_{L^1(\mathbf{R}^3)}^{-1} \, R^{-3} \Bigg(\mathscr{H}(f|\mu) - \int_{\langle v \rangle \geqslant R} f \log f \, \mathrm{d}v \\ - C' \int_{\langle v \rangle \geqslant R} \langle v \rangle^2 \, f \, \mathrm{d}v - C' \int_{\langle v \rangle \geqslant R} \mu \, \mathrm{d}v \Bigg), \end{aligned} \tag{1.25}$$

for some constant $C' > 0$.

As already explained, after obtaining this new entropy-dissipation estimates, we do not follow the usual arguments as in the entropy method presented in Section 1.4.3 because this would give us an inequality of the type (1.20) and hence would require a control of some weighted and high-order regularity bounds on the solution. However, in the case $-3 \leqslant \gamma \leqslant -2$ we do not know any a priori regularity estimate. The only estimate at hand, besides the natural physical a priori estimates appearing in the definition of weak or H-solutions, is the new estimate (1.23) which actually uses the entropy-dissipation.

We shall instead plug (1.25) in the differential inequality

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathscr{H}(f|\mu) \leqslant -\mathscr{D}(f)$$

which keeps the exponent 1 in the relative entropy $\mathscr{H}(f|\mu)$, at the price of the appearance of remainder terms, by choosing some $R = R(t)$ depending on time. These remainder terms are then controlled thanks to (1.23) and new bounds on the propagation of moments, and, finally, only at the very end shall we choose $R(t)$ in a suitable way to close a differential inequality for the relative entropy $\mathscr{H}(f|\mu)$. Solving this differential inequality gives us the decay in Theorem 1.B.

1.5 The spatially inhomogeneous equation

The celebrated work of DiPerna-Lions [84, 86] introduced the notion of renormalized solution for the Boltzmann equation (with cutoff collision kernels) and proved the global existence of renormalized solutions for large initial data. Inspired by this and by the work of Lions [147] which established some compactness properties for the Boltzmann and Landau operators, Villani [184] and Alexandre-Villani [5, 4] introduced the notion of renormalized solution with defect measure for the Landau equation and the Boltzmann equation for long-range interaction, i.e. for non-cutoff collision kernels, and proved the global existence of such solutions for large initial data. The question of existence of global smooth solutions for the Boltzmann and Landau equations is an outstanding open problem.

Concerning the trend-to-equilibrium problem, Desvillettes-Villani [83] proved the decay to equilibrium for a priori smooth solutions satisfying some uniform bounds for both Boltzmann and Landau equations.

The Cauchy theory for the Landau equation in a perturbative framework, that is, in a close-to-equilibrium regime, was initiated by the breakthrough work of Guo [115], which constructed a unique global classical solution to the Landau equation (1.1) in the torus $\mathbf{T}^3_x$ for all cases $-3 \leqslant \gamma \leqslant 1$, including thus the most physically interesting case of Coulomb potential $\gamma = -3$. For this Guo [115] introduced a novel nonlinear energy method, which has since been proven to be successful in several other situations, see for instance the work Guo [118] on the Vlasov-Poisson-Landau equation and the references therein. The solution constructed in Guo [115] was proven to be smooth by Chen-Desvillettes-He [65]. In regards to the decay to equilibrium, we shall mention the work of Mouhot-Neumann [160] and Yu [193], which proved the exponential convergence to equilibrium for the cases $-2 \leqslant \gamma \leqslant 1$. Furthermore, in the range $-3 \leqslant \gamma \leqslant -2$, Guo-Strain [174, 175] extended the method of Guo [115] and proved stretched exponential convergence to equilibrium. All the above results in a close-to-equilibrium framework hold in high-order Sobolev spaces with fast decay in the velocity variable. Concerning the well-posedness and the decay to equilibrium in a close-to-equilibrium setting for the cutoff Boltzmann equation we shall only refer to the pioneering work of Ukai [179], as well as to Ukai-Asano [180], Caflisch [37], and to the more recent works of Guo [116] and Gualdani-Mischler-Mouhot [114] and the references therein; whereas for the non-cutoff Boltzmann equation, whose structure shares similarities with the Landau equation, we refer to the pioneering works of Gressman-Strain [112] and Alexandre-Morimoto-Ukai-Xu-Yang [3, 2], as well as to the recent work Hérau-Tonon-Tristani [126] and the references therein.

Still in a close-to-equilibrium setting, we also mention the very recent results in which an initial boundary-value problem for Landau equation with Coulomb potential was solved and the unique solution was shown to converge to the equilibrium: the work of Guo-Hwang-Jang-Ouyang [119] which treats the case of a bounded domain with specular reflection boundary condition, and the work of Duan-Liu-Sakamoto-Strain [92] which considers the case of a finite channel domain with inflow or specular reflection boundary conditions.

Another perturbative framework in which one can study the Boltzmann and the Landau equation in the whole space $\mathbf{R}^3_x$ corresponds to a close-to-vacuum regime. Concerning the cutoff Boltzmann equation we refer to the work of Illner-Shinbrot [131]; and for the Landau equation we mention that the stability of the vacuum was recently established by Luk [149] for moderately soft potentials, and by Chaturvedi [60] for hard potentials.

It is worth mentioning that, very recently, a new research program has started aiming the study of conditional regularity properties of solutions to both Boltzmann and Landau equations. We only mention in this direction the works of Golse-Imbert-Mouhot-Vasseur [110] and Henderson-Snelson [124] concerning the smoothness of solutions to the Landau equation, and we refer the reader to the references therein as well as to the surveys of Mouhot [159] and Imbert-Silvestre [132].

We consider now the spatially inhomogeneous Landau equation (1.1) in a close-to-equilibrium setting. We shall always consider, without loss of generality, that the initial condition $f_{\rm in}$ satisfies the normalization

$$\int f_{\rm in}(x,v)\,\mathrm{d}x\,\mathrm{d}v = 1, \quad \int v f_{\rm in}(x,v)\,\mathrm{d}x\,\mathrm{d}v = 0, \quad \int |v|^2 f_{\rm in}(x,v)\,\mathrm{d}x\,\mathrm{d}v = 3,$$

and denote $\mu = (2\pi)^{-\frac{3}{2}} \mathrm{e}^{-\frac{|v|^2}{2}}$ the global Maxwellian equilibrium with same mass, momentum and energy than $f_{\rm in}$ (considering the normalization $|\mathbf{T}^3_x| = 1$).

The results that we shall present below concern the existence, uniqueness and the convergence to the equilibrium of solutions to (1.1) for initial data $f_{\rm in}$ sufficiently close to the equilibrium μ in some weighted Sobolev spaces. We split them into the cases $-2 \leqslant \gamma \leqslant 1$, in which exponential convergence is obtained, and $-3 \leqslant \gamma < -2$, for which algebraic and stretched exponential convergence is achieved.

1.5.1 Hard, Maxwellian, and moderately soft potentials

For the case of hard potentials $0 < \gamma \leqslant 1$ and Maxwellian molecules $\gamma = 0$, we consider weight functions $\omega = \omega(v) : \mathbf{R}^3 \to \mathbf{R}_+$ of the form: polynomial weight functions verifying

$$\omega(v) = \langle v \rangle^{\ell} \quad \text{with} \quad \ell > 3\gamma/2 + 7 + 3/2, \tag{1.26a}$$

and (stretched) exponential weight functions satisfying

$$\omega(v) = \mathrm{e}^{\kappa \langle v \rangle^s} \quad \text{with} \begin{cases} 0 < s < 2 \text{ and } \kappa > 0, & \text{or} \\ s = 2 \text{ and } 0 < \kappa < \frac{1}{2}. \end{cases} \tag{1.26b}$$

For moderately soft potentials $-2 \leqslant \gamma < 0$, we consider (stretched) exponential weight functions satisfying

$$\omega(v) = \mathrm{e}^{\kappa \langle v \rangle^s} \quad \text{with} \quad \begin{cases} -\gamma < s < 2 \text{ and } \kappa > 0, & \text{or} \\ s = 2 \text{ and } 0 < \kappa < \frac{1}{2}. \end{cases} \tag{1.26c}$$

We define the weighted Sobolev-type space $\mathcal{H}^3_x L^2_v(\omega)$ as the space associated to the norm

$$\|f\|_{\mathcal{H}^3_x L^2_v(\omega)} := \left(\sum_{\ell=0}^{3} \| \langle v \rangle^{-\ell(1-\frac{s}{2})} \, \nabla^{\ell}_x (\omega f) \|^2_{L^2_{x,v}} \right)^{1/2}$$

where $L^2_{x,v} = L^2(\mathbf{T}^3_x \times \mathbf{R}^3_v)$ is the usual Lebesgue space in $\mathbf{T}^3_x \times \mathbf{R}^3_v$, and we use hereafter the convention that $s = 0$ in the case where ω is a polynomial weight function. Let us denote by $\widetilde{\nabla}_v$ the anisotropic gradient

$$\widetilde{\nabla}_v f = \mathrm{pr}_v(\nabla_v f) + \langle v \rangle \, (\mathrm{id} - \mathrm{pr}_v) \nabla_v f$$

where pr_v denotes the projection onto the v-direction, that is, for any $\xi \in \mathbf{R}^3$,

$$\mathrm{pr}_v(\xi) = \left(\frac{v}{|v|} \cdot \xi \right) \frac{v}{|v|}.$$

Define the weighted Sobolev-type space $H^1_{v,\star}(\omega)$ in the velocity variable as the space associated to the norm

$$\|f\|_{H^1_{v,\star}(\omega)} := \left(\| \langle v \rangle^{\frac{\gamma+s}{2}} \omega f \|^2_{L^2_v} + \| \langle v \rangle^{\frac{\gamma}{2}} \widetilde{\nabla}_v (\omega f) \|^2_{L^2_v} \right)^{1/2}. \tag{1.27}$$

In a similar fashion as above, we define the weighted space $\mathcal{H}^3_x(H^1_{v,\star}(\omega))$ as the space associated to the norm

$$\|f\|^2_{\mathcal{H}^3_x(H^1_{v,\star}(\omega))} := \left(\sum_{\ell=0}^{3} \| \langle v \rangle^{-\ell(1-\frac{s}{2})} \, \nabla^{\ell}_x f \|^2_{L^2_x(H^1_{v,\star}(\omega))} \right)^{1/2},$$

where

$$\|g\|^2_{L^2_x(H^1_{v,\star}(\omega))} := \int_{\mathbf{T}^3_x} \|g\|^2_{H^1_{v,\star}(\omega)} \, \mathrm{d}x.$$

We are now able to state our main result in the case $-2 \leqslant \gamma \leqslant 1$ established in [57].

Theorem 1.C. *Assume that $-2 \leqslant \gamma \leqslant 1$. Let ω be a weight function verifying (1.26). There is a constant $\varepsilon_0 = \varepsilon_0(\omega) > 0$ such that if $\|f_{\mathrm{in}} - \mu\|_{\mathcal{H}^3_x L^2_v(\omega)} \leqslant \varepsilon_0$ then:*

(i) *There exists a unique global weak solution f to (1.1) that verifies*

$$\sup_{t \geqslant 0} \|f(t) - \mu\|^2_{\mathcal{H}^3_x L^2_v(\omega)} + \int_0^{\infty} \|f(t) - \mu\|^2_{\mathcal{H}^3_x(H^1_{v,\star}(\omega))} \, \mathrm{d}t \leqslant C \varepsilon_0^2,$$

for some constant $C > 0$.

(ii) *The solution satisfies the decay estimate, for all $t \geqslant 0$,*

$$\|f(t) - \mu\|_{\mathcal{H}^3_x L^2_v(\omega)} \leqslant C\, \mathrm{e}^{-\lambda t} \|f_{\mathrm{in}} - \mu\|_{\mathcal{H}^3_x L^2_v(\omega)},$$

for any $0 < \lambda < \lambda_0$ and some constant $C > 0$, where $\lambda_0 > 0$ is the spectral gap of the linearized collision operator (see Section 1.5.3 and estimate (1.33)).

It is worth mentioning that all the constants are constructive and that our theorem improves previous results on the well-posedness and decay in a close-to-equilibrium setting by considering larger spaces, that is, with less restrictive weights and less derivatives, in particular no derivatives in the velocity variable.

On the one hand, the work of Guo [115] has established, for all cases $-3 \leqslant \gamma \leqslant 1$, existence and uniqueness of solutions in high-order Sobolev space with fast decay in velocity

$$H^N_{x,v}(\mu^{-\frac{1}{2}}) = \left\{ f : \mathbf{T}^3_x \times \mathbf{R}^3_v \to \mathbf{R} \mid \mu^{-\frac{1}{2}} f \in H^N(\mathbf{T}^3_x \times \mathbf{R}^3_v) \right\}$$

for $N \geqslant 8$. On the other hand, we have also enlarged the spaces in which the exponential convergence to equilibrium occurs, improving them previous results of convergence from Yu [193] in $H^N_{x,v}(\mu^{-\frac{1}{2}})$ for $N \geqslant 8$, and in Mouhot-Neumann [160] in $H^N_{x,v}(\mu^{-\frac{1}{2}})$ for $N \geqslant 4$.

Our strategy is completely different from the nonlinear energy method developed by Guo [115]. It is based on simple nonlinear estimates for the operator Q and a trapping argument, which are then combined with spectral estimates for the linearized operator as well as decay and regularization estimates for the associated semigroup in the corresponding functional spaces. These new spectral and semigroup estimates are obtained by using the extension theory developed by Gualdani-Mischler-Mouhot [114], and introduced earlier by Mouhot [158] to study the spatially homogeneous Boltzmann equation. It is worth mentioning at this point that the issue of combining the linear estimates together with the nonlinear ones is subtle, insofar as the gain induced by the linear part is a priori not sufficiently strong to control the regularity loss coming from the nonlinear term.

The strategy behind the proof of Theorem 1.C follows the same philosophy of the proof of Theorem 1.D below, the main difference being the fact that, for potentials in the range $-2 \leqslant \gamma \leqslant 1$ as in the setting of Theorem 1.C, the linearized collision operator possesses a spectral gap (see Section 1.5.3 below), whereas in the case $-3 \leqslant \gamma \leqslant -2$ as in the framework of Theorem 1.D it does not. This fact also explains why we get exponential convergence in Theorem 1.C, but algebraic or stretched exponential convergence in Theorem 1.D.

Since the case where the linearized operator does not have a spectral gap is more difficult to handle, i.e. the case $-3 \leqslant \gamma \leqslant -2$, we shall therefore only present the proof of Theorem 1.D in Section 1.5.4 below.

1.5.2 Very soft and Coulomb potentials

For any $-3 \leqslant \gamma \leqslant -2$, we shall consider weight functions $\omega = \omega(v) : \mathbf{R}^3 \to \mathbf{R}_+$ of the form: polynomial weight functions verifying

$$\omega(v) = \langle v \rangle^{\ell} \quad \text{with} \quad \ell > 2 + \frac{3}{2}, \tag{1.28a}$$

and (stretched) exponential weight functions satisfying

$$\omega(v) = \mathrm{e}^{\kappa \langle v \rangle^s} \quad \text{with} \quad \begin{cases} 0 < s < 2 \text{ and } \kappa > 0, & \text{or} \\ s = 2 \text{ and } 0 < \kappa < \frac{1}{2}. \end{cases} \tag{1.28b}$$

For each of these classes of weight functions, we associate a (family of) decay functions $\Theta_\omega : \mathbf{R}_+ \to \mathbf{R}_+$ defined as: if $\omega = \langle v \rangle^{\ell}$ verifies (1.21a) we define

$$\Theta_\omega(t) = (1+t)^{-\frac{\ell - \ell_\star}{|\gamma|}} \tag{1.29a}$$

for any $0 < \ell_\star < 2 + \frac{3}{2}$, and if $\omega = e^{\kappa \langle v \rangle^s}$ satisfies (1.21b) we define

$$\Theta_\omega(t) = e^{-ct^{\frac{s}{|\gamma|}}} \tag{1.29b}$$

where $c > 0$ is some constant.

We define the weighted space $H^2_x L^2_v(\omega)$ as the Sobolev-type space associated to the norm

$$\|f\|_{H^2_x L^2_v(\omega)} := \left(\sum_{\ell=0}^{2} \|\nabla^\ell_x (\omega h)\|^2_{L^2_{x,v}} \right)^{1/2},$$

as well as the higher-order weighted space $H^2_x(H^1_{v,\star}(\omega))$ as the space associated to the norm

$$\|f\|^2_{H^2_x(H^1_{v,\star}(\omega))} := \left(\sum_{\ell=0}^{2} \|\nabla^\ell_x f\|^2_{L^2_x(H^1_{v,\star}(\omega))} \right)^{1/2},$$

where the space $H^1_{v,\star}(\omega)$ is defined in (1.27).

We can now state our main result in the case $-3 \leqslant \gamma \leqslant -2$ established in [55].

Theorem 1.D. *Assume $-3 \leqslant \gamma \leqslant -2$. Consider a weight ω satisfying (1.28). There exists $\varepsilon_0 = \varepsilon_0(\omega) > 0$ small enough such that if $\|f_{\mathrm{in}} - \mu\|_{H^2_x L^2_v(\omega)} \leqslant \varepsilon_0$ then:*

(i) *There exists a unique global weak solution f to (1.1) that verifies*

$$\sup_{t \geqslant 0} \|f(t) - \mu\|^2_{H^2_x L^2_v(\omega)} + \int_0^\infty \|f(t) - \mu\|^2_{H^2_x(H^1_{v,\star}(\omega))} \, dt \leqslant C\varepsilon_0^2,$$

for some constant $C > 0$.

(ii) *The solution satisfies the decay estimate: for all $t \geqslant 0$ there holds*

$$\|f(t) - \mu\|_{H^2_x L^2_v} \leqslant C\Theta_\omega(t) \|f_{\mathrm{in}} - \mu\|_{H^2_x L^2_v(\omega)}.$$

for some constant $C > 0$.

This result is constructive and improves to larger spaces $H^2_x L^2_v(\omega)$ the well-posedness theory of Guo [115] established in $H^N_{x,v}(\mu^{\frac{1}{2}})$ for $N \geqslant 8$, as well as the convergence to equilibrium of Guo-Strain [174, 175] established in the spaces $H^N_{x,v}(\mu^{-\theta})$ for $N \geqslant 8$ and $\theta \in (\frac{1}{2}, 1)$.

The proof involves two distinguished parts: simple nonlinear estimates for the Landau collision operator Q and a trapping argument; and stability/regularization estimates for the semigroup associated to the linearized operator in the corresponding spaces. Our strategy is mostly based on these semigroup stability/regularization estimates and in order to obtain such estimates we develop in [55] a method to prove non-uniform (non-exponential) stability estimates of semigroups in large functional spaces, by taking advantage of a *weak coercivity estimate* in one small reference space and developing then an enlargement trick for *weakly dissipative operators*.

This enlargement trick we develop is inspired by the extension theory developed by Gualdani-Mischler-Mouhot [114], and introduced earlier by Mouhot [158] to study the spatially homogeneous Boltzmann equation. This extension theory consists in enlarging the functional spaces in which spectral gap estimates as well as semigroup estimates are valid for operators satisfying a suitable factorization. It is worth emphasizing that the theory of Gualdani-Mischler-Mouhot [114] treats operators having a spectral gap, whereas in [55] we address the problem of operators that does not possess a spectral gap but only a *weak coercivity estimate*.

As a corollary of Theorem 1.D, we are able to improve the algebraic rate of convergence to equilibrium established by Desvillettes-Villani [83] in a non perturbative setting but assuming a priori bounds on the solution, in the following way:

Corollary. *Assume $-3 \leqslant \gamma \leqslant -2$. Consider a global strong solution f to the spatially inhomogeneous Landau equation* (1.1) *such that*

$$\sup_{t\geqslant 0}\Big(\|f(t)\|_{H^\ell_{x,v}} + \|\omega f(t)\|_{L^1_{x,v}}\Big) < +\infty,$$

for some explicit $\ell \geqslant 3$ large enough and some exponential weight function $\omega = \mathrm{e}^{\kappa\langle v\rangle^s}$ satisfying (1.28b)*. Assume further that the spatial density is uniformly positive on the torus, that is $\displaystyle\int_{\mathbf{R}^3} f(t,x,v)\,\mathrm{d}v > 0$ for all $t \geqslant 0$ and any $x \in \mathbf{T}^3$. Then this solution satisfies the following decay estimate: for all $t \geqslant 0$ one has*

$$\|f(t)-\mu\|_{H^2_x L^2_v} \leqslant C\,\mathrm{e}^{-ct^{\frac{s}{|\gamma|}}},$$

for some constants $c, C > 0$.

Let us now describe the main ideas behind the proof of Theorem 1.D. For f a solution to (1.1) we define therefore the perturbation

$$h := f - \mu.$$

Since the operator Q is bilinear and the Maxwellian μ is an equilibrium of the equation, one obtains that $Q(\mu+h,\mu+h) = Q(\mu,h) + Q(h,\mu) + Q(h,h)$. Therefore, defining the linearized collision operator L by

$$Lh := Q(\mu,h) + Q(h,\mu), \tag{1.30}$$

and defining the full linearized operator Λ by

$$\Lambda h := Lh - v\cdot\nabla_x h, \tag{1.31}$$

the perturbation h satisfies the equation

$$\partial_t h = \Lambda h + Q(h,h) \tag{1.32}$$

which is complemented with the initial datum $h_{\text{in}} = f_{\text{in}} - \mu$.

1.5.3 Linearized collision operator

Since the linearized collision operator L, defined in (1.30), will play an important role in our strategy, we gather here some known results concerning coercivity estimates.

The natural space for studying L is the Hilbert space $L^2_v(\mu^{-\frac12}) = \{f : \mathbf{R}^3 \to \mathbf{R} \mid \int_{\mathbf{R}^3} f^2\mu^{-1}\,\mathrm{d}v < \infty\}$ which is endowed with the norm

$$\|f\|_{L^2_v(\mu^{-1/2})} = \left(\int_{\mathbf{R}^3} f^2\mu^{-1}\,\mathrm{d}v\right)^{1/2}.$$

and the scalar product

$$\langle f,g\rangle_{L^2_v(\mu^{-1/2})} = \int_{\mathbf{R}^3} fg\,\mu^{-1}\,\mathrm{d}v.$$

Indeed, a straightforward computation gives

$$\begin{aligned}\langle Lf,g\rangle_{L^2_v(\mu^{-\frac12})} = -\frac12\int_{\mathbf{R}^3\times\mathbf{R}^3} a(v-v_*)&\Big\{\nabla_v(\mu^{-1}f)(v) - \nabla_{v_*}(\mu^{-1}f)(v_*)\Big\}\\ &\Big\{\nabla_v(\mu^{-1}g)(v) - \nabla_{v_*}(\mu^{-1}g)(v_*)\Big\}\,\mu(v_*)\mu(v)\,\mathrm{d}v_*\,\mathrm{d}v,\end{aligned}$$

which is a consequence of a linearization of estimate (1.10) in the H-theorem. Therefore the operator $L : D(L) \subseteq L^2_v(\mu^{-\frac{1}{2}}) \to L^2_v(\mu^{-\frac{1}{2}})$ is self-adjoint and nonnegative, so that its spectrum $\mathrm{sp}(L)$ is included in $\mathbf{R}_-$. Moreover, thanks to the conservation laws, its kernel is given by

$$\ker(L) = \mathrm{span}\{\mu, v_1\mu, v_2, \mu, v_3\mu, |v|^2\mu\}.$$

Several authors have then investigated (weak) coercivity properties for L. Summarizing results of Degond-Lemou [72], Guo [115], Baranger-Mouhot [12], Mouhot [157] and Mouhot-Strain [161], there is a constructive constant $\lambda_0 > 0$ such that for any f in the domain $D(L)$ of L one has

$$\langle -Lf, f\rangle_{L^2_v(\mu^{-1/2})} \geqslant \lambda_0 \|f - \pi f\|^2_{H^1_{v,\star}(\mu^{-1/2})} \tag{1.33}$$

where we recall that $H^1_{v,\star}(\mu^{-1/2})$ is defined in (1.27), and where π is the projection onto $\ker(L)$ given by

$$\pi f = \left(\int_{\mathbf{R}^3} f \,\mathrm{d}w\right)\mu + \left(\int_{\mathbf{R}^3} wf \,\mathrm{d}w\right)\cdot v\mu + \left(\int_{\mathbf{R}^3} \frac{(|w|^2-3)}{\sqrt{6}} f \,\mathrm{d}w\right)\frac{(|v|^2-3)}{\sqrt{6}}\,\mu.$$

One remarks that actually we also have the reverse inequality of (1.33) and therefore the operator L has a spectral gap, when acting on the space $L^2_v(\mu^{-\frac{1}{2}})$, if and only if $\gamma + 2 \geqslant 0$.

1.5.4 A semigroup-based approach

We shall focus our attention to the full linearized operator Λ defined in (1.31). Our aim is to obtain non-uniform (non-exponential) stability estimates of the semigroup S_Λ associated to Λ in various Hilbert spaces, as well as some regularization estimates. These estimates are a key ingredient of our method and they are obtained in several steps that we shall describe below.

Weak coercivity in a "small/reference" space

Our starting point is the weak coercivity estimate (1.33) satisfied by linearized collision operator L, when acting on the space $E_0 := L^2_v(\mu^{-1/2})$.

We then adapt the hypocoercive method developed by Mouhot-Neumann [160], for operators satisfying a spectral gap estimate, in order to deduce some weak coercivity estimate for the full linearized operator $\Lambda = L - v\cdot\nabla_x$. At first sight, the operator Λ possess coercivity properties only in the velocity variable v coming from the collision operator L. The method of Mouhot-Neumann [160] consists then in constructing a new norm in some H^1-type space for which the interplay between the collision and transport operators results in a gain of coercivity in the the spatial variable x. We can generalize this to operators satisfying a weak coercivity estimate as above, and we are able to obtain that the full linearized operator $\Lambda = L - v\cdot\nabla_x$ satisfies a weak coercivity estimate in some small/reference Hilbert space. More precisely, let us define the weighted Hilbert space $\mathcal{H}^1_{x,v}(\omega)$ as the space associated to the norm

$$\|f\|_{\mathcal{H}^1_{x,v}(\omega)} := \left(\|\omega f\|^2_{L^2_{x,v}} + \|\nabla_x(\omega f)\|^2_{L^2_{x,v}} + \|\langle v\rangle^{\frac{s}{4}-\frac{3}{2}}\,\nabla_v(\omega f)\|^2_{L^2_{x,v}}\right)^{1/2}. \tag{1.34}$$

Then on the space $E := \mathcal{H}^1_{x,v}(\mu^{-\frac{1}{2}})$ we can construct a new twisted norm $|||\cdot|||_E$, associated to a scalar product $\langle\langle\cdot,\rangle\rangle_E$, that is equivalent to the $\|\cdot\|_E$-norm defined in (1.34) and for which one has the following weak coercivity estimate

$$\langle\langle \Lambda h, h\rangle\rangle_E \lesssim -\|\Pi^\perp h\|^2_{E_\star} \quad \text{for any} \quad h \in D(\Lambda_{|E}), \tag{1.35}$$

where $D(\Lambda_{|E})$ is the domain of Λ when acting on E, $\Pi^\perp$ denotes the projection onto the orthogonal of $\ker(\Lambda)$, and $E_\star$ is another Hilbert space that is not included in E.

Factorization of the operator

For weight functions ω verifying (1.28), one consider the Hilbert spaces

$$X := H^2_x L^2_v(\omega) \subseteq X_0 := H^2_x L^2_v \quad \text{and} \quad X_\star := H^2_x(H^1_{v,\star}(\omega))$$

so that $X_\star$ is not contained in X, as well as

$$E_1 := \mathcal{H}^1_{x,v}(\mathrm{e}^{\kappa\langle v\rangle^2}) \quad \text{with} \quad \frac{1}{4} < \kappa < \frac{1}{2}$$

in such a way that $E_1 \subset E$.

We factorize the operator as $\Lambda = A + B$ with

$$Ah = Q(h,\mu) + R'\chi_R h \quad \text{and} \quad Bh = Q(\mu,h) - R'\chi_R h - v\cdot\nabla_x h,$$

where $R, R' > 0$ are constants to be chosen sufficiently large and χ_R is a smooth cutoff function. In this way the operators A and B satisfy the following properties:

— A is bounded from X_0 into X, and from E into E_1;

— B is *weakly dissipative* in X, in the sense

$$\langle Bh, h\rangle_X \lesssim -\|h\|^2_{X_\star} \quad \text{for any} \quad h \in D(B_{|X}), \tag{1.36a}$$

where $D(B_{|X})$ denotes the domain of B when acting on X;

— B is 0-*dissipative* in E_1, in the sense

$$\langle Bh, h\rangle_{E_1} \leqslant 0 \quad \text{for any} \quad h \in D(B_{|E_1}), \tag{1.36b}$$

where $D(B_{|E_1})$ denotes the domain of B when acting on E_1;

— AS_B and S_BA have some regularization properties, namely that there are $\ell, n \in \mathbf{N}^*$ such that

$$t \mapsto \|(AS_B)^{\star n}(t)\|_{\mathscr{L}(X,E_1)} \in L^1(\mathbf{R}_+)$$

and

$$t \mapsto \|(S_BA)^{\star \ell}(t)\|_{\mathscr{L}(E,X_0)} \in L^1(\mathbf{R}_+),$$

where the convolution product is defined by $S_1 \star S_2(t) = \int_0^t S_1(s)S_2(t-s)\,\mathrm{d}s$, $S^{\star 0} = \mathrm{id}$ and $S^{\star n} = S \star S^{\star(n-1)}$ for any $n \in \mathbf{N}^*$.

We observe here that one cannot deduce any decay estimate on the associated semigroup $\Pi^\perp S_\Lambda$ (resp. S_B) directly from inequality (1.35) (resp. inequality (1.36a)). This framework of weakly dissipative operators is hence more difficult to handle than the classical dissipative case, in which an analogous estimate is obtained with $E_\star = E$ (resp. $X_\star = X$), which in turn implies an exponential decay estimate for the associated semigroup.

Decay and regularization estimates for S_B

It is however possible to deduce the (non-uniform) stability of S_B by using estimate (1.36a) for different choices of Hilbert spaces X, that is with different choices of weight functions ω, together with an interpolation argument relating these different spaces. More precisely, one can first obtain the following stability estimate for S_B: for all $t \geqslant 0$ there holds

$$\|S_B(t)\|_{\mathscr{L}(X,X_0)} \lesssim \Theta(t) \tag{1.37}$$

where $\Theta = \Theta_{X,X_0} : \mathbf{R}^+ \to \mathbf{R}^+$ is a function that decays $\Theta(t) \to 0$ as $t \to \infty$ with an algebraic or a stretched exponential rate. Furthermore, one can obtain a regularization estimate: for all $t \geqslant 0$ there holds

$$\|S_B(t)\|_{\mathscr{L}(X'_\star, X_0)} \lesssim \frac{\Theta^\star(t)}{\min(1,\sqrt{t})} \tag{1.38}$$

where $\Theta^\star = \Theta^\star_{X'_\star, X_0} : \mathbf{R}^+ \to \mathbf{R}^+$ is a function that decays $\Theta(t) \to 0$ as $t \to \infty$ with an algebraic rate, and $X'_\star := H^2_x(H^{-1}_{v,\star}(\omega))$ is the dual space of $X_\star$ endowed with the norm

$$\|f\|_{H^2_x(H^{-1}_{v,\star}(\omega))} := \sup_{\|\phi\|_{H^2_x(H^1_{v,\star}(\omega))} \leqslant 1} \sum_{j=0}^{2} \iint_{\mathbf{T}^3 \times \mathbf{R}^3} \nabla^j_x f \nabla^j_x \phi \, \omega^2 \, \mathrm{d}v \, \mathrm{d}x \tag{1.39}$$

Extension argument

We next use an extension trick in order to deduce that $S_\Lambda \Pi^\perp$ enjoys the same decay and regularization properties of the semigroup S_B.

First of all, we deduce from the previous estimates (1.35) and (1.36b) together with an interpolation argument (as for the obtention of (1.37)), the stability of $S_\Lambda \Pi^\perp$ in the small/reference space E_1, more precisely that for all $t \geqslant 0$ there holds

$$\|S_\Lambda \Pi^\perp(t)\|_{\mathscr{L}(E_1,E)} \leqslant C \mathrm{e}^{-\kappa t^{\frac{2}{|\gamma|}}}, \tag{1.40}$$

for some constants $C, \kappa > 0$.

By using then the factorization $\Lambda = A + B$ and writing a truncated Dyson-Phillips series

$$\begin{aligned} S_\Lambda \Pi^\perp = &\sum_{0 \leqslant j \leqslant \ell-1} \Pi^\perp S_B \star (A S_B)^{(\star j)} + \sum_{0 \leqslant i \leqslant n-1} (S_B A)^{(\star i)} \star S_B \Pi^\perp \star (A S_B)^{(\star \ell)} \\ &+ (S_B A)^{(\star n)} \star S_\Lambda \Pi^\perp \star (A S_B)^{(\star \ell)}, \end{aligned}$$

one can deduce with the previous estimates that $S_\Lambda \Pi^\perp$ satisfies the the decay estimate (1.37) and the regularization estimate (1.38).

Weak dissipativity for Λ

From the above stability estimates for $S_\Lambda \Pi^\perp$ we now deduce the corresponding weak dissipative estimates for Λ. This is a crucial ingredient in order to treat the nonlinear equation, for with the stability estimate alone we cannot capture the regularization properties of the operator Λ that is needed to control the loss of regularity coming from the nonlinear term.

We define a new norm on $\Pi^\perp X$ by

$$|||h|||^2_X := \eta \|h\|^2_X + \int_0^\infty \|S_\Lambda(s) h\|^2_{X_0} \, \mathrm{d}s, \tag{1.41}$$

for a constant $\eta > 0$ small enough. This norm is equivalent to the $\|\cdot\|_X$-norm on $\Pi^\perp X$, and Λ satisfies the weak dissipativity estimate

$$\langle\!\langle \Lambda h, h \rangle\!\rangle_X \leqslant -K \|\Pi^\perp h\|^2_{X_\star} \quad \text{for any} \quad h \in D(\Lambda_{|X}), \tag{1.42}$$

for some constant $K > 0$ and where $\langle\!\langle \cdot, \cdot \rangle\!\rangle_X$ stands for the duality bracket associated to the $|||\cdot|||_X$-norm.

Nonlinear stability

We now come back to the nonlinear equation (1.32). By using (1.42) together with nonlinear estimates for controlling the term $\langle\!\langle Q(h,h), h\rangle\!\rangle_X$, one obtains the following a priori estimate

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\Pi^\perp h\|_X^2 \leqslant -K\|\Pi^\perp h\|_{X_\star}^2 + C\|\Pi^\perp h\|_{X_\star}^2\,\|\Pi^\perp h\|_X,$$

for some constants $K, C > 0$. The existence and uniqueness results are a consequence of this last estimate by supposing that $\|h_{\mathrm{in}}\|_X$ is sufficiently small and employing a standard iterative scheme, which thus gives us point (i) in Theorem 1.D.

We finally deduce the decay estimate of point (ii) in Theorem 1.D by using the above a priori estimate with different choices of weights ω and using an interpolation argument (as for the obtention of (1.37)).

1.6 Some perspectives

1.6.1 Regularity for the homogeneous Landau equation with Coulomb potential

As already explained in Section 1.4, uniqueness and regularity of solution to the spatially homogeneous Landau equation with Coulomb potential is still an open problem.

The new entropy-dissipation estimate of Desvillettes [76] was shown to produce new interesting information on the Landau equation. On the one hand, it already gave some better integrability bounds on the solutions. On the other hand, we were able in [50] to prove a variant of this entropy-dissipation that was suitable to attack the trend-to-equilibrium issue.

An interesting question would be to use some variant of this new entropy-dissipation of Desvillettes [76] together with the new techniques developed in Silvestre [173] and Golse-Gualdani-Imbert-Vasseur [109] in order to obtain some new results on the (partial) regularity of solutions.

1.6.2 The Vlasov–Maxwell–Landau system

The Vlasov–Maxwell–Landau system is a fundamental model in plasma physics that describes the evolution of the density of charged particles $f = f(t,x,v) \geqslant 0$ taking into account two phenomena: the collision between particles which is described by the Landau collision operator with Coulomb potential $\gamma = -3$; and the collective mean-field self-inducted force generated by the distribution of particles, which is described by the Lorentz force associated to the electromagnetic field produced by the mass density $\varrho_f = \int_{\mathbf{R}^3} f\,\mathrm{d}v$ and the current $j_f = \int_{\mathbf{R}^3} vf\,\mathrm{d}v$. More precisely it reads

$$\partial_t f + v\cdot\nabla_x f + (E + v\wedge B)\cdot\nabla_v f = Q(f,f), \tag{1.43}$$

where the electromagnetic field $(E(t,x), B(t,x))$ satisfies Maxwell's equations

$$\begin{cases}\partial_t E - \nabla_x \wedge B = -j_f,\\ \partial_t B + \nabla_x\wedge E = 0,\\ \nabla_x\cdot E = \rho_f - 1, \quad \nabla_x\cdot B = 0,\end{cases} \tag{1.44}$$

assuming that particles are confined in the torus $\mathbf{T}^3$.

One could hope that the methods and estimates of [55] could be applied in order to obtain existence, uniqueness and convergence to the equilibrium μ in a perturbative framework. The new difficult appearing here comes from the hyperbolic nature of Maxwell's equation, which would require to be compensated by the linearized estimates coming from the collision operator.

This issue has been handled in the case of high-order Sobolev spaces $H^N_{x,v}(\mu^{-\frac{1}{2}})$ with fast decay in velocity, see for instance the works of Duan-Strain [93] for the Vlasov–Maxell–Boltzmann equation, Duan [90] for the Vlasov–Maxwell–Landau equation, and the references therein. We believe an interesting question would be to combine the approach of Duan-Strain [93] and Duan [90] with the new stability estimates of [55], in order to extend these results to larger functional spaces.

1.6.3 Landau damping and entropy dissipation

The Vlasov–Poisson system is a classical model in kinetic theory describing the evolution of a plasma taking into account the self-generated electric field. It reads

$$\partial_t f + v \cdot \nabla_x f + E \cdot \nabla_v f = 0,$$

where $f = f(t, x, v)$ is the distribution of particles, $t \in \mathbf{R}_+$ the time variable, $x \in \mathbf{T}^3$ the spatial variable and $v \in \mathbf{R}^3$ the velocity variable. The electric field $E = E(t, x)$ obeys Poisson's equation

$$E = -\nabla_x \Phi * (\rho_f - 1) \quad \text{with} \quad \Phi(x) = \frac{1}{|x|}.$$

Although the Vlasov–Poisson system is Hamiltonian, Landau predicted (by looking to the linearized equation) that a damping phenomena can occur, known as *Landau damping*, impliying the stability of homogeneous equilibra of the system. Recently the nonlinear Landau damping was proved in the celebrated paper of Mouhot-Villani [162] for initial data in Gevrey class, see also Bedrossian-Masmoudi-Mouhot [18].

By considering the Vlasov–Poisson–Landau system

$$\partial_t f + v \cdot \nabla_x f + E \cdot \nabla_v f = \sigma Q(f, f),$$

where $\sigma > 0$ is a constant, an interesting question would be to understand how the Landau damping phenomenon interacts with the entropic dissipation coming from the collision operator Q. For instance, one could hope to enlarge the class of initial data in which the Landau damping occurs by exploiting the regularization properties of collisions. Moreover, since these two phenomena are supposed to happen in different time scales, one would like to quantify them with respect to the constant $\sigma > 0$ measuring the strength of collisions.

We mention that in fluid mechanics, an analogous *inviscid damping* phenomenon occurs for the two-dimensional incompressible Euler equation, as proved by Bedrossian-Masmoudi [17]. Later, Bedrossian-Masmoudi-Vicol [19] proved an *enhanced dissipation* phenomenon for the two-dimensional incompressible Navier–Stokes equation by combining the effect of the inviscid damping and the dissipation of the Navier–Stokes term.

Chapter 2

Linear kinetic equations with confining potential

In this chapter we present the works [51] in collaboration with J. Dolbeault, F. Hérau, S. Mischler, and C. Mouhot; and [52] jointly with J. Dolbeault, F. Hérau, S. Mischler, C. Mouhot, and C. Schmeiser.

2.1 Introduction

We are interested in this chapter in the evolution of a large system of particles in the whole space $\mathbf{R}^d$ whose interaction is modeled by a linear kinetic collision operator satisfying the natural local conservation laws of mass, momentum and energy, and which are confined by a given exterior potential $\phi = \phi(x) : \mathbf{R}^d \to \mathbf{R}$. The evolution of such a system is described by the following equation

$$\partial_t f + v \cdot \nabla_x f - \nabla_x \phi \cdot \nabla_v f = \mathscr{C} f. \tag{2.1}$$

which is complemented with an initial datum f_{in}. The unknown $f = f(t, x, v)$ represents the (perturbation of) distribution of particles that at time $t \in \mathbf{R}_+$ and position $x \in \mathbf{R}^3$ possesses velocity $v \in \mathbf{R}^d$. The operator $\mathscr{C}$ represents the collision operator and

$$\mathscr{T} := -v \cdot \nabla_x + \nabla_x \phi \cdot \nabla_v$$

the transport operator, so that one can rewrite equation (2.1) in the form

$$\partial_t f = \mathscr{C} f + \mathscr{T} f.$$

The *linear collision operator* $\mathscr{C}$ acts only on the velocity variable $v \in \mathbf{R}^d$, is a nonnegative self-adjoint operator in the space $L^2(\mathrm{d}\mu^{-1})$, where we denote

$$\mu(v) := \frac{1}{(2\pi)^{\frac{d}{2}}} \, \mathrm{e}^{-\frac{|v|^2}{2}},$$

and

$$L^2(\mathrm{d}\mu^{-1}) := \left\{ g : \mathbf{R}^d \to \mathbf{R} \;\middle|\; \int_{\mathbf{R}^d} g^2 \mu^{-1} \, \mathrm{d}v < \infty \right\},$$

and its kernel is a $(d+2)$-dimensional space given by

$$\ker(\mathscr{C}) = \operatorname{span} \left\{ \mu, v_1 \mu, \dots, v_d \mu, |v|^2 \mu \right\},$$

which is consequence of the local conservation laws for the mass, momentum and energy. One remarks that typical examples of collision operators satisfying those assumptions are the linearized Boltzmann and Landau operators around the Maxwellian μ.

We suppose that the collision operator $\mathscr{C}$ satisfies a quantitative version of the *spatially homogeneous linearized H-theorem*, namely that there exists a *spectral gap* constant $\lambda_{\mathscr{C}} > 0$ such that for any f in the domain $D(\mathscr{C})$ of the operator $\mathscr{C}$ one has

$$- \int_{\mathbf{R}^d} (\mathscr{C} f(v))\, f(v) \mu^{-1}(v)\, \mathrm{d}v \geqslant \lambda_{\mathscr{C}} \| f - \Pi f \|^2_{L^2(\mathrm{d}\mu^{-1})}, \tag{2.2a}$$

where Π denotes the $L^2(\mathrm{d}\mu^{-1})$-projection onto $\ker(\mathscr{C})$. We suppose moreover that any polynomial function $p(v) : \mathbf{R}^d \to \mathbf{R}$ of degree less or equal than 4 belongs to the domain $D(\mathscr{C})$ of $\mathscr{C}$, which implies in particular that there exists a constant $C_{\mathscr{C}}$ such that

$$\left| \int_{\mathbf{R}^d} (\mathscr{C} f(v))\, p(v) \mu^{-1}(v)\, \mathrm{d}v \right| \leqslant C_{\mathscr{C}} \| f - \Pi f \|_{L^2(\mu^{-1}\mathrm{d}v)} \tag{2.2b}$$

for any such polynomial.

Concerning the potential ϕ, we shall assume through the chapter that ϕ is confining, in the sense that $\mathrm{e}^{-\phi}\mathrm{d}x$ is a probability measure. Furthermore, we suppose that $\mathrm{e}^{-\phi}\mathrm{d}x$ satisfies the following *Poincaré inequality*, which can be seen as a spectral gap property in space, namely that there exists a positive constant $C_{\mathrm{P}} > 0$ such that

$$\int_{\mathbf{R}^d} |\varphi(x) - \langle \varphi \rangle|^2\, \mathrm{e}^{-\phi(x)}\, \mathrm{d}x \leqslant C_{\mathrm{P}} \int_{\mathbf{R}^d} |\nabla \varphi(x)|^2\, \mathrm{e}^{-\phi(x)}\, \mathrm{d}x, \tag{2.3a}$$

for all $\varphi \in H^1(\mathrm{e}^{-\phi}\mathrm{d}x)$ and where $\langle \varphi \rangle := \int \varphi(x) \mathrm{e}^{-\phi(x)}\, \mathrm{d}x$ denotes the mean of φ. We also suppose the following regularity and growth bounds

$$\phi \in \mathscr{C}^2(\mathbf{R}^d) \quad \text{and} \quad \lim_{|x| \to +\infty} \frac{|\nabla_x^2 \phi(x)|}{1 + |\nabla_x \phi(x)|^2} = 0 \tag{2.3b}$$

as well as that $\mathrm{e}^{-\phi}$ has enough bounded moments, more precisely that there exists a constant C_ϕ such that

$$\int_{\mathbf{R}^d} \left(|x|^4 + |\phi|^2 + |\nabla_x \phi|^4 + |\nabla_x^2 \phi|^2 + |\nabla_x \phi|^2 |\nabla_x^2 \phi|^2 \right) \mathrm{e}^{-\phi(x)}\, \mathrm{d}x \leqslant C_\phi; \tag{2.3c}$$

Without loss of generality, we normalize ϕ by affine transformations in such a way that

$$\int_{\mathbf{R}^d} x\, \mathrm{e}^{-\phi(x)}\, \mathrm{d}x = 0, \qquad \int_{\mathbf{R}^d} \nabla_x^2 \phi(x)\, \mathrm{e}^{-\phi(x)}\, \mathrm{d}x = \mathrm{I}_d. \tag{2.3d}$$

where $\nabla_x^2 \phi$ denotes the Hessian matrix of ϕ and I_d the identity matrix of size d.

In Section 2.2 we investigate some fundamental properties of the equation (2.1) and present its conservation laws as well as the associated equilibria and stationary solutions.

After that, in Section 2.3, we present our main result concerning the long-time behavior of solutions to (2.1). We shall state therein an exponential decay of solutions with quantitative estimates, which enters in the class of *hypocoercivity* results in kinetic theory. Loosely speaking, hypocoercivity refers to the study of evolution equations described by an operator which is the sum of a degenerate dissipative operator and a conservative one, and for which the mixing of the degenerate dissipative part and the conservative part lead to the convergence to an equilibrium state. In our setting, one observes that the collision operator $\mathscr{C}$ provides coercivity in the velocity variable v only, and the the conservative skew-symmetric transport operator $\mathscr{T}$ does not provide any dissipative property at all. The idea is that the interplay between the $\mathscr{C}$ and $\mathscr{T}$, which mixes the two variables of position and velocity, could provides us with coercivity in the missing spatial variable x. More precisely, we shall introduce new norms or spaces in which one can capture the missing coercivity. This is a subtle issue, for with the change of norms or spaces one could lose the good dissipativity properties of the collision operator $\mathscr{C}$.

In the above hypocoercivity result, three main functional tools will appear to be crucial in our strategy. The first of them consists in a set of Poincaré-type inequalities, the classical one being our assumption (2.3a) above. Related to these inequalities, the second important tool regards properties of a Witten-Laplace operator associated to the potential ϕ. Finally, the third ingredient concerns a set of new Korn-type inequalities in the whole space. All these ingredients have their own interest, and we shall finally present a complete study of them in Section 2.4.

2.2 Conservation laws, equilibria and stationary solutions

We gather in this section the conservation laws associated to (2.1) as well as the equilibria and stationary solutions associated to them. The first two of them correspond to the mass and energy conservation and are always valid. Another third conservation law appears when the potential ϕ possesses rotational invariance. Finally, some additional conditional conservation laws are present when the potential presents harmonicity in some or all directions.

Hereafter we hence assume that f is a solution of (2.1).

2.2.1 Mass

One first note that the conservation of mass reads

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbf{R}^{2d}} f(t,x,v)\,\mathrm{d}x\,\mathrm{d}v = 0. \tag{2.4}$$

We hence define the generalized Maxwellian

$$\mathscr{M}(x,v) := \mathrm{e}^{-\phi(x)}\mu(v) = \frac{1}{(2\pi)^{\frac{d}{2}}}\,\mathrm{e}^{-\phi(x)-\frac{|v|^2}{2}} \tag{2.5}$$

which is an equilibrium of (2.1), more precisely $\mathscr{T}\mathscr{M} = \mathscr{C}\mathscr{M} = 0$.

2.2.2 Energy

The second conservation law concerns the (centered) energy. Defining the Hamiltonian function

$$\mathcal{H}(x,v) := \frac{|v|^2 - d}{2} + \phi(x) - \langle\phi\rangle\,,$$

one easily computes

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbf{R}^{2d}} \mathcal{H}(x,v) f(t,x,v)\,\mathrm{d}x\,\mathrm{d}v = 0. \tag{2.6}$$

One can then check that the function

$$\mathcal{H}\mathscr{M} \tag{2.7}$$

is an equilibrium of (2.1), more precisely $\mathscr{T}(\mathcal{H}\mathscr{M}) = \mathscr{C}(\mathcal{H}\mathscr{M}) = 0$.

2.2.3 Rotational invariance of ϕ

When ϕ possesses rotational invariance then an additional conservation law appears. For any anti-symmetric $(d\times d)$-matrix with real coefficients $A \in \mathfrak{M}^a_{d,d}(\mathbf{R})$ we define the linear map $R_A : x \mapsto Ax$ from $\mathbf{R}^d$ into $\mathbf{R}^d$. We define the set of infinitesimal rotation linear maps

$$\mathcal{R} := \{R_A : A \in \mathfrak{M}^a_{d,d}(\mathbf{R})\} \tag{2.8}$$

as well as the subspace of infinitesimal rotation linear maps compatible with ϕ

$$\mathcal{R}_\phi := \{R_A \in \mathcal{R} \mid \nabla_x \phi(x) \cdot R_A(x) = 0 \text{ for any } x \in \mathbf{R}^d\}. \tag{2.9}$$

Therefore if $R_A \in \mathcal{R}_\phi$ then one obtains the conservation law

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbf{R}^{2d}} R_A(x) \cdot v f(t,x,v) \,\mathrm{d}x\,\mathrm{d}v = 0. \tag{2.10}$$

Hence defining the function

$$F_{\mathrm{rig}}(x,v) = R_A(x) \cdot v \mathscr{M}(x,v) = Ax \cdot v \mathscr{M}(x,v) \tag{2.11}$$

one easily verifies that F_{rig} is a equilibrium to (2.1). We then define the set of infinitesimal densities compatible with ϕ by

$$\mathfrak{R}_\phi := \{(x,v) \mapsto R_A(x) \cdot v \mathscr{M}(x,v) : R_A \in \mathcal{R}_\phi\}.$$

2.2.4 Harmonicity of ϕ

At this stage we shall consider three different cases depending on harmonicity properties of the potential ϕ. More precisely, defining

$$\mathfrak{H}_\phi := \mathrm{span}\{\nabla_x \phi(x) - x : x \in \mathbf{R}^d\}, \quad d_\phi := \dim(\mathfrak{H}_\phi),$$

we hence split our analysis into the three following cases.

The fully non-harmonic case

The *fully non-harmonic case* is given by $d_\phi = d$, which corresponds to the situation where there is no direction in which ϕ is harmonic. In this case, there are no more conservation laws besides the ones presented above.

The partially harmonic case

The *partially harmonic case* is given by $1 \leqslant d_\phi \leqslant d-1$, which corresponds to the situation where there are $d - d_\phi$ directions in which ϕ is harmonic, more precisely $\partial_{x_i}\phi = x_i$ in those directions. In this case we shall work with a coordinate system where x_i for $i \in \{d_\phi + 1, \dots, d\}$ denote the harmonic coordinates.

This means that the confinement is *harmonic* in the direction x_i, and there is an additional 2-*cycle (almost) conservation law* given by

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\mathbf{R}^{2d}} x_i f \,\mathrm{d}x\,\mathrm{d}v\right) &= \left(\int_{\mathbf{R}^{2d}} v_i f \,\mathrm{d}x\,\mathrm{d}v\right), \\ \frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\mathbf{R}^{2d}} v_i f \,\mathrm{d}x\,\mathrm{d}v\right) &= -\left(\int_{\mathbf{R}^{2d}} x_i f \,\mathrm{d}x\,\mathrm{d}v\right), \end{aligned}$$

which implies that these two global quantities evolve as a scalar harmonic oscillator with period 1. In this case, some harmonic directional modes may appear. More precisely, defining the constants

$$\gamma_i = \int_{\mathbf{R}^{2d}} x_i f_{\mathrm{in}} \,\mathrm{d}x\,\mathrm{d}v \quad \text{and} \quad \bar{\gamma}_i = \int_{\mathbf{R}^{2d}} v_i f_{\mathrm{in}} \,\mathrm{d}x\,\mathrm{d}v$$

for any $i \in \{d_\phi + 1, \cdots, d\}$, we introduce the function

$$F_{\mathrm{dir}}(t,x,v) = \sum_{i=d_\phi+1}^{d} \gamma_i (x\cos(t) + v\sin(t))\mathscr{M} + \bar{\gamma}_i (v_i \cos(t) - x_i \sin(t))\mathscr{M}. \tag{2.12}$$

One easily observes that F_{dir} is a solution of (2.1) and one can check that $f - F_{\mathrm{dir}}$ satisfies the following additional conservation law, for any $i \in \{d_\phi + 1, \dots, d\}$ and any $t \geqslant 0$, there holds

$$\int_{\mathbf{R}^{2d}} x_i (f(t,x,v) - F_{\mathrm{dir}}(t,x,v)) \,\mathrm{d}x \,\mathrm{d}v = \int_{\mathbf{R}^{2d}} v_i (f(t,x,v) - F_{\mathrm{dir}}(t,x,v)) \,\mathrm{d}x \,\mathrm{d}v = 0. \tag{2.13}$$

We then define the set of harmonic modes

$$\begin{aligned} \mathfrak{D}_\phi = \operatorname{span} \Big\{ &(t,x,v) \mapsto (x_i \cos t + v_i \sin t) \mathscr{M}(x,v), \\ &\qquad (t,x,v) \mapsto (v_i \cos t - x_i \sin t) \mathscr{M}(x,v) \;:\; i \in \{d_\phi + 1, \cdots, d\} \Big\}, \end{aligned}$$

with the convention $\mathfrak{D}_\phi = \varnothing$ if $d_\phi = d$. On observes that these modes appear as stationary solutions corresponding to an inertia-driven oscillation of the particles in the potential well, where the local density oscillates around the mean in a direction in $\ker(\mathfrak{H}_\phi)$ with period 1. These modes can be superposed independently along different harmonic coordinates, and independently of the previous stationary solutions.

The fully harmonic case

The *fully harmonic case* corresponds to $d_\phi = 0$, that is when $\phi(x) = \frac{1}{2}|x|^2 + \frac{d}{2} \log 2\pi$ is the harmonic potential.

In this case, all the coordinates are harmonic so that the almost conservation law (2.13) holds for any $i \in \{1, \dots, d\}$ with F_{dir} defined by (2.12) with $d_\phi = 0$.

Furthermore, there is an additional 2-*cycle (almost) conservation law* given by

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbf{R}^{2d}} (x \cdot v) f \,\mathrm{d}x \,\mathrm{d}v &= -2 \int_{\mathbf{R}^{2d}} \frac{(|x|^2 - |v|^2)}{2} f \,\mathrm{d}x \,\mathrm{d}v, \\ \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbf{R}^{2d}} \frac{(|x|^2 - |v|^2)}{2} f \,\mathrm{d}x \,\mathrm{d}v &= 2 \int_{\mathbf{R}^{2d}} (x \cdot v) f \,\mathrm{d}x \,\mathrm{d}v, \end{aligned}$$

which implies that these two global quantities evolves as a scalar harmonic oscillator with period 2. We hence introduce the constants

$$\delta = \frac{1}{d} \int_{\mathbf{R}^{2d}} (x \cdot v) f_{\mathrm{in}} \,\mathrm{d}x \,\mathrm{d}v \quad \text{and} \quad \bar{\delta} = \frac{1}{d} \int_{\mathbf{R}^{2d}} \frac{1}{2} \left(|x|^2 - |v|^2 \right) f_{\mathrm{in}} \,\mathrm{d}x \,\mathrm{d}v,$$

as well as the function

$$\begin{aligned} F_{\mathrm{pul}}(t,x,v) = \delta &\left(x \cdot v \cos(2t) + \frac{(|x|^2 - |v|^2)}{2} \sin(2t) \right) \mathscr{M}(x,v) \\ &+ \bar{\delta} \left(\frac{(|x|^2 - |v|^2)}{2} \cos(2t) - x \cdot v \sin(2t) \right) \mathscr{M}(x,v). \end{aligned} \tag{2.14}$$

One easily verifies that this function is a solution of (2.1) and one checks that $f - F_{\mathrm{pul}}$ satisfies the additional conservation law, for any $t \geqslant 0$,

$$\begin{aligned} &\int_{\mathbf{R}^{2d}} (x \cdot v)(f(t,x,v) - F_{\mathrm{pul}}(t,x,v)) \,\mathrm{d}x \,\mathrm{d}v \\ &\qquad = \int_{\mathbf{R}^{2d}} \frac{1}{2} \left(|x|^2 - |v|^2 \right) (f(t,x,v) - F_{\mathrm{pul}}(t,x,v)) \,\mathrm{d}x \,\mathrm{d}v = 0. \end{aligned} \tag{2.15}$$

We then define the set of pulsating modes

$$\mathfrak{P}_\phi = \operatorname{span} \Big\{ (t,x,v) \mapsto \left(x \cdot v \cos(2t) + \frac{|x|^2 - |v|^2}{2} \sin(2t) \right) \mathscr{M}(x,v),$$

$$(t,x,v) \mapsto \left(\frac{|x|^2 - |v|^2}{2} \cos(2t) - x \cdot v \sin(2t) \right) \mathscr{M}(x,v) \Big\},$$

with the convention $\mathfrak{P}_\phi = \varnothing$ if $d_\phi \neq 0$. Remark that these modes corresponds to a radially symmetric pulsation of the particles in the potential well with period 2.

2.3 Long-time behavior

We shall now present in this section our result concerning the long-time behavior of solutions to (2.1). As already explained in the introduction, we shall obtain a hypocoercivity result showing that the solution to (2.1) converges exponentially fast to the equilibrium and stationary solutions presented in Section 2.2.

Hypocoercivity theory for kinetic equations has received a lot of attention in recent years and there is a quite extensive literature on the subject. We do not attempt in giving a exhaustive list of works here but we shall only mention some of them and we refer to the references therein.

We start by referring to the work of Villani [189] which started a systematic study of hypocoercivity in an abstract framework. We also mention that earlier works by Hérau-Nier [125], Eckmann-Hairer [95], and Helffer-Nier [123] had established the exponential convergence to equilibrium for spatially inhomogeneous Fokker-Planck equations and more general linear hypoelliptic equations. Still in the case of Fokker-Planck equations, new results were recently obtained by Mischler-Mouhot [154] using the factorization method developed in Gualdani-Mischler-Mouhot [114].

We also mention the celebrated paper by Desvillettes-Villani [81, 83] which developed a nonlinear method in order to prove the convergence to equilibrium for the spatially inhomogeneous linear Fokker-Planck equation as well as for the spatially inhomogeneous nonlinear Boltzmann and Landau equations for a priori smooth solutions. We also mention that, in a different context of solutions in a close-to-equilibrium framework, Guo [115] developed a new method for studying the Landau equation with Coulomb potential, which was then extended by Guo-Strain [174, 175] who obtained quantitative rates of convergences for both Boltzmann and Landau equations with soft potentials.

Moreover, we shall mention the work of Mouhot-Neumann [160] which developed a hypocoercive method to prove the exponential convergence to equilibrium for Boltzmann and Landau equations in the torus. Furthermore, Dolbeault-Mouhot-Schmeiser [88] have recently introduced a new abstract method for obtaining hypocoercive results for linear kinetic equations that conserves only the mass.

It is worth mentioning that none of the above methods is capable to handle our framework of a linear kinetic operator with all physical local conservation laws (mass, momentum, and energy) in a confining potential. Indeed, the above results dealing with a confining potential in the whole space $\mathbf{R}^3_x$ are only valid for linear kinetic equations with only one conservation law, as for instance in Dolbeault-Mouhot-Schmeiser [88].

The only results on the equation (2.1) we are aware of correspond to the result of Duan [89] who proved the exponential decay for the linear Boltzmann operator with confinement giving by the harmonic potential $\phi(x) = |x|^2/2 + d\log(2\pi)$ and for well-prepared initial data, and the work of Duan-Li [91] who extended the previous result for the linear Boltzmann equation by considering more general potentials satisfying some rigidity properties. It is worth mentioning that both results of Duan [89] and Duan-Li [91] are non-constructive.

Let us now describe our result. We denote by $L^2(\mathrm{d}\mathscr{M}^{-1})$ the Lebesgue space

$$L^2(\mathrm{d}\mathscr{M}^{-1}) = \left\{ f : \mathbf{R}^d \times \mathbf{R}^d \to \mathbf{R} \text{ measurable} \;\Big|\; \int_{\mathbf{R}^{2d}} |f(x,v)|^2 \mathscr{M}^{-1}(x,v)\,\mathrm{d}x\,\mathrm{d}v < \infty \right\}$$

endowed with the norm

$$\|f\|_{L^2(\mathrm{d}\mathscr{M}^{-1})} := \left(\int_{\mathbf{R}^{2d}} |f(x,v)|^2 \mathscr{M}^{-1}(x,v)\,\mathrm{d}x\,\mathrm{d}v\right)^{1/2}.$$

We obtain in [52] the following result.

Theorem 2.A. *Assume that the potential ϕ and the collision operator $\mathscr{C}$ satisfy hypothesis (2.3) and (2.2). There exists positive constants $\lambda, C > 0$ such that if $f \in \mathscr{C}(\mathbf{R}^+; L^2(\mathrm{d}\mathscr{M}^{-1}))$ is the solution to (2.1) associated to the initial datum $f_{\mathrm{in}} \in L^2(\mathrm{d}\mathscr{M}^{-1})$, then there are constants $\alpha, \beta \in \mathbf{R}$ and functions $F_{\mathrm{rig}} \in \mathfrak{R}_\phi$, $F_{\mathrm{dir}} \in \mathfrak{D}_\phi$ and $F_{\mathrm{pul}} \in \mathfrak{P}_\phi$ depending on f_{in} such that, for all $t \geqslant 0$, one has*

$$\|f(t) - \{\alpha\mathscr{M} + \beta\mathcal{H}\mathscr{M} + F_{\mathrm{rig}} + F_{\mathrm{dir}}(t) + F_{\mathrm{pul}}(t)\}\|_{L^2(\mathrm{d}\mathscr{M}^{-1})} \leqslant C\,\mathrm{e}^{-\lambda t}\|f_{\mathrm{in}}\|_{L^2(\mathrm{d}\mathscr{M}^{-1})}.$$

As a byproduct of our result, one also deduces that, up to normalization, the only equilibria or stationary solutions of (2.1) that cancels the collision operator $\mathscr{C}$ are the Maxwellian $\mathscr{M}$, the Hamiltonian function $\mathcal{H}\mathscr{M}$, and the functions belonging to $\mathfrak{R}_\phi$, $\mathfrak{D}_\phi$ and $\mathfrak{P}_\phi$ when these sets are not empty.

Here the constant $\lambda > 0$ only depends on the constants appearing in our assumptions and (2.2) and (2.3), as well as on geometric properties of the potential ϕ as the constants appearing in Poincaré-type inequalities (see Proposition 2.1) and Korn-type inequalities (see Theorem 2.B).

We shall now present the main ideas of the proof.

2.3.1 Rescaling

It is convenient to work on the probability space

$$L^2(\mathrm{d}\mathscr{M}) = \left\{h : \mathbf{R}^d \times \mathbf{R}^d \to \mathbf{R} \text{ measurable} \;\middle|\; \int_{\mathbf{R}^{2d}} |h(x,v)|^2 \mathscr{M}(x,v)\,\mathrm{d}x\,\mathrm{d}v < \infty\right\}$$

endowed with the scalar product

$$\langle h_1, h_2\rangle := \int_{\mathbf{R}^{2d}} h_1(x,v)h_2(x,v)\mathscr{M}(x,v)\,\mathrm{d}x\,\mathrm{d}v$$

and the associated norm

$$\|h\| := \left(\int_{\mathbf{R}^{2d}} |h(x,v)|^2 \mathscr{M}(x,v)\,\mathrm{d}x\,\mathrm{d}v\right)^{1/2}.$$

Note that when considering a function of x only, respectively v only, the norms $L^2(\mathrm{e}^{-\phi}\mathrm{d}x)$, respectively $L^2(\mathrm{d}\mu)$, coincide with $L^2(\mathrm{d}\mathscr{M})$ via a unitary embedding. We shall denote $\langle\cdot\rangle$ the mean in $L^2(\mathrm{e}^{-\phi}\mathrm{d}x)$.

Therefore for a given solution f to (2.1) with initial data f_{in}, using the equilibria and special modes built in (2.5)–(2.7)–(2.11)–(2.12)–(2.14) we define

$$h := \frac{f - \alpha\mathscr{M} - \beta\mathcal{H}\mathscr{M} - F_{\mathrm{rig}} - F_{\mathrm{dir}} - F_{\mathrm{pul}}}{\mathscr{M}} \in L^2(\mathrm{d}\mathscr{M}). \tag{2.16}$$

Due to the fact that the equation (2.1) is linear and that all special modes satisfy it, we obtain that h satisfies the new equation

$$\partial_t h = \mathcal{L}h := \mathcal{T}h + \mathcal{C}h, \tag{2.17}$$

complemented with the initial conditions h_{in}, defined via (2.16), and where the linear operators are given by

$$\mathcal{T}h := \mathscr{T}h = \nabla_x\phi\cdot\nabla_v h - v\cdot\nabla_x h \quad\text{ and }\quad \mathcal{C}h := \mu^{-1}\mathscr{C}(\mu h)\,.$$

From the assumptions on $\mathscr{C}$, one obtains that the operator $\mathcal{C}$ is non-negative and self-adjoint in $L^2(\mathrm{d}\mu)$, acts only on the velocity variable, and its kernel is given by

$$\ker(\mathcal{C}) = \operatorname{span}\left\{1, v_1, \dots, v_d, |v|^2\right\}.$$

We decompose h in the following orthogonal way in $L^2(\mathrm{d}\mathscr{M})$:

$$h = \pi h + h^{\perp} \quad \text{with} \quad \pi h := r + m \cdot v + e\,\frac{(|v|^2 - d)}{\sqrt{2d}},$$

where $h^{\perp} = h^{\perp}(t,x,v)$ is the *microscopic part* of h and the *macroscopic part* πh is composed by the mass r, the momentum m and the energy e, defined as

$$r(t,x) := \int_{\mathbf{R}^d} h(t,x,v)\mu(v)\,\mathrm{d}v, \tag{2.18a}$$

$$m(t,x) := \int_{\mathbf{R}^d} v h(t,x,v)\mu(v)\,\mathrm{d}v, \tag{2.18b}$$

$$e(t,x) := \int_{\mathbf{R}^d} \frac{(|v|^2 - d)}{\sqrt{2d}}\, h(t,x,v)\mu(v)\,\mathrm{d}v. \tag{2.18c}$$

Remark that by construction one has

$$\|h\|^2 = \|r\|^2 + \|m\|^2 + \|e\|^2 + \|h^{\perp}\|^2.$$

As consequence of (2.2a) that we integrate with respect to the spatial variable, one obtains

$$-\langle \mathcal{C}h, h\rangle \geqslant \lambda_{\mathscr{C}} \|h^{\perp}\|^2. \tag{2.19}$$

According to the definition of h, the properties of all special modes described above and the (conditional) conservation laws (2.4)–(2.6)–(2.10)–(2.13)–(2.15), we can deduce the conservation laws associated to the new unknown h. First of all, mass and energy conservation read here

$$\langle r\rangle = 0 \quad \text{and} \quad \sqrt{\frac{d}{2}}\langle e\rangle + \langle(\phi - \langle\phi\rangle)r\rangle = 0. \tag{2.20}$$

Then the possible conservation law associated to the rotational invariance of the potential ϕ now corresponds to

$$\mathbb{P}(m) \in \mathcal{R}_{\phi}^{c}, \tag{2.21}$$

where, recalling that $\mathcal{R}_\phi$ is the set of infinitesimal rotation linear maps compatible with ϕ defined in (2.9), we denote $\mathbb{P}$ the $L^2(\mathrm{e}^{-\phi}\mathrm{d}x)$-projection onto the set of infinitesimal rotation linear maps $\mathcal{R}$ defined in (2.9), and $\mathcal{R}_\phi^c$ stands for the orthogonal of $\mathcal{R}_\phi$ with respect to $L^2(\mathrm{e}^{-\phi}\mathrm{d}x)$ inside $\mathcal{R}$.

In the presence of harmonic directions, that is when $0 \leqslant d_\phi \leqslant d-1$, the conservation law associated to harmonic directional invariance writes

$$\langle r x_i\rangle = 0 \quad \text{and} \quad \langle m_i\rangle = 0 \quad \text{for any} \quad i \in \{d_\phi + 1, \cdots, d\}. \tag{2.22}$$

Finally, when $d_\phi = 0$ and thus $\phi(x) = \frac{1}{2}|x|^2 + \frac{d}{2}\log 2\pi$, the last conservation law associated to harmonic pulsating invariance reads

$$\langle m \cdot x\rangle = 0 \quad \text{and} \quad \sqrt{\frac{d}{2}}\,\langle e\rangle - \langle(\phi - \langle\phi\rangle)r\rangle = 0. \tag{2.23}$$

2.3.2 A Lyapunov strategy

The spectral gap estimate (2.19) gives directly a control on the microscopic part $h^\perp$ of a solution h to (2.17). Indeed, since $\mathcal{T} = -\mathcal{T}^*$, one obtains

$$\frac{\mathrm{d}}{\mathrm{d}t}\|h\|^2 \leqslant -\kappa_0 \|h^\perp\|^2. \tag{2.24}$$

for some constant $\kappa_0 > 0$. We are hence missing the macroscopic part πh, and the idea is then to construct a functional $\mathscr{F}(h)$ that is a Lyapunov functional for (2.17), is equivalent to $\|h\|^2$ and, furthermore, for which we obtain an inequality of the type

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{F}(h) \leqslant -\lambda \mathscr{F}(h)$$

for some positive constant $\lambda > 0$, which in turn implies our main result in Theorem 2.A by Grönwall lemma and coming back to the original unknown f.

This suitable entropy functional $\mathscr{F}(h)$ is constructed by starting from the norm $\|h\|^2$ to which we add, stet by step, new partial Lyapunov functionals in order to control the missing terms appearing on the macroscopic part πh.

As a first step, we look to the evolution of the macroscopic quantities r, m and e defined in (2.18). More precisely, the evolution of the mass r, the momentum m, the energy e and some suitable high-order moments of h is given by

$$\begin{cases}
\partial_t r = \nabla_x^* \cdot m & \text{(2.25a)}\\
\partial_t m = -\nabla_x r + \sqrt{\dfrac{2}{d}}\nabla_x^* e + \nabla_x^* \cdot E[h^\perp] & \text{(2.25b)}\\
\partial_t e = -\sqrt{\dfrac{2}{d}}\nabla_x \cdot m + \nabla_x^* \cdot \Theta[h^\perp] & \text{(2.25c)}\\
\partial_t E[h] = -2\mathrm{D}_x^s m + E[(\mathcal{C}+\mathcal{T})h^\perp] & \text{(2.25d)}\\
\partial_t \Theta[h] = -\left(1 + \dfrac{2}{d}\right)\nabla_x e + \Theta[(\mathcal{C}+\mathcal{T})h^\perp], & \text{(2.25e)}
\end{cases}$$

where $\mathrm{D}_x^s m = \frac{1}{2}(\nabla_x m + \nabla_x m^\top)$ denotes the symmetric part of $\nabla_x m$, $\nabla_x^* := -\nabla_x + \nabla_x \phi$, and the matrix $E[g]$ and the vector $\Theta[g]$ are defined by

$$E[g] = \int_{\mathbf{R}^d} (v \otimes v - \mathrm{I}_d) g\mu(v)\,\mathrm{d}v$$

and

$$\Theta[g] = \int_{\mathbf{R}^d} v\,\frac{(|v|^2 - d - 2)}{\sqrt{2d}}\,g\mu(v)\,\mathrm{d}v.$$

We observe from this that we could expect to construct a functional to obtain a control of $\nabla_x e$ by using equation (2.25e), since the remainder term depends only on $h^\perp$ which is already controlled thanks to (2.24), and then we would obtain a control of $e - \langle e\rangle$ thanks to Poincaré inequality.

We shall follow this heuristics and for this we need to introduce the following *error* quantities:

$$e_s := e - \langle e\rangle \tag{2.26a}$$

$$m_s := m - \langle \mathrm{D}_x^a m\rangle x - \frac{1}{d}\langle \nabla_x \cdot m\rangle x - \langle m\rangle \tag{2.26b}$$

$$w_s := r_s - \sqrt{\frac{2}{d}}\langle e\rangle \phi_s. \tag{2.26c}$$

where $\mathrm{D}_x^a m = \frac{1}{2}(\nabla_x m - \nabla_x m^\top)$ denotes the anti-symmetric part of $\nabla_x m$,

$$r_s := r - \langle \nabla_x r \rangle \cdot x - \frac{1}{2d} \langle \Delta_x r \rangle (|x|^2 - \langle |x|^2 \rangle)$$

and

$$\phi_s = \phi - \langle \phi \rangle - \frac{1}{2d} \langle \Delta_x \phi \rangle (|x|^2 - \langle |x|^2 \rangle).$$

In what follows we shall use the properties of the Laplace-Witten operator associated to ϕ given by

$$\Delta_\phi f = -\Delta_x f + \nabla_x \phi \cdot \nabla_x f$$

as well as the positive operator $\Lambda := -\Delta_\phi + \mathrm{id}$. More precisely, we will use the operators Λ^{-1} and $\Lambda^{-\frac{1}{2}}$, whose properties are detailed in Section 2.4.1 below, as well as Poincaré-type inequalities presented in Proposition 2.1 and Korn-type inequalities in Theorem 2.B.

The first construction is aimed to control the e_s term. More precisely using (2.25e) one obtains

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \left\langle \Lambda^{-1} \nabla_x e, \Theta[h] \right\rangle &\leqslant -\kappa \|\Lambda^{-\frac{1}{2}} \nabla_x e\|^2 + C \|h^\perp\| \|h\| \\ &\leqslant -\kappa_1 \|e_s\|^2 + C \|h^\perp\| \|h\| \end{aligned} \tag{2.27}$$

for some positive constant $\kappa_1 > 0$ and where we have used the Poincaré-Lions inequality (2.38).

Once we can control the term e_s with the above functional, we turn our attention to the moment m by using equation (2.25d). We are then able to control the error term m_s by computing

$$\frac{\mathrm{d}}{\mathrm{d}t} \left\langle \Lambda^{-1} \mathrm{D}_x^s m_s, E[h] - \sqrt{\frac{2}{d}} \langle e \rangle \mathrm{I}_d \right\rangle \leqslant -\kappa_2 \|m_s\|^2 + C \|e_s\| \|h\| + C \|h^\perp\| \|h\| \tag{2.28}$$

for some positive constant $\kappa_2 > 0$ and where now we need to use the zeroth order Poincaré-Korn inequality (2.42). Remark now that the remainder term in e_s is harmless since it has been controlled by the previous functional in (2.27).

We can control now the error quantity w_s related to the inhomogeneous part of the mass r defined in (2.26c) by using the equation (2.25b). More precisely there holds

$$\frac{\mathrm{d}}{\mathrm{d}t} \left\langle \Lambda^{-1} \nabla_x w_s, m_s \right\rangle \leqslant -\kappa_3 \|w_s\|^2 + C \|e_s\|^2 + C \|h^\perp\|^2 + C \|m_s\| \|h\| \tag{2.29}$$

for some positive constant $\kappa_3 > 0$ and where we have used again the Poincaré-Lions inequality (2.38). It turns out that we shall also need in the sequel the control of $\Lambda^{-\frac{1}{2}} \partial_t w_s$ that can be easily obtained by

$$\frac{\mathrm{d}}{\mathrm{d}t} \left\langle -\Lambda^{-1} \partial_t w_s, w_s \right\rangle \leqslant -\|\Lambda^{-\frac{1}{2}} \partial_t w_s\|^2 + C \|w_s\| \|h\|. \tag{2.30}$$

In order to complete the proof, we need to control the time depending quantities $\langle e \rangle$, $\langle \mathrm{D}_x^a m \rangle$, $\langle \nabla_x \cdot m \rangle$, $\langle m \rangle$, $\langle \nabla_x r \rangle$, $\langle \Delta_x r \rangle$ involved in the definition of e_s, m_s and w_s in (2.26). To simplify the notation, we introduce

$$A(t) := \langle \mathrm{D}_x^a m \rangle, \quad b(t) := \langle m \rangle, \quad c(t) := \langle e \rangle,$$

and we rewrite the expression of the macroscopic quantity in terms of these new functions, which thus only dependent of the time variable, and we obtain

$$r(t,x) = -b'(t) \cdot x + c''(t) \frac{1}{2\sqrt{2d}} (|x|^2 - \langle |x|^2 \rangle) + c(t) \sqrt{\frac{2}{d}} (\phi - \langle \phi \rangle) + z(t,x),$$

$$
\begin{aligned}
m(t,x) &= A(t)x + b(t) - \frac{c'(t)}{\sqrt{2d}}x + m_s(t,x),\\
e(t,x) &= c(t) + e_s(t,x),
\end{aligned}
$$

We observe that the new quantity z is controlled by the previous controlled macroscopic quantities, more precisely

$$
\begin{aligned}
\|z\|^2 &\lesssim \|w_s\|^2 + \|e_s\|^2 + \|h^\perp\|^2,\\
\|\Lambda^{-\frac{1}{2}}\partial_t z\|^2 &\lesssim \|\Lambda^{-\frac{1}{2}}\partial_t w_s\|^2 + \|m_s\|^2 + \|h^\perp\|^2.
\end{aligned}
$$

We deduce from this then the main differential equation relating the quantities A, b, b'', c' and c''', namely

$$
\begin{aligned}
&\frac{1}{\sqrt{2d}}[2(\phi - \langle\phi\rangle) + \nabla_x\phi\cdot x - d]c' + \frac{1}{2\sqrt{2d}}(|x|^2 - \langle|x|^2\rangle)c'''\\
&\qquad\qquad - \nabla_x\phi\cdot b - x\cdot b'' - \nabla_x\phi Ax = \nabla_x^*\cdot m_s - \partial_t z.
\end{aligned} \tag{2.31}
$$

From this rigidity estimate (2.31) we are able to control the skew-symmetric matrix A. More precisely, defining

$$
X := \frac{1}{\sqrt{2d}}[2(\phi - \langle\phi\rangle) + \nabla_x\phi\cdot x - d]c + \frac{1}{2\sqrt{2d}}(|x|^2 - \langle|x|^2\rangle)c'' - x\cdot b'
$$

and

$$
Y := \frac{1}{\sqrt{2d}}\langle 2\phi x\rangle c + \frac{1}{2\sqrt{2d}}\langle|x|^2 x\rangle c'' - \langle x\otimes x\rangle b',
$$

we obtain that

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left\langle -\left(X - Y\cdot\nabla_x\phi\right), \Lambda^{-1}\nabla_x\phi Ax\right\rangle\\
&\qquad \leqslant -\kappa\|\Lambda^{-\frac{1}{2}}\nabla_x\phi Ax\|^2 + C\|m_s\|^2 + C\|\Lambda^{-\frac{1}{2}}\partial_t w_s\|^2 + C\|h^\perp\|\,\|h\|\\
&\qquad \leqslant -\kappa_5|A|^2 + C\|m_s\|^2 + C\|\Lambda^{-\frac{1}{2}}\partial_t w_s\|^2 + C\|h^\perp\|\,\|h\|
\end{aligned} \tag{2.32}
$$

for some positive constant $\kappa_5 > 0$, where we have used the Korn-type inequality (2.42) together with the conservation law (2.21).

Up to this stage, all quantities have been controlled by using a partial Lyapunov functional. We can now control the quantities b, b'', c', c''' thanks to a rigidity property implied by equation (2.31). This is the point in which the three different cases taking into account the harmonicity of the potential (fully non-harmonic $d_\phi = d$; partially harmonic $1 \leqslant d_\phi \leqslant d-1$; fully harmonic $d_\phi = 0$;) come into play. Observe that (2.31) can be rewritten as

$$
\alpha_1(x)c' + \alpha_2(x)c''' + \beta_1(x)\cdot b + \beta_2(x)\cdot b'' = S
$$

and therefore we recognize a harmonic oscillator for b and for c'. This is responsible for the appearance of the harmonic and pulsating modes described in Section 2.2.4. Since we have already taken this into account in the definition of h, using the conservation laws (2.22) and (2.23) and splitting the argument into the three different cases along d_ϕ, we can obtain that

$$
|b| + |b''| + |c'| + |c'''| \lesssim |A| + \|\Lambda^{-\frac{1}{2}}\partial_t w_s\| + \|m_s\| + \|h^\perp\|. \tag{2.33}
$$

We construct the final part of our functional by remarking that

$$
\frac{\mathrm{d}}{\mathrm{d}t}\langle -b, b'\rangle \leqslant -|b'|^2 + C|A|^2 + C\|m_s\|^2 + C\|\Lambda^{-\frac{1}{2}}\partial_t w_s\|^2 + C\|h^\perp\|^2 \tag{2.34}
$$

and

$$\frac{d}{dt}\left\langle -c', c''\right\rangle \leqslant -|c''|^2 + C|A|^2 + C\|m_s\|^2 + C\|\Lambda^{-\frac{1}{2}}\partial_t w_s\|^2 + C\|h^\perp\|^2. \tag{2.35}$$

In order to complete the scheme we finally need to control c and r, and thanks to (2.20) we obtain

$$|c| + \|r\| \lesssim |b'| + |c''| + \|w_s\| + \|e_s\| + \|h^\perp\|. \tag{2.36}$$

Finally we introduce the functional

$$\begin{aligned}\mathscr{F}(h) := \|h\|^2 &+ \varepsilon_1 \left\langle \Lambda^{-1}\nabla_x e, \Theta[h]\right\rangle + \varepsilon_2 \left\langle \Lambda^{-1}\mathrm{D}^s_x m_s, E[h] - \sqrt{\frac{2}{d}}\langle e\rangle \mathrm{I}_d\right\rangle\\ &+ \varepsilon_3 \left\langle \Lambda^{-1}\nabla_x w_s, m_s\right\rangle + \varepsilon_4 \left\langle -\Lambda^{-1}\partial_t w_s, w_s\right\rangle + \varepsilon_5 \left\langle -\left(X - Y\cdot\nabla_x\phi\right), \Lambda^{-1}\nabla_x\phi A x\right\rangle\\ &+ \varepsilon_6 \left\langle -b, b'\right\rangle + \varepsilon_6 \left\langle -c', c''\right\rangle\end{aligned}$$

with a suitable choice of the constants $0 \ll \varepsilon_6 \ll \varepsilon_5 \ll \varepsilon_4 \ll \varepsilon_3 \ll \varepsilon_2 \ll \varepsilon_1 \ll 1$. Therefore, gathering estimates (2.24)–(2.27)–(2.28)–(2.29)–(2.30)–(2.32)–(2.34)–(2.35) yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{F}(h) \leqslant -\kappa \mathscr{D}(h)$$

for some constant $\kappa > 0$ and where

$$\mathscr{D}(h) := \|h^\perp\|^2 + \|e_s\|^2 + \|m_s\|^2 + \|w_s\|^2 + \|\Lambda^{-\frac{1}{2}}\partial_t w_s\|^2 + |A|^2 + |b'|^2 + |c''|^2.$$

We can conclude the proof of Theorem 2.A by remarking that, thanks to (2.33) and (2.36), one has the equivalence

$$\|h\|^2 \lesssim \mathscr{F}(h) \lesssim \mathscr{D}(h) \lesssim \|h\|^2.$$

2.4 Korn inequalities in the whole space

In this section we present Korn-type inequalities verified by the measure $\mathrm{e}^{-\phi}\mathrm{d}x$ where we recall that the confining potential $\phi : \mathbf{R}^d \to \mathbf{R}^d$ satisfies (2.3).

We start our discussion by recalling that the classical Korn inequality states that, in a bounded domain, one can control the L^2-norm of the gradient of a vector-field by the norm of its symmetric part. More precisely, thanks to the works of Korn [138, 139, 140], for any bounded smooth domain Ω in $\mathbf{R}^d$ one has that

$$\|\nabla u\|^2_{L^2(\Omega)} \leqslant 2\|\mathrm{D}^s u\|^2_{L^2(\Omega)}$$

for any vector-field $u \in \mathscr{C}^2(\overline{\Omega}; \mathbf{R}^d)$ such that $u = 0$ on $\partial\Omega$, where we recall that $\mathrm{D}^s u = \frac{1}{2}(\nabla u + \nabla u^\top)$ denotes the symmetric part of the gradient ∇u and $\mathrm{D}^a u = \frac{1}{2}(\nabla u - \nabla u^\top)$ its anti-symmetric part.

Motivated by kinetic equations in which situation the constraint $u = 0$ on $\partial\Omega$ is too restrictive, Desvillettes-Villani [82] established a similar Korn inequality in a bounded domain Ω for vector fields u with Neumann homogeneous boundary conditions $u \cdot n = 0$ on $\partial\Omega$, where n denotes the outward unit normal vector of $\partial\Omega$. More precisely, recalling that for any anti-symmetric matrix $A \in \mathfrak{M}^a_{d,d}(\mathbf{R})$ we denote $R_A : x \mapsto Ax$ and that we have defined in (2.8) the set of infinitesimal rotations by $\mathcal{R} = \{R_A : A \in \mathfrak{M}^a_{d,d}(\mathbf{R})\}$, they proved that

$$\inf_{R\in\mathcal{R}_\Omega} \|\nabla(u - R)\|^2_{L^2(\Omega)} \leqslant C_\Omega \|\mathrm{D}^s u\|^2_{L^2(\Omega)}$$

for any vector-field $u \in \mathscr{C}^2(\overline{\Omega}; \mathbf{R}^d)$ such that $u \cdot n = 0$ on $\partial\Omega$, where $\mathcal{R}_\Omega := \{R \in \mathcal{R} \mid n(x) \cdot R(x) = 0 \text{ for any } x \in \partial\Omega\} \subset \mathcal{R}$ is the set of infinitesimal rotations preserving Ω. One remarks that the above Korn inequality takes into account rotational invariance of the domain Ω and gives quantitative estimates for the constant C_Ω.

In a similar fashion as in the work of Desvillettes-Villani [82], and motivated by the study of kinetic equation in the whole space with confining potential as described in Section 2.3, our aim in this section is to present constructive versions of Korn's inequality in the whole space $\mathbf{R}^d$ in presence of a confining potential in weighted L^2 spaces. Here constructive means that the constants are estimated in term of classical or explicit constants which account for geometrical properties of the potential ϕ. Although the motivation comes from kinetic equations, we believe that the results we shall present below have their own interest and actually they account for more than it was really needed in Sections 2.3.

From now on we shall only focus on Korn inequalities on the whole space and, concerning the classical Korn inequality in a bounded domain with Dirichlet boundary condition, we refer to the results of Friedrichs [105], Duvaut-Lions [94], Ciarlet [67] and Horgan [130], as well as the more recent works of Lewicka-Müller [144], Bauer-Pauly [14, 13], Neff-Pauly-Witsch [164], and the references therein.

2.4.1 Properties of the Witten-Laplace operator

As a preliminary step, we shall investigated properties of a Witten-Laplace operator associated to the potential ϕ, which can be seen as the replacement of the Laplace operator with Neumann boundary conditions in the case of a bounded domain, as well as some Poincaré-type inequalities satisfied by the measure $\mathrm{e}^{-\phi}\mathrm{d}x$.

Hereafter we shall work on the space $L^2(\mathrm{e}^{-\phi}\mathrm{d}x)$ defined by

$$L^2(\mathrm{e}^{-\phi}\mathrm{d}x) = \left\{ f : \mathbf{R}^d \to \mathbf{R} \;\middle|\; \int_{\mathbf{R}^d} f^2 \mathrm{e}^{-\phi}\,\mathrm{d}x < \infty \right\}$$

endowed with the norm

$$\|f\| := \left(\int_{\mathbf{R}^d} f^2 \mathrm{e}^{-\phi}\,\mathrm{d}x \right)^{\frac{1}{2}}$$

and we denote by $\langle f \rangle = \int_{\mathbf{R}^d} f \mathrm{e}^{-\phi}\,\mathrm{d}x$ the mean.

We define the Witten-Laplace operator Δ_ϕ associated to ϕ and the associated positive operator $\Lambda := -\Delta_\phi + \mathrm{id}$ in the following way

$$\Delta_\phi f := -\Delta f + \nabla\phi \cdot \nabla f \quad \text{and} \quad \Lambda f := -\Delta_\phi f + f.$$

The operators $-\Delta_\phi$ and Λ are essentially self-adjoint and one can show that their common domain is given by $D(\Lambda) = H^2(\mathrm{e}^{-\phi}\mathrm{d}x)$, where one has

$$H^2(\mathrm{e}^{-\phi}\mathrm{d}x) = \left\{ f \in L^2(\mathrm{e}^{-\phi}\mathrm{d}x) \;\middle|\; \|(1+|\nabla\phi|^2)f\| + \|(1+|\nabla\phi|^2)^{\frac{1}{2}}\nabla f\| + \|\nabla^2 f\| < \infty \right\}.$$

One can therefore define the bijective operators

$$\Lambda^{-\frac{1}{2}} : L^2(\mathrm{e}^{-\phi}\mathrm{d}x) \to H^1(\mathrm{e}^{-\phi}\mathrm{d}x) \quad \text{and} \quad \Lambda^{-1} : L^2(\mathrm{e}^{-\phi}\mathrm{d}x) \to H^2(\mathrm{e}^{-\phi}\mathrm{d}x),$$

where

$$H^1(\mathrm{e}^{-\phi}\mathrm{d}x) = \left\{ f \in L^2(\mathrm{e}^{-\phi}\mathrm{d}x) \;\middle|\; \|(1+|\nabla\phi|^2)^{\frac{1}{2}} f\| + \|\nabla f\| < \infty \right\},$$

which satisfy the following bounds: for any $f \in L^2(\mathrm{e}^{-\phi}\mathrm{d}x)$ one has

$$\|\nabla\Lambda^{-\frac{1}{2}} f\| + \|(1+|\nabla\phi|^2)^{\frac{1}{2}}\Lambda^{-\frac{1}{2}} f\| \lesssim \|f\|$$

and

$$\|(1+|\nabla\phi|^2)\Lambda^{-1} f\| + \|(1+|\nabla\phi|^2)^{\frac{1}{2}}\nabla\Lambda^{-1} f\| + \|\nabla^2\Lambda^{-1} f\| \lesssim \|f\|$$

with quantitative and constructive constants. Hereafter we shall use the convention that the operators $\Lambda^{-\frac{1}{2}}$ and Λ^{-1} act component-wisely when applied to vector-fields or matrix-valued functions.

We recall that the measure $\mathrm{e}^{-\phi}\mathrm{d}x$ satisfies a Poincaré inequality (2.3a), and we prove that it also satisfies the following related inequalities:

Proposition 2.1. (i) Strong Poincaré inequality: *There is a constant $C_{\mathrm{SP}} > 0$ such that for any $f \in H^1(\mathrm{e}^{-\phi}\mathrm{d}x)$ there holds*

$$\|(1 + |\nabla\phi|^2)^{\frac{1}{2}}(f - \langle f\rangle)\|^2 \leqslant C_{\mathrm{SP}}\|\nabla f\|^2. \tag{2.37}$$

(ii) Poincaré-Lions inequality: *There is a constant $C_{\mathrm{PL}} > 0$ such that for any $f \in L^2(\mathrm{e}^{-\phi}\mathrm{d}x)$ there holds*

$$C_{\mathrm{PL}}^{-1}\|f - \langle f\rangle\| \leqslant \|\Lambda^{-\frac{1}{2}}\nabla f\| \leqslant \|f - \langle f\rangle\|. \tag{2.38}$$

(iii) *There is a constant $C_{\mathrm{LPL}} > 0$ such that for any $f \in H^{-1}(\mathrm{e}^{-\phi}\mathrm{d}x)$ there holds*

$$C_{\mathrm{LPL}}^{-1}\|\Lambda^{-\frac{1}{2}}(f - \langle f\rangle)\| \leqslant \|\Lambda^{-1}\nabla f\| \leqslant \|\Lambda^{-\frac{1}{2}}(f - \langle f\rangle)\|, \tag{2.39}$$

All the above estimates are proven in [51] with quantitative bounds. Although some of them are classical and belong to folklore, estimates (2.38) and (2.39) are new. They are obtained by using the spectral theorem with the property

$$-\Delta_\phi \geqslant C_{\mathrm{P}}^{-1}\,\mathrm{id} \quad \text{on} \quad \left\{f \in H^2(\mathrm{e}^{-\phi}\mathrm{d}x) \mid \langle f\rangle = 0\right\}$$

that is a consequence of the Poincaré inequality (2.3a) satisfied by $\mathrm{e}^{-\phi}$, together with commutator estimates and the bounds listed above.

2.4.2 Weighted Korn inequalities in the whole space

We are now able to state our main result concerning Korn-type inequalities in the whole space.

Recall that for any anti-symmetric matrix $A \in \mathfrak{M}^a_{d,d}(\mathbf{R})$ we denote $R_A : x \mapsto Ax$ and we have defined in (2.8) the the set of infinitesimal rotation linear maps by $\mathcal{R} = \{R_A : A \in \mathfrak{M}^a_{d,d}(\mathbf{R})\}$, as well as the the subspace of infinitesimal rotation linear maps compatible with ϕ, defined in (2.9), by $\mathcal{R}_\phi = \{R_A \in \mathcal{R} \mid \nabla_x\phi(x) \cdot R_A(x) = 0 \text{ for any } x \in \mathbf{R}^d\}$.

Theorem 2.B. *Assume that the potential ϕ satisfies (2.3).*

(i) Korn inequality: *There exist positive constants $C_{\mathrm{K}}, C'_{\mathrm{K}}, C_{\mathrm{RD}} > 0$ such that for any vector-field $u \in H^1(\mathrm{e}^{-\phi}\mathrm{d}x)$ there holds*

$$\inf_{R\in\mathcal{R}} \|\nabla(u - R)\|^2 \leqslant C_{\mathrm{K}}\|\mathrm{D}^s u\|^2 \tag{2.40a}$$

$$\inf_{R\in\mathcal{R}_\phi} \|\nabla(u - R)\|^2 \leqslant C'_{\mathrm{K}}\|\mathrm{D}^s u\|^2 + 2C_{\mathrm{RD}}\|\nabla_x\phi \cdot u\|^2. \tag{2.40b}$$

(ii) Poincaré-Korn inequality: *There exist positive constants $C_{\mathrm{PK}}, C'_{\mathrm{PK}}, C_{\mathrm{RV}} > 0$ such that for any vector-field $u \in H^1(\mathrm{e}^{-\phi}\mathrm{d}x)$ there holds*

$$\inf_{R\in\mathcal{R}} \|u - \langle u\rangle - R\|^2 \leqslant C_{\mathrm{PK}}\|\mathrm{D}^s u\|^2 \tag{2.41a}$$

$$\inf_{R\in\mathcal{R}_\phi} \|u - \langle u\rangle - R\|^2 \leqslant C'_{\mathrm{PK}}\|\mathrm{D}^s u\|^2 + 2C_{\mathrm{RV}}\|\nabla_x\phi \cdot u\|^2. \tag{2.41b}$$

(iii) zeroth order Poincaré-Korn inequality: *There exist positive constants* $C_{\mathrm{PK},0}$*,* $C'_{\mathrm{PK},0}$*,* $C_{\mathrm{RV},0} > 0$ *such that for any vector-field* $u \in L^2(\mathrm{e}^{-\phi}\mathrm{d}x)$ *there holds*

$$\inf_{R\in\mathcal{R}} \|u - \langle u\rangle - R\|^2 \leqslant C_{\mathrm{PK},0}\|\Lambda^{-\frac{1}{2}}\,\mathrm{D}^s u\|^2 \tag{2.42a}$$

$$\inf_{R\in\mathcal{R}_\phi} \|u - \langle u\rangle - R\|^2 \leqslant C'_{\mathrm{PK},0}\|\Lambda^{-\frac{1}{2}}\,\mathrm{D}^s u\|^2 + 2C_{\mathrm{RV},0}\|\Lambda^{-\frac{1}{2}}\nabla_x\phi\cdot u\|^2. \tag{2.42b}$$

All the above constants are quantitative and constructive in the sense that they can be estimated in terms of classical or explicit geometrical constants associated to the potential ϕ.

We shall mention that the only result of this type we are aware of is the work by Duan [89] where inequality (2.41a) is established by non-constructive arguments.

Let us now present the main ideas of the proof, and we only focus on point (i) of Theorem 2.B. We first mention that a crucial ingredient corresponds to the estimates on the operators $\Lambda^{-\frac{1}{2}}$ and Λ^{-1} as well as on the Poincaré-type inequalities gathered in Proposition 2.1.

We start by (2.40a) and hence consider $u \in H^1(\mathrm{e}^{-\phi}\mathrm{d}x)$ such that $\langle u\rangle = \langle \mathrm{D}^a u\rangle = 0$. We decompose

$$\|\nabla u\|^2 = \|\mathrm{D}^s u\|^2 + \|\mathrm{D}^a u\|^2$$

so that we can conclude by controlling the norm of the anti-symmetric part of the gradient $\mathrm{D}^a u$ by its symmetric part $\mathrm{D}^s u$. At this point we shall invoke the second main ingredient of our analysis, which is a consequence of the Schwarz Lemma and allows us to write all components of a the second-order differential of a vector field u thanks to its symmetric components, namely

$$\partial_k\left(\mathrm{D}^a u\right)_{ij} = \partial_j\left(\mathrm{D}^s u\right)_{ik} - \partial_i\left(\mathrm{D}^s u\right)_{jk} \quad \text{for any} \quad i,j,k \in \{1,\cdots,d\}. \tag{2.43}$$

Therefore, applying the left-hand side of Poincaré-Lions inequality (2.38), one obtains first that

$$\|\mathrm{D}^a u\|^2 = \sum_{ij}\|(\mathrm{D}^a u)_{ij}\|^2 \lesssim \sum_{ij}\|\Lambda^{-\frac{1}{2}}\nabla(\mathrm{D}^a u)_{ij}\|^2,$$

then identity (2.43) implies

$$\sum_{ij}\|\Lambda^{-\frac{1}{2}}\nabla(\mathrm{D}^a u)_{ij}\|^2 \lesssim \sum_{jk}\|\Lambda^{-\frac{1}{2}}\nabla(\mathrm{D}^s u)_{jk}\|^2.$$

Finally, applying the right-hand side of the Poincaré-Lions inequality (2.38) yields

$$\sum_{jk}\|\Lambda^{-\frac{1}{2}}\nabla(\mathrm{D}^s u)_{jk}\|^2 \lesssim \sum_{jk}\|(\mathrm{D}^s u)_{jk}\|^2 = \|\mathrm{D}^s u\|^2.$$

Once this is achieved, the second inequality (2.40b) is then a consequence of (2.40a) and our third main tool, which correspond to the rigidity constant C_{RD} that measures the possible defaults of (centered) axi-symmetry of the potential ϕ. This rigidity constant is positive since it is obtained as the following finite-dimensional minimization problem depending on ϕ:

$$C_{\mathrm{RD}}^{-1} = \min_{(A,b)\in(\mathfrak{M}_\phi^c\otimes\mathbf{R}^d)\setminus\{(0,0)\}} \frac{\|\nabla\phi\cdot(Ax+b)\|^2}{|A|^2+|b|^2} > 0,$$

where $\mathfrak{M}_\phi = \{A \in \mathfrak{M}^a_{d,d}(\mathbf{R}) \mid \nabla\phi(x)\cdot Ax = 0 \text{ for any } x \in \mathbf{R}^d\}$ and $\mathfrak{M}_\phi^c$ is the orthogonal of $\mathfrak{M}_\phi$ inside $\mathfrak{M}^a_{d,d}(\mathbf{R})$.

We remark that the rigidity constants associated to points (ii) and (iii) are obtained as the following finite-dimensional minimization problems depending on ϕ:

$$C_{\mathrm{RV}}^{-1} = \min_{(R,b)\in(\mathcal{R}_\phi^c\otimes\mathbf{R}^d)\setminus\{(0,0)\}} \frac{\|\nabla\phi\cdot(R+b)\|^2}{\|R+b\|^2} > 0,$$

where $\mathcal{R}_\phi^c$ is the orthogonal of $\mathcal{R}_\phi$ inside $\mathcal{R}$, and finally

$$C_{\mathrm{RV},0}^{-1} = \min_{(R,b)\in(\mathcal{R}_\phi^c\otimes\mathbf{R}^d)\setminus\{(0,0)\}} \frac{\|\Lambda^{-\frac{1}{2}}\nabla\phi\cdot(R+b)\|^2}{\|R+b\|^2} > 0.$$

2.5 Some perspectives

2.5.1 Weak coercivity

A first interesting question would be to investigate the large-time behavior of solutions to the equation (2.1) in the situation which the *coercivity* properties are replaced by a weaker condition.

More precisely, one could consider the case in which the linear collision operator $\mathscr{C}$ does not possesses a spectral gap (2.2a) but only a *weak coercivity* property.

In a similar fashion, one could also consider the situation in which the probability measure $\mathrm{e}^{-\phi}\mathrm{d}x$ does not satisfy Poincaré inequality (2.3a) but only a *weak Poincaré inequality* (see for instance Röckner-Wang [168] and Wang [190]). This would give us two directions to investigate: new Korn-type inequalities associated to weak Poincaré inequalities; and then the large-time behavior of solutions to (2.1).

In the above situations, one then would expect to obtain rates of convergence that are not anymore exponential.

2.5.2 Kinetic equations in a bounded domain

A very important class of kinetic equations consists in the case of a bounded domain with appropriated boundary conditions. More precisely consider

$$\partial_t f + v\cdot\nabla_x f = \mathscr{C}(f) \tag{2.44}$$

where the spatial variable x lies in a bounded domain $\Omega\subseteq\mathbf{R}^d$. This equation is complemented with boundary conditions, modelling the interaction of the particles with the boundary of the domain.

For $x\in\partial\Omega$ we denote by $n(x)$ the outward unit normal vector of $\partial\Omega$ at x. We then define the sets

$$\Gamma_\pm := \left\{(x,v)\in\partial\Omega\times\mathbf{R}^d \mid \pm n(x)\cdot v>0\right\},$$

and consider the following different boundary conditions:

— *Specular reflection boundary condition*: there holds

$$\forall(x,v)\in\Gamma_-,\quad f(t,x,v) = f(t,x,\mathcal{R}_x(v))$$

where $\mathcal{R}_x(v) = v-2(n(x)\cdot v)n(x)$. This condition corresponds to the situation in which the particles bounce at the boundary as billiard balls;

— *Diffusive boundary condition*: there holds

$$\forall(x,v)\in\Gamma_-,\quad f(t,x,v) = c_\mu\,\mu(v)\int_{n(x)\cdot w>0} f(t,x,w)\,n(x)\cdot w\,\mathrm{d}w$$

where $\mu(v) = (2\pi)^{-\frac{d}{2}}\mathrm{e}^{-\frac{|v|^2}{2}}$ and $c_\mu = \int_{n(x)\cdot v>0}\mu(v)\,n(x)\cdot v\,\mathrm{d}v$. This condition corresponds to the situation in which particles are absorbed by the boundary and then are emitted into the domain following the equilibrium;

as well as a combination of them.

This problem has been solved for cutoff nonlinear Boltzmann equations, see Guo [117], Briant-Guo [36], Kim-Lee [137] and the references therein. The only result we are aware of in the case of singular collision operators in a bounded domain, that is for non-cutoff Boltzmann and Landau operators, are the very recent works of Guo-Hwang-Jang-Ouyang [119] for the Landau equation with specular reflection, and Duan-Liu-Sakamoto-Strain [92] for non-cutoff Boltzmann and Landau equations in a finite channel with inflow or specular reflection boundary conditions.

An interesting question would to apply the method developed in this chapter, in particular using Korn-type inequalities in order to obtain coercivity for the macroscopic momentum, in order to prove the convergence to equilibrium for the linear equation (2.44), the main objective being to treat the purely specular reflection case. This would gives us a constructive proof of the convergence and we would also be able to treat non-cutoff Boltzmann and Landau linearized equations. Once this is achieved, one could be interested in obtaining perturbative results for the associated nonlinear Boltzmann and Landau equations.

2.5.3 Nonlinear kinetic equations with confining potential

A very interesting problem, which to our knowledge is completely open, would be to investigate a nonlinear kinetic equation in the whole space with confining potential in a close-to-equilibrium framework. More precisely, one considers the equation

$$\partial_t F + v \cdot \nabla_x F - \nabla_x \phi \cdot \nabla_x F = Q(F, F) \tag{2.45}$$

for the distribution of particles $F = F(t, x, v)$, and where Q represents the Boltzmann or Landau collision operator.

The perturbation $f = F - \mathscr{M}$, where $\mathscr{M} = \mathrm{e}^{-\phi(x)}\mu(v)$ is an equilibrium of (2.45), verifies

$$\partial_t f + v \cdot \nabla_x f - \nabla_x \phi \cdot \nabla_v f = \mathrm{e}^{-\phi}\mathscr{C}(f) + Q(f, f), \tag{2.46}$$

where $\mathscr{C}(f) = Q(\mu, f) + Q(f, \mu)$ is the linearized collision operator associated to Q.

One remarks here that, in comparison to (2.1), the coercive term in the linear part of the equation (2.46) is now degenerated because of the factor $\mathrm{e}^{-\phi}$ in front of the linearized collision operator. On the one hand, we believe that the methods of this chapter could also handle the linear equation

$$\partial_t f + v \cdot \nabla_x f - \nabla_x \phi \cdot \nabla_v f = \mathrm{e}^{-\phi}\mathscr{C}(f),$$

providing us with a very slow convergence to the equilibrium. However, at first sight, the good dissipative properties of the operator $\mathrm{e}^{-\phi}\mathscr{C}(f)$ are not strong enough to handle the nonlinear term in a perturbative way. In order to circumvent this issue, we could try to complement the above estimates with new dispersion-like estimates.

Chapter 3

Isothermal fluid equations

In this chapter we present the works [44] and [43] in collaboration with R. Carles and M. Hillairet.

3.1 Introduction

This chapter is devoted to the study of isothermal compressible fluid equations in the whole space $\mathbf{R}^d$ with $d \geqslant 1$. We consider the fluid system

$$\begin{cases} \partial_t \varrho + \operatorname{div}(\varrho u) = 0 & (3.1a) \\ \partial_t(\varrho u) + \operatorname{div}(\varrho u \otimes u) + \nabla P(\varrho) = \dfrac{\varepsilon^2}{2}\varrho \nabla \left(\dfrac{\Delta\sqrt{\varrho}}{\sqrt{\varrho}} \right) + \nu \operatorname{div}(\varrho \mathrm{D}^s u) & (3.1b) \end{cases}$$

complemented with an initial data $(\varrho_{\text{in}}, u_{\text{in}})$. Here $\varrho = \varrho(t,x) : \mathbf{R}_+ \times \mathbf{R}^d \to \mathbf{R}$ denotes the density of the fluid, $u = u(t,x) : \mathbf{R}_+ \times \mathbf{R}^d \to \mathbf{R}^d$ represents its velocity field, $P = P(\varrho)$ its pressure, $t \in \mathbf{R}_+$ is the time variable and $x \in \mathbf{R}^d$ the spatial variable. An *isothermal fluid* corresponds to the case in which the pressure law is affine with respect to the density, more precisely we shall consider that

$$P(\varrho) = \varrho.$$

The notation $\mathrm{D}^s u := \frac{1}{2}(\nabla u + \nabla u^\top)$ stands for the symmetric part of the gradient ∇u, and the parameters ε and ν are nonnegative constants in such a way that system (3.1) encodes the following fundamental fluid equations:

— the *Euler equation* corresponds to the case $\varepsilon = \nu = 0$;

— the *quantum Euler* or *Euler–Korteweg equation* corresponds to the case $\varepsilon > 0$ and $\nu = 0$;

— the *Navier–Stokes equation* (with degenerate viscosity) corresponds to the case $\varepsilon = 0$ and $\nu > 0$;

— the *quantum Navier–Stokes* or *Navier–Stokes–Korteweg equation* (with degenerate viscosity) corresponds to the case $\varepsilon, \nu > 0$.

We observe that the dissipative term we consider here corresponds to a degenerate Navier–Stokes term, for the diffusivity vanishes for null densities. We shall mention that another common fluid model corresponds to *polytropic fluids*, in which case the pressure law is given by

$$P(\varrho) = \rho^\gamma \quad \text{with} \quad \gamma > 1.$$

One easily obtains that system (3.1) enjoys some fundamental properties, at least formally. First of all, one has the conservation of mass: for all $t \geqslant 0$ one has

$$\int_{\mathbf{R}^d} \varrho(t,x)\,\mathrm{d}x = \int_{\mathbf{R}^d} \varrho_{\text{in}}(x)\,\mathrm{d}x, \tag{3.2}$$

as well as the non-negativity of the density: if $\varrho_{\text{in}} \geqslant 0$ then $\varrho(t,\cdot) \geqslant 0$ for all $t \geqslant 0$. Furthermore, defining the energy functional

$$E[\varrho,u] := \frac{1}{2}\int_{\mathbf{R}^d} \varrho|u|^2\,\mathrm{d}x + \frac{\varepsilon^2}{2}\int_{\mathbf{R}^d} |\nabla\sqrt{\varrho}|^2\,\mathrm{d}x + \int_{\mathbf{R}^d} \varrho\log\varrho\,\mathrm{d}x \tag{3.3}$$

one formally obtains the following energy identity, for all $t \geqslant 0$ one has

$$\frac{\mathrm{d}}{\mathrm{d}t}E[\varrho,u] + \nu\int_{\mathbf{R}^d} \varrho|\mathrm{D}^s u|^2\,\mathrm{d}x = 0$$

which implies

$$E[\varrho,u](t) + \int_0^t \nu\int_{\mathbf{R}^d} \varrho|\mathrm{D}^s u|^2\,\mathrm{d}x\,\mathrm{d}s = E_{\text{in}}, \tag{3.4}$$

where E_{in} denotes the energy of the initial data.

An important property of the isothermal fluid system $\gamma = 1$ is that the pressure term $\int_{\mathbf{R}^d} \varrho\log\varrho\,\mathrm{d}x$ appearing in the energy E has no definite sign, which prevent us to obtain a priori estimates from the above energy identity, as opposed to the pressure term

$$\frac{1}{\gamma-1}\int_{\mathbf{R}^d} \varrho^\gamma\,\mathrm{d}x$$

that appears in the energy of a polytropic fluid $\gamma > 1$. This is one of the reasons for which isothermal fluids were marginally studied in the literature.

In Section 3.2 we shall first recall some results on the large-time behavior of some special type of global solutions to the polytropic Euler equation, that is, system (3.1) with $\varepsilon = \nu = 0$ and $\gamma > 1$. We shall then construct a special type of global solutions to the isothermal Euler equation, and we will see the difference in the large-time behavior between the isothermal and polytropic cases.

We shall introduce in Section 3.3 a suitable time-dependent scaling of the unknowns (ϱ, u) defining thus the new unknowns (R, U) by (3.10). This will provides us with a rescaled isothermal system (3.11) in terms of (R, U), with which we shall be able to circumvent the difficulty related to a energy with no definite sign as described above. We then gather the new properties for the new unknowns (R, U). On the one hand, this new scaling is inspired by the work of Carles-Gallagher [45] on the logarithmic nonlinear Schrödinger equation. Indeed, thanks to the Madelung transform one can, at least formally, link a nonlinear Schrödinger equation to a Euler–Korteweg system, i.e. system (3.1) with $\varepsilon > 0$ and $\nu = 0$, with a pressure law depending on the nonlinearity of the Schrödinger equation. In the case of a logarithmic nonlinearity the corresponding Euler–Korteweg system is the isothermal one. It was proven in Carles-Gallagher [45] that the dispersion in the logarithmic nonlinear Schrödinger equation follows a universal behavior, which differs from the case of power-like nonlinearities. On the other hand, this new scaling appears naturally in the construction of the special type of global solutions for the isothermal Euler equation in Section 3.2.

We shall be interested in the sequel in two results for the isothermal system (3.1): existence of global weak solutions and their large-time behavior. Both results are actually stated in terms of the new unknowns (R, U) satisfying the rescaled isothermal system (3.11), since as explained above it is with these unknowns that one can obtain good energy estimates, and only then we obtain the corresponding results for the system (3.1) by coming back to the original unknowns (ϱ, u).

The first result we present in Section 3.4 concerns the large-time dynamics of global weak solutions to the isothermal system for all types of fluids described above: Euler ($\varepsilon = \nu = 0$), Euler–Korteweg ($\varepsilon > 0$ and $\nu = 0$), Navier–Stokes ($\varepsilon = 0$ and $\nu > 0$), and Navier–Stokes–Korteweg ($\varepsilon, \nu > 0$) equations.

In Section 3.5 we present our second result concerning the existence of global weak solutions to the isothermal system for Navier–Stokes ($\varepsilon = 0$ and $\nu > 0$) and Navier–Stokes–Korteweg ($\varepsilon, \nu > 0$) fluids.

3.2 Special global solutions to the Euler equation

In this section we consider the Euler equation (3.1) $\varepsilon = \nu = 0$ in both isothermal $\gamma = 1$ and polytropic $\gamma > 1$ cases, that is

$$\begin{cases} \partial_t \varrho + \operatorname{div}(\varrho u) = 0 & (3.5a) \\ \partial_t(\varrho u) + \operatorname{div}(\varrho u \otimes u) + \nabla(\varrho^\gamma) = 0. & (3.5b) \end{cases}$$

We shall present particular families of global solutions to (3.5) and then we will observe a contrast in their asymptotic behavior: in the polytropic case the density ϱ disperses with any profile, whereas in the isothermal case the density disperses with a faster rate and with a universal Gaussian profile.

3.2.1 Polytropic case $\gamma > 1$

The local well-posedness of smooth solutions to (3.5) is obtained in the work of Makino-Ukai-Kawashima [150] (see also the works of Chemin [61] and Xin [192]). More precisely, considering the new unknowns

$$(a, u) = \left(\varrho^{\frac{\gamma-1}{2}}, u\right)$$

one turns system (3.5) into a hyperbolic symmetric system

$$\begin{cases} \partial_t a + u \cdot \nabla a + \dfrac{\gamma-1}{2}\, a \operatorname{div} u = 0, & (3.6a) \\ \partial_t u + u \cdot \nabla u + \dfrac{2\gamma}{\gamma-1}\, a \nabla a = 0, & (3.6b) \end{cases}$$

with initial data $(a_{\text{in}}, u_{\text{in}}) = (\varrho_{\text{in}}^{\frac{\gamma-1}{2}}, u_{\text{in}})$, so that there exists a unique local solution $(a, u) \in \mathscr{C}(0, T^\star; H^s(\mathbf{R}^d))^{1+d}$, provided that $s > 1 + \frac{d}{2}$ and $(\varrho_{\text{in}}^{\frac{\gamma-1}{2}}, u_{\text{in}}) \in H^s(\mathbf{R}^d)^{1+d}$.

Furthermore, it is also proven in Makino-Ukai-Kawashima [150] that singularities appear in finite time (see also Xin [192]). More precisely, for any non-zero smooth initial data $(a_{\text{in}}, u_{\text{in}})$ with compact support, the solution to (3.6) develops a singularity in finite time, which is a consequence of the following two ingredients: As long as the solution is smooth, its speed of propagation is zero; and a virial computation shows that if the solution is global, then it is dispersive in the sense

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} \int_{\mathbf{R}^d} |x|^2 \varrho(t,x)\,\mathrm{d}x \geqslant E_\gamma[\varrho, u](t) \min\left(2, 3(\gamma-1)\right) = E_{\gamma,\text{in}} \min\left(2, 3(\gamma-1)\right) > 0,$$

where we have used the assumption $\gamma > 1$ and the conservation of the energy E_γ in the polytropic case, defined by

$$E_\gamma[\varrho, u] = \frac{1}{2} \int_{\mathbf{R}^d} \varrho |u|^2 \,\mathrm{d}x + \frac{1}{\gamma-1} \int_{\mathbf{R}^d} \varrho^\gamma \,\mathrm{d}x,$$

which is satisfied for smooth solutions. Therefore, supposing that the solution remains smooth for all times and then integrating the above estimate twice yields

$$\int_{\mathbf{R}^d} |x|^2 \varrho(t,x)\,\mathrm{d}x \gtrsim t^2,$$

which is incompatible with the fixed compact support of ϱ and the conservation of mass, since for some $K > 0$ independent of time there holds

$$\int_{\mathbf{R}^d} |x|^2 \varrho(t,x)\,\mathrm{d}x \leqslant K^{2d} \int_{\mathbf{R}^d} \varrho_{\text{in}}\,\mathrm{d}x.$$

Although the previous result on the appearance of singularities in finite time, a first global existence of smooth solutions to (3.5) for polytropic fluids was obtained by D. Serre [172] (see also the work of Grassin [111]), under an extra geometric assumption involving a special structure for the initial velocity. We consider $1 < \gamma \leqslant 1 + \frac{2}{d}$, then we define the new unknowns $(\overline{R}, \overline{U})$ defined by

$$\varrho(t,x) = \frac{1}{(1+t)^d}\, \overline{R}\left(\frac{t}{1+t}, \frac{x}{1+t}\right)$$

and

$$u(t,x) = \frac{1}{1+t}\, \overline{U}\left(\frac{t}{1+t}, \frac{x}{1+t}\right), + \frac{x}{1+t},$$

and assume that $R_{\mathrm{in}}^{\frac{\gamma-1}{2}}, U_{\mathrm{in}} \in H^s(\mathbf{R}^d)$ for some $s > 1 + \frac{d}{2}$. Remark that this means $\varrho_{\mathrm{in}}^{\frac{\gamma-1}{2}} \in H^s(\mathbf{R}^d)$ and $u_{\mathrm{in}} - x \in H^s(\mathbf{R}^d)$, thus u_{in} does not belong to $L^2(\mathbf{R}^d)$, and that the time variable has been compactified.

It is proven in D. Serre [172] that there exists $\eta > 0$ such that if $\|(\varrho_{\mathrm{in}}^{\frac{\gamma-1}{2}}, U_{\mathrm{in}})\|_{H^s(\mathbf{R}^d)} \leqslant \eta$, then there is a unique global solution to (3.5), in the sense that $(\overline{R}, \overline{U}) \in \mathscr{C}([0,1]; H^s(\mathbf{R}^d))^{1+d}$, and moreover its asymptotic behavior is given by

$$\lim_{t\to\infty} \left\| \left(\varrho(t,x) - \frac{1}{t^d} R_\infty\left(\frac{x}{t}\right), u(t,x) - \frac{x}{1+t} - \frac{1}{1+t} U_\infty\left(\frac{x}{t}\right)\right)\right\|_{L^\infty(\mathbf{R}^d)} = 0. \tag{3.7}$$

for some profile $R_\infty, U_\infty \in H^s(\mathbf{R}^d)$. Conversely, if $R_\infty, U_\infty \in H^s(\mathbf{R}^d)$ are such that $\|(R_\infty^{\frac{\gamma-1}{2}}, U_\infty)\|_{H^s(\mathbf{R}^d)} \leqslant \eta$, then there exists Cauchy data $(\varrho_{\mathrm{in}}^{\frac{\gamma-1}{2}}, u_{\mathrm{in}}) \in H^s(\mathbf{R}^d)^{1+d}$ such that the associated solution is global in time in the same sense as above and verifies (3.7).

It is important to observe that in the above result, one obtains that the density disperses with a universal rate

$$\|\varrho(t)\|_{L^\infty(\mathbf{R}^d)} \lesssim \frac{1}{t^d}$$

and an arbitrary (smooth and small enough) asymptotic profile R_∞.

3.2.2 Isothermal case $\gamma = 1$

We shall construct explicit global solutions with Gaussian densities and affine velocities to the isothermal Euler equation (3.5).

We remark that it was first observed by Yuen [194] that a particular structure of the initial data is preserved by the flow: a Gaussian density and an affine velocity centered at the same point. Here we shall consider initial Gaussian density and affine velocities with different centers in the form

$$\varrho_{\mathrm{in}}(x) = b_0 \mathrm{e}^{-\sum_{j=1}^d \alpha_{0j} x_j^2}, \quad u_{\mathrm{in},j}(x) = \beta_{0j} x_j + c_{0j}, \quad j \in \{1, \dots, d\},$$

with $b_0, \alpha_{0j} > 0$, $\beta_{0,j}, c_{0,j} \in \mathbf{R}$. We plug the ansatz

$$\varrho(t,x) = b(t) \mathrm{e}^{-\sum_{j=1}^d \alpha_j(t)(x_j - \overline{x}_j)^2}, \quad u_j(t,x) = \beta_j(t) x_j + c_j(t)$$

into (3.5) and we obtain a set of ordinary differential equations to solve. The solution is finally given by

$$\alpha_j(t) = \frac{\alpha_{0j}}{\tau_j(t)^2}, \quad \beta_j(t) = \frac{\dot{\tau}_j(t)}{\tau_j(t)},$$

where $\tau_j \in \mathscr{C}^2(\mathbf{R}_+, \mathbf{R}_+)$ is the unique solution to

$$\ddot{\tau}_j = \frac{2\alpha_{0j}}{\tau_j}, \quad \tau_j(0) = 1, \quad \dot{\tau}_j(0) = \beta_{0j}, \tag{3.8}$$

that satisfies moreover $\tau_j(t) \underset{t\to\infty}{\sim} 2t\sqrt{\alpha_{0j}\log t}$ and $\dot\tau_j(t) \underset{t\to\infty}{\sim} 2\sqrt{\alpha_{0j}\log t}$.

We have hence constructed a global solution (ϱ, u) to the isothermal Euler equation (3.5) that satisfies moreover the following asymptotic behavior

$$\varrho(t,x) \underset{t\to\infty}{\sim} \frac{\|\varrho_{\mathrm{in}}\|}{\pi^{d/2}} \frac{\mathrm{e}^{-|x|^2/(2t\sqrt{\log t})^2}}{(2t\sqrt{\log t})^d} \quad \text{and} \quad u(t,x) \underset{t\to\infty}{\sim} \frac{x}{t}. \tag{3.9}$$

It is worth noticing the difference in the asymptotic behaviors of the density ϱ of the fluid with respect to the polytropic case: the dispersion rate is faster (by a logarithmic correction) and the asymptotic profile is a fixed Gaussian function, which is different to the polytropic case where the profile is arbitrary.

Actually the above construction also works for the other isothermal fluid equations, that is, for Euler–Korteweg ($\varepsilon > 0$ and $\nu = 0$), Navier–Stokes ($\varepsilon = 0$ and $\nu > 0$), and Navier–Stokes–Korteweg ($\varepsilon, \nu > 0$), and hence provides us with a global solution with the same asymptotic behavior (3.9). In these cases, new terms depending on ε, ν appear in the equation (3.8), but its solution satisfies the same asymptotic as above.

3.3 Rescaled system

From now on in this chapter we shall only consider the case of isothermal fluids $\gamma = 1$.

As explained in the introduction, inspired by the global solutions constructed in Section 3.2 as well as by the work of Carles-Gallagher [45] on the dispersion of solutions to the nonlinear logarithmic Schrödinger equation, if (ϱ, u) is a solution to the isothermal system (3.1) we then consider the new rescaled unknowns (R, U) defined by

$$\varrho(t,x) = \frac{1}{\tau(t)^d} R\left(t, \frac{x}{\tau(t)}\right) \frac{\|\varrho_{\mathrm{in}}\|_{L^1}}{\|\Gamma\|_{L^1}} \tag{3.10a}$$

and

$$u(t,x) = \frac{1}{\tau(t)} U\left(t, \frac{x}{\tau(t)}\right) + \frac{\dot\tau(t)}{\tau(t)} x, \tag{3.10b}$$

where $\Gamma(y) = \mathrm{e}^{-|y|^2}$ is a Gaussian function and the function $\tau \in \mathscr{C}^2(\mathbf{R}_+, \mathbf{R}_+)$ is the global solution to the nonlinear ODE

$$\ddot\tau = \frac{2}{\tau}, \quad \tau(0) = 1, \quad \dot\tau(0) = 0$$

that remains uniformly bounded from below by a strictly positive constant and its large time behavior is given by

$$\tau(t) \underset{t\to\infty}{\sim} 2t\sqrt{\log t}, \quad \dot\tau(t) \underset{t\to\infty}{\sim} 2\sqrt{\log t}.$$

We rewrite the original isothermal system (3.1) in the terms of the new unknown $(R, U) = (R(t,y), U(t,y))$, and we obtain the following rescaled isothermal system

$$\begin{cases} \partial_t R + \dfrac{1}{\tau^2} \operatorname{div}(RU) = 0 & (3.11\mathrm{a}) \\ \partial_t (RU) + \dfrac{1}{\tau^2} \operatorname{div}(RU \otimes U) + 2yR + \nabla R & (3.11\mathrm{b}) \\ \qquad = \dfrac{\varepsilon^2}{2\tau^2} R \nabla\left(\dfrac{\Delta\sqrt{R}}{\sqrt{R}}\right) + \dfrac{\nu}{\tau^2}\operatorname{div}(R\mathrm{D}^s U) + \dfrac{\nu\dot\tau}{\tau}\nabla R. \end{cases}$$

where we shall denote by $y \in \mathbf{R}^d$ the spatial variable in the new unknowns (R, U). Since the change of unknown (3.10) preserves the integrability properties of density and velocity unknowns locally in time, we shall focus only on system (3.11) from now on and, as already explained in the introduction, our results will be stated in terms of the new unknowns (R, U).

3.3.1 New a priori estimates

We first observe that system (3.11) still conserves mass and thanks to (3.10) we now have, for all $t \geqslant 0$,

$$\int_{\mathbf{R}^d} R(t,y)\,\mathrm{d}y = \int_{\mathbf{R}^d} \Gamma(y)\,\mathrm{d}y = \pi^{\frac{d}{2}}.$$

An interesting and important feature of system (3.11) is that it possesses an energy functional which is now sign-definite and still verifies an energy-dissipation type estimate. As a consequence, we shall extract a priori estimates from it. More precisely, we define the energy functional associated to (3.11) as

$$\mathscr{E}[R,U] = \frac{1}{2\tau^2}\int_{\mathbf{R}^d} \left(R|U|^2 + \varepsilon^2 |\nabla\sqrt{R}|^2 \right) \mathrm{d}y + \int_{\mathbf{R}^d} \left(R|y|^2 + R\log R \right) \mathrm{d}y. \tag{3.12}$$

Therefore, at least formally, solutions to (3.11) verify the following energy identity: for all $t \geqslant 0$, there holds

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{E}[R,U] + \mathscr{D}[R,U] = -\nu \frac{\dot{\tau}}{\tau^3}\int_{\mathbf{R}^d} R \operatorname{div} U \,\mathrm{d}y$$

which yields

$$\mathscr{E}[R,U](t) + \int_0^t \mathscr{D}[R,U](s)\,\mathrm{d}s = \mathscr{E}_{\mathrm{in}} - \nu \int_0^t \frac{\dot{\tau}}{\tau^3}\int_{\mathbf{R}^d} R \operatorname{div} U \,\mathrm{d}y\,\mathrm{d}s, \tag{3.13}$$

where $\mathscr{E}_{\mathrm{in}}$ denotes the energy of the initial data, and the nonnegative dissipation is given by

$$\mathscr{D}[R,U] = \frac{\dot{\tau}}{\tau^3}\int_{\mathbf{R}^d} \left(R|U|^2 + \varepsilon^2 |\nabla\sqrt{R}|^2 \right) \mathrm{d}y + \frac{\nu}{\tau^4}\int_{\mathbf{R}^d} R|\mathrm{D}^s U|^2\,\mathrm{d}y. \tag{3.14}$$

We remark that, thanks to the conservation of mass, the functional $\mathscr{E}$ is nonnegative by observing that the last term in (3.12) corresponds to the relative entropy of R with respect to Γ, that is

$$\int_{\mathbf{R}^d} \left(R|y|^2 + R\log R \right) \mathrm{d}y = \int_{\mathbf{R}^d} R \log\left(\frac{R}{\Gamma} \right) \mathrm{d}y \geqslant 0.$$

Finally, we observe that identity (3.13) provides a priori estimates for solutions (R,U) to (3.11), assuming that the initial energy $\mathscr{E}_{\mathrm{in}}$ is finite. Indeed, on the one hand, the last term in the right-hand side of (3.13) can de controlled by the dissipation $\mathscr{D}$ using that $\int_0^\infty \frac{\dot{\tau}^2}{\tau^2} < \infty$, more precisely

$$\nu \int_0^t \frac{\dot{\tau}}{\tau^3}\int_{\mathbf{R}^d} R|\operatorname{div} U|\,\mathrm{d}y\,\mathrm{d}s \leqslant C\nu \int_0^t \frac{\dot{\tau}^2}{\tau^2}\int_{\mathbf{R}^d} R\,\mathrm{d}y\,\mathrm{d}s + \frac{1}{2}\int_0^t \mathscr{D}[R,U](s)\,\mathrm{d}s.$$

On the other hand, a standard argument also gives the control of $\int_{\mathbf{R}^d} R|\log R|\mathrm{d}y$, since we obtain a control of $\int_{\mathbf{R}^d} R\,(1+|y|^2+\log R)\,\mathrm{d}y$.

The construction of a positive-definite energy which is dissipated with time is a first building-block to construct solutions to (3.11). However, it is classical in compressible fluid mechanics, more precisely when dealing with Navier–Stokes equations with degenerate viscosity, that (3.13) must be completed. For instance, studies on compactness of finite-energy solution to (3.11) require to handle the viscous stress $R\mathrm{D}^s U$. Yet, when $\varepsilon = 0$ the information provided by (3.13) is insufficient to pass to the limit in this term, see for instance the works Bresch-Desjardins-Lin [31] and Vasseur-Yu [182], because we lack information on the regularity of the density R. More specifically, in the case of (3.11), with (3.13) alone, it is not clear also how to define the Korteweg term when $\varepsilon > 0$.

In order to handle the above difficulties, another important functional, known as BD-entropy, was introduced in Bresch-Desjardins [30] and Bresch-Desjardins-Lin [31], and has now become a standard tool in the study of degenerate compressible Navier–Stokes equations. In the case of system (3.11), the BD-entropy reads

$$\mathscr{E}_{\mathrm{BD}}[R,U]=\frac{1}{2\tau^2}\int_{\mathbf{R}^d}\Big(R|U+\nu\nabla\log R|^2+\varepsilon^2|\nabla\sqrt{R}|^2\Big)\,\mathrm{d}y+\int_{\mathbf{R}^d}\Big(R|y|^2+R\log R\Big)\,\mathrm{d}y \tag{3.15}$$

where we observe that, as before, the last integral defines a nonnegative term. By computing the evolution of this BD-entropy, one also obtains that, at least formally, for all $t\geqslant 0$ there holds

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{E}_{\mathrm{BD}}[R,U]+\mathscr{D}_{\mathrm{BD}}[R,U]=\nu\frac{2d}{\tau^2}\int_{\mathbf{R}^d}R\,\mathrm{d}y+\nu\frac{\dot\tau}{\tau^3}\int_{\mathbf{R}^d}R\,\mathrm{div}\,U\,\mathrm{d}y$$

which implies

$$\begin{aligned}\mathscr{E}_{\mathrm{BD}}[R,U](t)&+\int_0^t\mathscr{D}_{\mathrm{BD}}[R,U](s)\,\mathrm{d}s\\&=\mathscr{E}_{\mathrm{BD,in}}+\nu\int_0^t\frac{2d}{\tau^2}\int_{\mathbf{R}^d}R\,\mathrm{d}y\,\mathrm{d}s+\nu\int_0^t\frac{\dot\tau}{\tau^3}\int_{\mathbf{R}^d}R\,\mathrm{div}\,U\,\mathrm{d}y\,\mathrm{d}s\end{aligned} \tag{3.16}$$

where $\mathscr{E}_{\mathrm{BD,in}}$ denotes the BD-entropy of the initial data and the nonnegative dissipation associated to the BD-entropy is defined as

$$\begin{aligned}\mathscr{D}_{\mathrm{BD}}[R,U]=\frac{\dot\tau}{\tau^3}\int_{\mathbf{R}^d}\Big(R|U|^2+\varepsilon^2|\nabla\sqrt{R}|^2\Big)\,\mathrm{d}y+\frac{\nu}{\tau^4}\int_{\mathbf{R}^d}R|\mathrm{D}^aU|^2\,\mathrm{d}y\\+\frac{\nu\varepsilon^2}{\tau^4}\int_{\mathbf{R}^d}R|\nabla^2\log R|^2\,\mathrm{d}y+\frac{4\nu}{\tau^2}\int_{\mathbf{R}^d}|\nabla\sqrt{R}|^2\,\mathrm{d}y,\end{aligned} \tag{3.17}$$

where $\mathrm{D}^aU:=\frac12(\nabla U-\nabla U^\top)$ denotes the anti-symmetric part of ∇U. Therefore gathering together the energy and the BD-entropy estimates yields that, for all $t\geqslant 0$, one has

$$\begin{aligned}\mathscr{E}[R,U](t)+\mathscr{E}_{\mathrm{BD}}[R,U](t)+\int_0^t\mathscr{D}[R,U](s)\,\mathrm{d}s+\int_0^t\mathscr{D}_{\mathrm{BD}}[R,U](s)\,\mathrm{d}s\\=\mathscr{E}_{\mathrm{in}}+\mathscr{E}_{\mathrm{BD,in}}+\nu\int_0^t\frac{2d}{\tau^2}\int_{\mathbf{R}^d}R\,\mathrm{d}y\,\mathrm{d}s\end{aligned} \tag{3.18}$$

and thanks to the conservation of mass and the fact that $\int_0^\infty\tau^{-2}(t)\,\mathrm{d}t<\infty$, the last term is uniformly bounded. We note that, in view of (3.16), we gain information on the regularity of R when $\nu>0$ which may help in the compactness issue of weak solutions to (3.11). To define the Korteweg term, we may also apply the classical identity

$$R\nabla\left(\frac{\Delta\sqrt{R}}{\sqrt{R}}\right)=\mathrm{div}(\sqrt{R}\nabla^2\sqrt{R}-\nabla\sqrt{R}\otimes\nabla\sqrt{R}),$$

and remark that, thanks to Jungel [135] and Vasseur-Yu [182], one has

$$\int_\Omega|\nabla^2\sqrt{R}|^2+\int_\Omega|\nabla R^{1/4}|^4\lesssim\int_\Omega R|\nabla^2\log R|^2\lesssim\int_\Omega|\nabla^2\sqrt{R}|^2+\int_\Omega|\nabla R^{1/4}|^4,$$

which holds for $\Omega=\mathbf{R}^d$ or $\mathbf{T}^d$.

3.3.2 Notion of weak solutions

Motivated by the a priori estimates provided above, we introduce the following notion of weak solutions to (3.11).

Definition 3.1. Assume $\varepsilon,\nu\geqslant 0$. Let $(\sqrt{R}_{\mathrm{in}},(\sqrt{R}U)_{\mathrm{in}})\in L^2(\mathbf{R}^d)\times L^2(\mathbf{R}^d)$. One says that (R,U) is a *global weak solution* to (3.11) associated to the initial data $(\sqrt{R}_{\mathrm{in}},(\sqrt{R}U)_{\mathrm{in}})$ if there exists a collection $(\sqrt{R},\sqrt{R}U,\mathbb{S}_K,\mathbb{T}_N)$ such that:

(i) The following regularities are satisfied

$$
\begin{aligned}
(1+|y|^2)^{1/2}\sqrt{R},\ \sqrt{R}U &\in L^\infty_{\mathrm{loc}}(\mathbf{R}_+;L^2(\mathbf{R}^d))\\
\nabla\sqrt{R} &\in L^\infty_{\mathrm{loc}}(\mathbf{R}_+;L^2(\mathbf{R}^d)) \text{ if } \varepsilon>0 \text{ or } \nu>0,\\
\nabla^2\sqrt{R} &\in L^2_{\mathrm{loc}}(\mathbf{R}_+;L^2(\mathbf{R}^d)) \text{ if } \varepsilon>0,\\
\mathbb{T}_N &\in L^2_{\mathrm{loc}}(\mathbf{R}_+;L^2(\mathbf{R}^d)) \text{ if } \nu>0,\\
\mathbb{S}_K &\in L^2_{\mathrm{loc}}(\mathbf{R}_+;L^1(\mathbf{R}^d)) \text{ if } \varepsilon>0,
\end{aligned}
$$

with the compatibility conditions $\sqrt{R}\geqslant 0$ a.e. on $(0,\infty)\times\mathbf{R}^d$ and $\sqrt{R}U=0$ a.e. on $\{\sqrt{R}=0\}$.

(ii) **Euler equation ($\varepsilon=\nu=0$):** The following system is satisfied in $\mathscr{D}'((0,\infty)\times\mathbf{R}^d)$

$$
\begin{cases}
\partial_t(|\sqrt{R}|^2)+\dfrac{1}{\tau^2}\operatorname{div}(\sqrt{R}\sqrt{R}U)=0, & (3.19a)\\
\partial_t(\sqrt{R}\sqrt{R}U)+\dfrac{1}{\tau^2}\operatorname{div}(\sqrt{R}U\otimes\sqrt{R}U)+2y|\sqrt{R}|^2+\nabla\left(|\sqrt{R}|^2\right)=0. & (3.19b)
\end{cases}
$$

(iii) **Korteweg and Navier–Stokes equations ($\varepsilon>0$ or $\nu>0$):** The following system is satisfied in $\mathscr{D}'((0,\infty)\times\mathbf{R}^d)$

$$
\begin{cases}
\partial_t\sqrt{R}+\dfrac{1}{\tau^2}\operatorname{div}(\sqrt{R}U)=\dfrac{1}{2\tau^2}\operatorname{trace}(\mathbb{T}_N) & (3.20a)\\
\partial_t(\sqrt{R}\sqrt{R}U)+\dfrac{1}{\tau^2}\operatorname{div}(\sqrt{R}U\otimes\sqrt{R}U)+2y|\sqrt{R}|^2+\nabla\left(|\sqrt{R}|^2\right) & (3.20b)\\
\qquad=\dfrac{1}{\tau^2}\operatorname{div}\left(\nu\sqrt{R}\,\mathbb{S}_N+\dfrac{\varepsilon^2}{2}\mathbb{S}_K\right)+\dfrac{\nu\dot{\tau}}{\tau}\nabla\left(|\sqrt{R}|^2\right), &
\end{cases}
$$

where $\mathbb{S}_N$ is the symmetric part of the tensor $\mathbb{T}_N$ and the following compatibility conditions hold

$$
\sqrt{R}\mathbb{T}_N=\nabla(\sqrt{R}\sqrt{R}U)-2\sqrt{R}U\otimes\nabla\sqrt{R}, \tag{3.21a}
$$

$$
\mathbb{S}_K=\sqrt{R}\nabla^2\sqrt{R}-\nabla\sqrt{R}\otimes\nabla\sqrt{R}. \tag{3.21b}
$$

(iv) The initial condition is satisfied in the sense of distributions.

Remark that weak solutions to (3.11) are defined in terms of $\sqrt{R}$ and $\sqrt{R}U$, since these are the natural quantities appearing in the energy and BD-entropy estimates. Therefore R has to be understood as $|\sqrt{R}|^2$, and whenever U appears alone it should be understood as $U=\frac{\sqrt{R}U}{\sqrt{R}}\mathbf{1}_{\sqrt{R}>0}$, which is well-defined thanks to the compatibility condition. In the energy and BD-entropy estimates, the term $R|\mathrm{D}^sU|^2$ has to be understood as $|\mathbb{S}_N|^2$ and, similarly, the term $R|\mathrm{D}^aU|^2$ as $|\mathbb{T}_N-\mathbb{S}_N|^2$ and the term $R|\nabla U|^2$ as $|\mathbb{T}_N|^2$.

Furthermore, note that in the case $\varepsilon>0$ or $\nu>0$, we do not ask for the continuity equation (3.11a) in terms of R to hold but rather the continuity equation (3.20a) in terms of $\sqrt{R}$. One can however check that our definition implies the usual continuity equation (3.11a). Also, the fact that the mass is conserved is implied by our definition.

3.4 Universal large-time dynamics

This section is devoted to the large-time behavior of the rescaled isothermal fluid system (3.11) for all types of fluids: Euler ($\varepsilon=\nu=0$), Euler–Korteweg ($\varepsilon>0$ and $\nu=0$), Navier–Stokes ($\varepsilon=0$ and $\nu>0$), and Navier–Stokes–Korteweg ($\varepsilon,\nu>0$).

We can now state our result on the universal behavior for global weak solutions established in [44].

Theorem 3.A. *Assume $\varepsilon, \nu \geqslant 0$. Let (R,U) be a global weak solution to the (3.11) in the sense of Definition 3.1, then there holds:*

(i) *If $\int_0^\infty \mathscr{D}[R,U](t)\,\mathrm{d}t < \infty$, then*

$$\int_{\mathbf{R}^d} yR(t,y)\,\mathrm{d}y \longrightarrow 0 \quad \text{and} \quad \left|\int_{\mathbf{R}^d} (RU)(t,y)\,\mathrm{d}y\right| \longrightarrow \infty \quad \text{as} \quad t \to +\infty,$$

unless initially one has $\int_{\mathbf{R}^d} yR_{\mathrm{in}}(y)\,\mathrm{d}y = \int_{\mathbf{R}^d} \sqrt{R}_{\mathrm{in}}(\sqrt{R}U)_{\mathrm{in}}(y)\,\mathrm{d}y = 0$, in which case these quantities remain zero for any time.

(ii) *If $\sup_{t\geqslant 0} \mathscr{E}[R,U](t) < \infty$ and the energy $E[\varrho,u]$ of the original system (defined by (3.2)) satisfies $E(t) = o(\log t)$ as $t \to +\infty$, then*

$$\int_{\mathbf{R}^d} |y|^2 R(t,y)\,\mathrm{d}y \longrightarrow \int_{\mathbf{R}^d} |y|^2 \Gamma(y)\,\mathrm{d}y \quad \text{as} \quad t \to +\infty.$$

(iii) *If $\sup_{t\geqslant 0} \mathscr{E}[R,U](t) + \int_0^\infty \mathscr{D}[R,U](t)\,\mathrm{d}t < \infty$, then*

$$R(t,\cdot) \rightharpoonup \Gamma \text{ weakly in } L^1(\mathbf{R}^d) \text{ as } t \to +\infty.$$

Remark 3.1. The results in Theorem 3.A actually hold in a more general setting of pressure laws satisfying $P \in \mathscr{C}^1(\mathbf{R}_+, \mathbf{R}_+) \cap \mathscr{C}^2(\mathbf{R}_+^*, \mathbf{R}_+)$ convex with $P'(0) > 0$. Observe that the isothermal case $P(\varrho) = \varrho$ falls in this class, but the polytropic fluids $P(\varrho) = \varrho^\gamma$ with $\gamma > 1$ do not.

Coming back to the original unknowns (ϱ, u) by (3.10), this result yields that if (ϱ, u) is a global weak solution to the isothermal system (3.1), for a suitable notion of weak solutions analogous to Definition 3.1, that satisfies further global estimates, then one has

$$\varrho(t,x) \underset{t\to\infty}{\sim} \frac{\|\varrho_{\mathrm{in}}\|_{L^1(\mathbf{R}^d)}}{\pi^{\frac{d}{2}}} \frac{1}{\tau(t)^d}\, \mathrm{e}^{-\frac{|x|^2}{\tau(t)^2}} \quad \text{weakly in} \quad L^1(\mathbf{R}^d).$$

Points (i) and (ii) in Theorem 3.A follow from a straightforward computation and using the appropriated assumptions.

Let us now explain the proof of point (iii) in Theorem 3.A in more details. At a very formal level, discarding terms which seem negligible for large time in (3.11), we get

$$\begin{cases} \partial_t R + \dfrac{1}{\tau^2} \operatorname{div}(RU) = 0, \\ \partial_t(RU) + 2yR + \nabla R = 0, \end{cases}$$

hence

$$\partial_t \left(\tau^2 \partial_t R\right) = LR := \Delta R + 2\operatorname{div}(yR),$$

where L is a Fokker–Planck operator. Since $\tau^2 \ll (\dot{\tau}\tau)^2$ as $t \to \infty$, we expect that

$$\partial_t \left(\tau^2 \partial_t R\right) = \tau^2 \partial_t^2 R + 2\dot{\tau}\tau \partial_t R \approx 2\dot{\tau}\tau\partial_t R \ \text{ as } \ t \to \infty.$$

Therefore introducing the new time-variable $s(t) = \frac{1}{2}\log \dot{\tau}(t)$ that satisfies $s(t) \approx \frac{1}{4}\log\log t$ as $t \to \infty$, one gets $\partial_s R \approx LR$. The large time behavior is thus expected to be dictated by the Fokker–Planck equation $\partial_s R_\infty = LR_\infty$, for which we know by the work Arnold-Markowich-Toscani-Unterreiter [9] that $R(s) \to \Gamma$ in $L^1(\mathbf{R}^d)$ as $s \to \infty$.

In order to make this heuristic rigorous, we need to work with weak solutions and exploit the assumption $\sup_{t\geqslant 0} \mathscr{E}[R,U](t) + \int_0^\infty \mathscr{D}[R,U](t)\,\mathrm{d}t < \infty$ that shall provides us with enough compactness to conclude. Let us denote by $\alpha : s \mapsto \alpha(s) = t$ the inverse map of s. Let

$s \in [0,1]$, consider a sequence $s_n \to \infty$ when $n \to \infty$ and define $\alpha_n(s) := \alpha(s+s_n)$. We define the sequences $\bar{R}_n(s,y) := R(\alpha_n(s),y)$, $\bar{U}_n(s,y) := U(\alpha_n(s),y)$, $\bar{\mathbb{T}}_{N,n} := \mathbb{T}_N(\alpha_n(s),y)$, and $\bar{\mathbb{S}}_{K,n} := \mathbb{S}_K(\alpha_n(s),y)$, in such a way that

$$\partial_s \bar{R}_n - \frac{2}{(\dot{\tau}\circ\alpha_n)^2}\partial_s \bar{R}_n + \frac{1}{(\dot{\tau}\circ\alpha_n)^2}\partial_s^2 \bar{R}_n = L\bar{R}_n + \mathcal{N}_{\alpha_n}[\bar{R}_n, \bar{U}_n, \bar{\mathbb{T}}_{N,n}, \bar{\mathbb{S}}_{K,n}] \tag{3.22}$$

where $\mathcal{N}_{\alpha_n}$ denotes a nonlinear term depending on $\bar{R}_n, \bar{U}_n, \bar{\mathbb{T}}_{N,n}, \bar{\mathbb{S}}_{K,n}$. Thanks to the assumption $\sup_{t\geqslant 0} \mathscr{E}[R,U](t) + \int_0^\infty \mathscr{D}[R,U](t)\,\mathrm{d}t < \infty$, one obtains the estimates

$$\sup_{n\in\mathbf{N}} \sup_{s\in[0,1]} \int_{\mathbf{R}^d} \bar{R}_n(1+|y|^2+|\log \bar{R}_n|)\mathrm{d}y \leqslant C, \tag{3.23a}$$

$$\lim_{n\to\infty} \int_0^1 \left(\frac{\dot{\tau}\circ\alpha_n}{\tau\circ\alpha_n}\right)^2 \left(\|\sqrt{\bar{R}_n}\bar{U}_n\|^2_{L^2(\mathbf{R}^d)} + \varepsilon^2 \|\nabla\sqrt{\bar{R}_n}\|^2_{L^2(\mathbf{R}^d)}\right) \mathrm{d}s = 0, \tag{3.23b}$$

and

$$\lim_{n\to\infty} \nu \int_0^1 \frac{\dot{\tau}\circ\alpha_n}{(\tau\circ\alpha_n)^3} \|\bar{\mathbb{S}}_{N,n}\|^2_{L^2(\mathbf{R}^d)}\,\mathrm{d}s = 0. \tag{3.23c}$$

Thanks to estimate (3.23a) and the Dunford-Pettis theorem, we deduce that there exists $R_\infty \in L^1((0,1)\times\mathbf{R}^d)$ such that, up to extracting a subsequence, one has

$$\bar{R}_n \rightharpoonup R_\infty \text{ weakly in } L^1((0,1)\times\mathbf{R}^d) \text{ as } n\to\infty,$$

Therefore, passing to the limit $n\to\infty$ in equation (3.22), we obtain

$$\partial_s R_\infty = LR_\infty \quad \text{in} \quad \mathscr{D}'((0,1)\times\mathbf{R}^d),$$

the other terms vanishing in the limit thanks to the estimates (3.23). By remarking that $\bar{R}_n$ cannot lose mass in the limit, we finally conclude that $R_\infty = \Gamma$ thanks to the work of Arnold-Markowich-Toscani-Unterreiter [9].

3.5 Existence of global weak solutions

In this section we are interested in the construction of global weak solutions for the rescaled isothermal Navier–Stokes and Navier–Stokes–Korteweg equations, that is system (3.11) with $\nu > 0$ and $\varepsilon \geqslant 0$.

As already mentioned in the introduction, isothermal fluids are marginally studied in the literature. As a matter of fact, we are only aware of a result of Jungel [135] that establishes the existence of a particular type of weak solutions for the two-dimensional Navier–Stokes–Korteweg system (3.1) on the torus $\mathbf{T}^2$, and of the work by Plotnikov-Weigant [166, 183] which proves existence of solutions to the two-dimensional Navier–Stokes equation with fixed positive viscosity coefficients in a bounded domain, more precisely replacing the degenerate viscous stress tensor $\varrho \mathrm{D}^s u$ in (3.1b) by $2\mu\mathrm{D}^s u + \lambda \operatorname{div} u I_d$ with $\mu > 0$ and $2\mu+\lambda \geqslant 0$. Furthermore, the fact that we are working on the whole space $\mathbf{R}^d$ excludes the possibility of using a relative-entropy based approach since non-zero constant densities do not provide a reference finite-energy solution.

On the other hand, the literature concerning compressible Navier–Stokes type equations in the polytropic case $\gamma > 1$ is quite extensive. We shall only mention a few works and we refer the reader to the references therein.

The pioneering work of P.-L. Lions [148] proved the existence of global weak solutions to the three-dimensional Navier–Stokes equation with fixed positive viscosity coefficients for large initial data in the case $\gamma > \frac{9}{5}$, and it was later extended by Fereisl-Novotný-Petzeltová [96] to the case $\gamma > \frac{3}{2}$. Still concerning the Navier–Stokes equation with fixed positive viscosity, we refer to the very recent result of Bresch-Jabin [32], in which global weak solutions are constructed in the case of anisotropic viscous stress tensor and more general pressure laws, and the references therein.

Regarding the Navier–Stokes–Korteweg system (3.1) in the polytropic case $\gamma > 1$ on the torus $\mathbf{T}^d$, we mention the work of Jungel [135] who constructed a particular type of weak solutions in the range $\nu < \varepsilon$, Antonelli-Spirito [7, 8] who constructed global weak solutions in the range $\varepsilon < \nu < \alpha\varepsilon$ for some $\alpha > 1$, and Vasseur-Yu [182] who constructed global weak solutions when adding supplementary damping terms to the system. We also mention the work of Lacroix-Violet and Vasseur [141], which introduced the notion of renormalized solutions and proved the existence of global weak solutions in the torus $\mathbf{T}^3$. Since their method allows one to consider the semi-classical limit $\varepsilon \to 0$, it also provides a global weak solution to the corresponding Navier–Stokes equation.

Furthermore, we mention the recent results of Vasseur-Yu [181] and Li-Xin [146], obtained independently and by different methods, which solved a longstanding open problem by proving the existence of global weak solutions to the three-dimensional Navier–Stokes equation with degenerate viscosity (3.1) on the torus $\mathbf{T}^3$ for $1 < \gamma < 3$ (see also the associated Bourbaki's seminar by Rousset [169]). These results were recently extended by Bresch-Vasseur-Yu [34] where more general density dependent viscous stress tensors are considered in the case $\gamma > 1$, and we refer the reader to the references therein.

By working with the new unknowns (R, U) we circumvent the difficulties of the isothermal fluids (3.1) in which the energy has no definite sign. We are able then to construct global weak solutions for the rescaled isothermal Navier–Stokes and Navier–Stokes–Korteweg equations, that is system (3.11) with $\nu > 0$ and $\varepsilon \geqslant 0$.

The following existence result is established in [43].

Theorem 3.B. *Assume $\nu > 0$ and $\varepsilon \geqslant 0$. Consider an initial data $(\sqrt{R}_{\rm in}, (\sqrt{R}U)_{\rm in}) \in L^2(\mathbf{R}^d) \times L^2(\mathbf{R}^d)$ with finite energy $\mathscr{E}_{\rm in} < \infty$, finite BD-entropy $\mathscr{E}_{\rm BD,in} < \infty$, and satisfying the compatibility conditions $\sqrt{R}_{\rm in} \geqslant 0$ a.e. on $\mathbf{R}^d$ and $(\sqrt{R}U)_{\rm in} = 0$ a.e. on $\{\sqrt{R}_{\rm in} = 0\}$.*

Then there exists at least one global weak solution to (3.11) in the sense of Definition 3.1, which satisfies moreover the following energy and BD-entropy inequalities: for all $t \geqslant 0$ one has

$$\mathscr{E}[R, U](t) + \int_0^t \mathscr{D}[R, U](s)\,\mathrm{d}s \leqslant C(\mathscr{E}_{\rm in}), \tag{3.24}$$

$$\mathscr{E}_{\rm BD}[R, U](t) + \int_0^t \mathscr{D}_{\rm BD}[R, U](s)\,\mathrm{d}s \leqslant C'(\mathscr{E}_{\rm in}, \mathscr{E}_{\rm BD,in}), \tag{3.25}$$

for some positive constants $C, C' > 0$.

Remark 3.2. We also establish a similar result in the case of the rescaled isothermal Euler–Korteweg equation (3.11) with $\varepsilon > 0$ and $\nu = 0$. Its proof is quite different from the case $\nu > 0$, since it is based on the nonlinear logarithmic Schrödinger equation and uses the Madelung transform, and requires the initial data to be well-prepared.

Coming back to the original unknowns (ϱ, u) via (3.10), we hence obtain the existence of global weak solutions (ϱ, u) to the isothermal system (3.1) with $\nu > 0$ and $\varepsilon \geqslant 0$, for a suitable notion of weak solutions analogous to Definition 3.1. It is worth remarking that a shortcoming of this construction is that we do not obtain the energy inequality associated to the energy $E[\varrho, u]$ (see (3.3) and (3.4)) of the original system. However, the regularity of (R, U) implies that the energy $E[\varrho, u](t)$ is well-defined for any $t \geqslant 0$. More precisely, from (3.24) one obtains that, for all $t \geqslant 0$,

$$\frac{1}{2}\int_{\mathbf{R}^d} \varrho(t,x) \left| u(t,x) - \frac{\dot\tau(t)}{\tau(t)} x \right|^2 \mathrm{d}x + \frac{1}{2}\int_{\mathbf{R}^d} |\nabla\sqrt{\varrho}|^2(t,x)\,\mathrm{d}x$$

$$+\int_{\mathbf{R}^d}\rho(t,x)\log(\varrho(t,x))\,\mathrm{d}x + d\left(\log(\tau(t))+\frac{1}{\tau(t)^2}\right)\int_{\mathbf{R}^d}\varrho(t,x)\,\mathrm{d}x$$
$$+\int_0^t\left\{\int_{\mathbf{R}^d}\frac{\dot\tau(s)}{\tau(s)}\varrho(s,x)\left|u(s,x)-\frac{\dot\tau(s)}{\tau(s)}x\right|^2\mathrm{d}x+\nu\int_{\mathbf{R}^d}\varrho(s,x)\left|\mathrm{D}^s u(s,x)-\frac{\dot\tau(s)}{\tau(s)}\right|^2\mathrm{d}x\right\}\mathrm{d}s\leqslant C.$$

The construction of global weak solutions to (3.11) is based on several levels of approximations, inspired by the method employed by the works of Vasseur-Yu [181, 182] for Navier–Stokes and Navier–Stokes–Korteweg equations (3.1) (with $\nu>0$ and $\varepsilon\geqslant 0$) in the polytropic case $\gamma>1$, together with the notion of renormalized solutions of Lacroix-Violet and Vasseur [141].

The first level of approximation consists in the addition of drag forces to the left-hand side of (3.11b), that is, the terms
$$\frac{r_0}{\tau^2}U+\frac{r_1}{\tau^2}R|U|^2U,$$
with $r_0,r_1\geqslant 0$, aiming at providing more dissipation. This gives us the following approximated system, called the rescaled isothermal system with drag forces,

$$\begin{cases}\partial_t R+\dfrac{1}{\tau^2}\operatorname{div}(RU)=0, & (3.26\text{a})\\[2mm] \partial_t(RU)+\dfrac{1}{\tau^2}\operatorname{div}(RU\otimes U)+2yR+\nabla R+\dfrac{r_0}{\tau^2}U+\dfrac{r_1}{\tau^2}R|U|^2U & (3.26\text{b})\\[2mm] \qquad\qquad=\dfrac{\varepsilon^2}{2\tau^2}R\nabla\left(\dfrac{\Delta\sqrt{R}}{\sqrt{R}}\right)+\dfrac{\nu}{\tau^2}\operatorname{div}(R\mathrm{D}^sU)+\dfrac{\nu\dot\tau}{\tau}\nabla R, & \end{cases}$$

In the second level we replace the whole space $\mathbf{R}^d$ by a periodic box $\mathbf{T}^d_\ell$ of size $\ell>0$, where ℓ is aimed to go to infinity at the very last step of the construction of solutions. Remark that we need in this step to take into account the adaptation of the initial data, given on $\mathbf{R}^d$, in order to fit in the periodic framework. This step is motivated by the fact that the construction of weak solutions in the context of compressible fluid mechanics is often performed in the periodic setting $\mathbf{T}^d$, for this geometry provides compactness in space more easily and integrations by parts are harmless. The periodic case is also convenient for the approximation of the initial density by a density bounded away from zero, property which is propagated by the equation when we consider a regularization of the continuity equation (3.11a) and is needed when we add cold pressure and regularizing terms, which will be done in the remainder two levels of approximation.

The third step of approximation consists in adding a regularizing term to (3.11a) as well as a cold pressure and regularizing terms to (3.11b). This provides us with the following regularized system on the torus $\mathbf{T}^d_\ell$:

$$\begin{cases}\partial_t R+\dfrac{1}{\tau^2}\operatorname{div}(RU)=\dfrac{\delta_1}{\tau^2}\Delta R, & (3.27\text{a})\\[2mm] \partial_t(RU)+\dfrac{1}{\tau^2}\operatorname{div}(RU\otimes U)+2yR+\nabla R-\eta_1\nabla R^{-\alpha} & (3.27\text{b})\\[2mm] \qquad+\dfrac{r_0}{\tau^2}U+\dfrac{r_1}{\tau^2}R|U|^2U+\dfrac{\delta_1}{\tau^2}(\nabla R\cdot\nabla)U & \\[2mm] \qquad=\dfrac{\varepsilon^2}{2\tau^2}R\nabla\left(\dfrac{\Delta\sqrt{R}}{\sqrt{R}}\right)+\dfrac{\nu}{\tau^2}\operatorname{div}(R\mathrm{D}^sU)+\dfrac{\nu\dot\tau}{\tau}\nabla R & \\[2mm] \qquad+\dfrac{\delta_2}{\tau^2}\Delta^2U+\dfrac{\eta_2}{\tau^2}R\nabla\Delta^{2s+1}R, & \end{cases}$$

where the regularization parameters verify $0<\delta_1,\delta_2,\eta_1,\eta_2<1$, the parameters $\alpha,s>0$ are chosen sufficiently large, and the drag forces parameters r_0,r_1 as well as the Korteweg parameter ε are positive $r_0,r_1,\varepsilon>0$.

In the fourth and final level of approximation, we introduce an approximation of the initial density for the system in the periodic box $\mathbf{T}_\ell^d$ by initial densities $R_{\mathrm{in},\theta}$ bounded by below

$$\inf_{y\in\mathbf{T}_\ell^d} R_{\mathrm{in},\theta}(y) \geqslant \theta > 0.$$

We first construct global weak solutions to (3.27) for initial data $R_{\mathrm{in},\theta}$ bounded by below. This is done by a standard Faedo-Galerkin approximation and provides us with a global weak solution satisfying uniform estimates associated to regularized versions of the energy and the BD-entropy.

In the next step we fix the drag parameters $r_0, r_1 > 0$ and the size of the torus $\ell > 0$. Then, considering sequence of solutions constructed above, we are then able to pass to the limit with respect to the regularization parameters: first the limit $\delta_1, \delta_2 \to 0$ and only then $\eta_1, \eta_2 \to 0$. This procedure provides us with a global weak solution to the isothermal system with drag forces (3.26) on the torus $\mathbf{T}_\ell^d$ and with $r_0, r_1, \varepsilon > 0$. These solutions also verify a family of estimates provided by the energy and the BD-entropy associated to the system (3.26).

In the final step, we shall pass to the limits $\theta \to 0$, $r_0, r_1 \to 0$ and $\ell \to \infty$ simultaneously. In order to do this we proceed as in the work of Lacroix-Violet and Vasseur [141] and consider an adapted notion of renormalized solutions. The notion of renormalized solution of Lacroix-Violet and Vasseur [141], which can be easily adapted to our framework, is slightly stronger than the notion of weak solutions and it is based on a renormalized version of the momentum equation. The main interest of the notion of renormalized solutions is that it is easier to construct them. More precisely one has

— For $r_0, r_1 \geqslant 0$, any renormalized weak solution is also a weak solution;

— In the presence of drag forces $r_0, r_1 > 0$ and the Korteweg term $\varepsilon > 0$, the notions of renormalized and weak solutions are equivalent.

We then first prove that the global weak solutions to the isothermal system with drag forces (3.26) on the torus $\mathbf{T}_\ell^d$ and with $r_0, r_1, \varepsilon > 0$ constructed previously are indeed renormalized solutions. After that, we prove compactness of these renormalized solutions with respect to the parameters r_0, r_1, ε and ℓ. By passing to the limit $\theta \to 0$, $r_0, r_1 \to 0$ and $\ell \to \infty$, we hence obtain renormalized solutions of the isothermal system (3.11) in the whole space $\mathbf{R}^d$, which also gives us a global weak solution.

It is worth mentioning that this step has to be the last one, since the presence of the drag forces requires to control $r_0 (\log R)_-$ in $L^1(\mathbf{R}^d)$, which is inconsistent with the property $\sqrt{R} \in H^1(\mathbf{R}^d)$.

3.6 Some perspectives

3.6.1 Large-time behavior for polytropic fluids $\gamma > 1$

We presented in Section 3.2 some special class of global solutions to the Euler equation for polytropic fluids $\gamma > 1$, established by D. Serre [172] and, in a different geometric context, by Grassin [111].

An interest question would be to investigate the large-time dynamics for polytropic fluids $\gamma > 1$, for all cases $\varepsilon, \nu \geqslant 0$, by adapting the method presented in this chapter.

More precisely, define the new unknowns (R, U) as in (3.11) but with a different time-scale $\tau = \tau(t)$ which will depend on γ. Then, deriving the energy and BD-entropy estimates associated to the rescaled unknowns (R, U) in a similar way as done in this chapter, we should be able to obtain compactness properties which would in turn yields a conditional result as in Theorem 3.A: if (R, U) is a global wek solution verifying some global bounds, then the rescaled density R weakly converges to some asymptotic profile in large-time. After that we would like to obtain the converse of this result as in D. Serre [172] and Grassin [111], showing then that the asymptotic profile can be arbitrary, which in turn is in contrast with the results for isothermal fluids.

3.6.2 From Schrodinger to Euler/Korteweg and to Vlasov

On the one hand, as already explained before, starting from solutions ψ^ε to the nonlinear logarithmic Schödinger equation with parameter ε in the semi-classical scaling, by performing a Madelung transform to ψ^ε one formally obtains the isothermal Euler-Korteweg equation (3.1) with parameter $\varepsilon > 0$. Therefore, in the semi-classical limit $\varepsilon \to 0$ one formally recovers the isothermal Euler equation (3.1).

On the other hand, by considering the Wigner transform W^ε of the solution ψ^ε to the nonlinear logarithmic Schödinger equation, one formally obtains in the semi-classical limit $\varepsilon \to 0$ that W^ε converges to a solution of the following Vlasov-type equation:

$$\partial_t f + v \cdot \nabla_x f - \nabla_x \log \varrho \cdot \nabla_v f = 0 \tag{3.28}$$

where $f = f(t,x,v)$ and $\varrho(t,x) = \int_{\mathbf{R}^d} f(t,x,v)\,\mathrm{d}v$, which is called the *kinetic isothermal Euler system*. We remark that a mono-kinetic density

$$f(t,x,v) = \varrho(t,x) \otimes \delta_{v=u(t,x)}$$

is a solution to (3.28) if and only if (ϱ, u) is a solution to the isothermal Euler equation (3.1), relating therefore the isothermal Euler system to a Vlasov-type equation.

The large-time results of Carles–Gallagher [45] and [44] cover the Schrödinger and the fluid sides of the above formal link. A very interesting question would be to study the large-time behavior of global solutions to (3.28) inspired by the methods developed in this chapter.

It is worth mentioning that, although the kinetic isothermal Euler equation and other related singular Vlasov-type equations are known to be strongly ill-posed, see for instance the work of Han-Kwan and Nguyen [120], Ferriere [99] proved the existence of a special form of global solutions to (3.28) by starting from the analysis of the logarithmic Schrödinger in Carles–Gallagher [45] and making rigorous the above formal limit of the Wigner transform.

Chapter 4

Mean-field limit for the Stokes equation around random spheres

In this chapter we present the work [54] in collaboration with M. Hillairet.

4.1 Introduction

The main motivation of the work presented in this chapter comes from the interest in a rigorous derivation of mesoscopic equations for the dynamics of a cloud of solid particles immersed in a viscous incompressible fluid, in a suitable asymptotic limit. As explained in Desvillettes [75], there are different ways of modeling the particles in suspension.

On the one hand, if the cloud is composed by few particles, their behavior can be described by a finite-dimensional system and the coupling with the fluid equations gives us a problem of solid-fluid interaction type, similar for instance of Desjardins-Esteban [73, 74], Glass-Sueur [108], and Takahashi [176].

On the other hand, if the number of particles is very high, the description of each particle is irrelevant and one is rather interested, depending on the fractional volume of particles, in a kinetic-fluid description as, for example, in Baranger-Desvillettes [11, 28] and Boudin-Desvillettes-Grandmont-Moussa [28]; or in a multiphasic description as for instance in Ishii-Hibiki [133].

We are interested in a kinetic-fluid description corresponding to the following situation. Consider a cloud of N particles $\{B_i\}_{i=1,\dots N}$, described by their center of mass X_i and their velocity V_i, immersed in a viscous incompressible fluid. The motion of the coupled system in given by Newton's law for the particles, that is, for any $i = 1, \dots N$ one has

$$\dot{X}_i = V_i, \quad \dot{V}_i = \int_{\partial B_i} \Big(\nabla u + \nabla u^\top - p \mathrm{I}_3 \Big) n \mathrm{d}\sigma \tag{4.1a}$$

where n denotes the unit normal vector on ∂B_i directed inward the particle, I_3 the identity matrix of size 3×3. The unknowns $u(t, x) \in \mathbf{R}^3$ and $p(t, x) \in \mathbf{R}$ corresponds, respectively, to the velocity field and the pressure of the fluid that satisfies the Navier-Stokes equation in the domain $\Omega^N := \mathbf{R}^3 \setminus \bigcup_{i=1}^N \overline{B}_i$, corresponding to the whole space deprived of the particles,

$$\begin{cases} \partial_t u + u \cdot \nabla_x u = \Delta_x u - \nabla_x p, & (4.1b) \\ \operatorname{div}_x u = 0. & (4.1c) \end{cases}$$

which is complemented with the following boundary and limit conditions

$$u_{|\partial B_i} = V_i \quad \text{for any} \quad i = 1, \dots N, \tag{4.1d}$$

$$\lim_{|x| \to \infty} u(x) = 0 \tag{4.1e}$$

When the number of particles N is very large, one could hope then to describe the cloud behavior by means of the unknown $f(t,x,v) \geqslant 0$ that represents the density of particles that at time $t \geqslant 0$ and position $x \in \mathbf{R}^3$ posses velocity $v \in \mathbf{R}^3$. When the particle phase has negligible volume fraction, one expects a kinetic-fluid description of the system given by coupled system consisting in a Vlasov-type equation together with a Navier-Stokes equation to which one adds a friction term, more precisely

$$\begin{cases} \partial_t f + v \cdot \nabla_x f + 6\pi \operatorname{div}_v[(u-v)f] = 0, & (4.2a) \\ \partial_t u + u \cdot \nabla_x u = \Delta_x u - \nabla_x p - 6\pi \displaystyle\int_{\mathbf{R}^3} f(u-v) \mathrm{d}v, & (4.2b) \\ \operatorname{div}_x u = 0. & (4.2c) \end{cases}$$

One observes the presence of a drag-force term

$$-6\pi \int_{\mathbf{R}^3} f(u-v) \mathrm{d}v$$

in the right-hand side of the fluid equation (4.2b), called the Brinkman force, which models the exchange of momenta between the cloud of particles and the fluid.

Two different issues appear when dealing with the above formal limit from (4.1) to (4.2) when the number of particles goes to infinity:

— the derivation of a Vlasov-type equation for the cloud of particles from a many-particle system;

— the derivation of the macroscopic equation verified by the fluid, which correspond to a homogenization problem for the fluid equation in a perforated domain.

Making this formal limit rigorous in full generality is still an open problem, and we shall present in this chapter a result concerning the second issue described above.

More precisely, starting from a Stokes fluid and supposing the behavior of the cloud of particles given, we shall obtain, in the limit where the number of particles goes to infinity, the new term appearing in the macroscopic equation satisfied by the fluid, i.e. the Brinkman force, which describes to the influence of the cloud of particles into the fluid. This corresponds to an homogenization problem for the Stokes equation in a randomly perforated domain with non-vanishing boundary conditions, that we shall address by a Liouville-type approach in the spirit of Rubinstein [170].

It is worth mentioning that another approach for deriving the Vlasov–Navier–Stokes system (4.2) is developed in Bernard-Desvillettes-Golse-Ricci [21, 22] by starting from a multiphasic Boltzmann system for a binary gaseous mixture under a suitable scaling limit.

4.2 A homogenization problem

Let us describe in details the homogenization problem that we shall address in this chapter.

4.2.1 Stokes equation in a perforated domain

Let $N \in \mathbf{N}^*$ arbitrary large and consider N particles/obstacles given by spheres of radius $\frac{1}{N}$ in the whole space $\mathbf{R}^3$, each of them being characterized by its center X_i^N and its velocity V_i^N. One observes that the volume fraction occupied by the spheres is typically of size $\frac{1}{N^2}$.

We denote by $Z_i^N = (X_i^N, V_i^N)$ the state variable for each particle $i \in \{1, \dots, N\}$ and define the set of admissible configurations

$$\mathcal{O}^N := \left\{ \mathcal{Z}^N = (Z_1^N, \dots, Z_N^N) \in (\mathbf{R}^3 \times \mathbf{R}^3)^N \;\middle|\; |X_i^N - X_j^N| > \tfrac{2}{N} \quad \text{for any} \quad i \neq j \right\}.$$

Given an admissible configuration $\mathcal{Z}^N \in \mathcal{O}^N$, we denote the particles/obstacles by $B_i^N = B(X_i^N, \frac{1}{N})$ for any $i = 1, \dots, N$, and consider the following Stokes problem in a perforated domain

$$\left\{ \begin{aligned} -\Delta u + \nabla p &= 0 \\ \operatorname{div} u &= 0 \end{aligned} \right. \quad \text{in} \quad \Omega^N = \mathbf{R}^3 \setminus \bigcup_{i=1}^N \overline{B_i^N}, \tag{4.3a}$$

with boundary conditions

$$\begin{aligned} u(x) &= V_i^N \quad \text{on} \quad \partial B_i^N \text{ for } i = 1, \dots, N, \\ \lim_{|x| \to \infty} |u(x)| &= 0, \end{aligned} \tag{4.3b}$$

which therefore corresponds to a three-dimensional stationary exterior problem for the Stokes equation. This type of problem is extensively studied in Galdi [106] where it is proven that there exists a unique solution (u, p) to (4.3). We may then construct the extended velocity-field $U_N[\mathcal{Z}^N]$ on the whole space $\mathbf{R}^3$ by

$$U_N[\mathcal{Z}^N](x) = \left\{ \begin{array}{ll} u(x) & \text{if } x \in \Omega^N \\ V_i^N & \text{if } x \in B_i^N \text{ for } i = 1, \dots, N. \end{array} \right. \tag{4.4}$$

in such a way that $U_N[\mathcal{Z}^N] \in \dot{H}^1(\mathbf{R}^3)$

It is worth mentioning that the scaling we work with, that is the choice for the radius of particles equal to $\frac{1}{N}$, corresponds to the good scaling for which one expect to obtain the Brinkman force at the limit $N \to \infty$. This can be easily seen, at least formally, by recalling Stokes formula, which says that the drag force corresponding to a viscous flow with density ϱ_∞ and velocity u_∞ around a spherical obstacle of radius $\varepsilon > 0$ and with boundary condition V is given by the formula

$$\text{drag force} = 6\pi\varepsilon\varrho_\infty(u_\infty - V).$$

Therefore, with the scaling $\varepsilon = \frac{1}{N}$ the force exerted on the fluid by one particle is of order $O(\frac{1}{N})$, and assuming that one could superpose all these forces, one would obtain that the total force exerted on the fluid is of order $O(1)$, and thus one could expect to obtain a new term in the limit. On the other hand, if the radius of particles verify $\varepsilon \ll \frac{1}{N}$, the volume fraction occupied by the particles is very small and in the limit $N \to \infty$ we recover the Stokes equation in the whole space. Finally, if $\varepsilon \gg \frac{1}{N}$, then in the limit $N \to \infty$ we expect to recover Darcy's law.

4.2.2 Stokes-Brinkman equation

For a given nonnegative function $\varrho \in L^{3/2}(\mathbf{R}^3)$ and a vector-field $j \in L^{6/5}(\mathbf{R}^3)$, we consider the associated Stokes-Brinkman problem in the whole space

$$\left\{ \begin{aligned} -\Delta u + \nabla p + 6\pi\varrho u &= 6\pi j \\ \operatorname{div} u &= 0 \end{aligned} \right. \quad \text{in} \quad \mathbf{R}^3, \tag{4.5a}$$

with vanishing condition at infinity

$$\lim_{|x| \to \infty} |u(x)| = 0. \tag{4.5b}$$

Under the above assumptions and by a standard Lax-Milgram argument one can obtain, for instance as in Mecherbet-Hillairet [152] in the case of a bounded domain, that the equation (4.5) admits a unique weak solution $(u[\varrho, j], p[\varrho, j]) \in \dot{H}^1(\mathbf{R}^3) \times L^2(\mathbf{R}^3)$. Moreover, thanks to elliptic regularity estimates for Stokes-type equations of Galdi [106], one can recover regularity estimates for the weak solution and prove that $u[\varrho, j] \in H^2(\mathbf{R}^3)$.

4.3 Mean-field limit

The work of Allaire [6] was the first to prove that the sequence of solutions to the Stokes problem (4.3) in a periodic perforated domain and with with vanishing boundary conditions, that is assuming that $(V_1^N, \dots, V_N^N) = 0$, converges to the corresponding solution to the Stokes–Brinkman equation (4.5).

In the work of Desvillettes-Golse-Ricci [77] this result was extended to a non-periodic framework and with arbitrary boundary conditions. More precisely, for a bounded domain $\Omega \subseteq \mathbf{R}^3$ and for a given sequence of configurations $(\mathcal{Z}^N)_{N\geqslant 1}$ satisfying the following dilute regime

$$\min_{i\neq j} |X_i^N - X_j^N| \geqslant \frac{C}{N^{\frac{1}{3}}} \quad \text{and} \quad \min_{i=1,\dots N} \operatorname{dist}(X_i^N, \partial\Omega) \geqslant \frac{C}{N^{\frac{1}{3}}} \tag{4.6}$$

for some positive constant $C > 0$, they considered the Stokes equation in the perforated domain $\Omega \setminus \bigcup_{i=1}^N B(X_i^N, \frac{1}{N})$. They assumed moreover that the normalized energy

$$\frac{1}{N}\sum_{i=1}^N |V_i^N|^2 \leqslant C'$$

is uniformly bounded with respect to N, and that the empirical density $\varrho^N[\mathcal{Z}^N]$ and empirical flux $j^N[\mathcal{Z}^N]$ associated to the sequence $(\mathcal{Z}^N)_{N\in\mathbf{N}^*}$ converge in the limit $N \to \infty$, namely that there are $\varrho \in \mathscr{C}(\overline{\Omega})$ and $j \in \mathscr{C}(\overline{\Omega}, \mathbf{R}^3)$ such that

$$\varrho^N[\mathcal{Z}^N] := \frac{1}{N}\sum_{i=1}^N \delta_{X_i^N} \rightharpoonup \varrho \quad \text{weakly in the sense of measures}$$

and

$$j^N[\mathcal{Z}^N] := \frac{1}{N}\sum_{i=1}^N V_i^N \delta_{X_i^N} \rightharpoonup j \quad \text{weakly in the sense of measures.}$$

Then it was proven in Desvillettes-Golse-Ricci [77] that the extended field $U_N[\mathcal{Z}^N]$ defined in (4.4) via the solution to the Stokes problem (4.3) converges weakly in $H_0^1(\Omega)$ to the unique solution $u[\varrho, j]$ to the Stokes–Brinkman equation (4.5). The proof of the above results was based on compactness methods and relied on the resolution of the Stokes equation around a spherical obstacle.

In the work of Hillairet [128] the above result was extended to a less restrictive regime in comparison to (4.6), which will be detailed in (4.10) in Theorem 4.1 below, by developing a new method based on the variational formulation of the Stokes (4.3) and the Stokes–Brinkman (4.5) equations.

Motivated by the work of Hillairet [128], our idea was then then to extend the above results by considering random sequences of configurations and also providing quantitative estimates of convergence. It is worth mentioning that one could not hope to obtain this type of result, that is choosing the spherical obstacles in a random manner, from the work Desvillettes-Golse-Ricci [77] since the weight of configurations satisfying the dilute regime (4.6) would vanish in the limit $N \to \infty$. However this will be possible to achieve under the less restrictive regime given by (4.10).

We shall now present our result regarding the limit from the Stokes equation around spherical particles (4.3), chosen randomly, to the limit Stokes–Brinkman equation (4.5).

Consider then a sequence $(\mathcal{Z}^N)_{N\in\mathbf{N}^*}$ of exchangeable $\mathcal{O}^N$-valued random variables, and let $(F^N)_{N\in\mathbf{N}^*}$ be the sequence of their associated laws, that is each F^N is a symmetric probability measures on $\mathcal{O}^N$, and we further suppose that $F^N \in L^1(\mathcal{O}^N)$ for all $N \in \mathbf{N}^*$. Assume that the sequence $(F^N)_{N\in\mathbf{N}^*}$ satisfies the following properties:

— The cloud of particles occupies a bounded region: there exists some bounded open subset Ω_0 of $\mathbf{R}^3$ such that, for any $N \in \mathbf{N}^*$, there holds

$$\operatorname{supp}(F^N) \subset (\Omega_0 \times \mathbf{R}^3)^N; \tag{4.7a}$$

— Control on the growth of the marginals of $(F^N)_{N\in\mathbf{N}^*}$: there exists a constant $C_1 \geqslant 1$ such that, for any $N \in \mathbf{N}^*$ and any integer $1 \leqslant m \leqslant N$, there holds

$$\sup_{(x_1,\ldots,x_m)\in\mathbf{R}^{3m}} \int_{\mathbf{R}^{3m}} \mathbf{1}_{\mathcal{O}^m[\frac{1}{N}]}(z_1,\ldots,z_m)\, F_m^N(z_1,\ldots,z_m)\,\mathrm{d}v_1\ldots\mathrm{d}v_m \leqslant (C_1)^m \tag{4.7b}$$

where

$$\mathcal{O}^m[\tfrac{1}{N}] = \Big\{((X_1,V_1),\ldots,(X_m,V_m)) \in (\mathbf{R}^3\times\mathbf{R}^3)^m \;\Big|\; |X_i - X_j| > \tfrac{2}{N} \quad \forall\, i \neq j\Big\}$$

and $F_m^N \in L^1(\mathcal{O}^m[\frac{1}{N}])$ denotes the m-th marginal of F^N defined by

$$F_m^N(z_1,\ldots,z_m) = \int_{\mathbf{R}^{6(N-m)}} \mathbf{1}_{\mathcal{O}^N}(z_1,\ldots,z_N)\, F^N(z_1,\ldots,z_N)\,\mathrm{d}z_{m+1}\ldots\mathrm{d}z_N;$$

— Bounded moments: there exists $k_0 \geqslant 5$ and a constant $C_2 > 0$ such that, for any $N \in \mathbf{N}^*$, there holds

$$\int_{\mathbf{R}^3\times\mathbf{R}^3} |z_1|^{k_0} F_1^N(z_1)\,\mathrm{d}z_1 \leqslant C_2; \tag{4.7c}$$

— Control on the interaction of particles that are too close: there exists a constant $C_3 > 0$ such that, for any $N \in \mathbf{N}^*$, there holds

$$\sup_{(x_1,x_2)\in\mathbf{R}^3\times\mathbf{R}^3} \int_{\mathbf{R}^3\times\mathbf{R}^3} \mathbf{1}_{\mathcal{O}^2[\frac{1}{N}]}(z_1,z_2)\,|v_1| F_2^N(z_1,z_2)\,\mathrm{d}v_1\mathrm{d}v_2 \leqslant C_3. \tag{4.7d}$$

For each configuration $\mathcal{Z}^N$, we define the associated empirical density and the empirical flux respectively by

$$\varrho^N[\mathcal{Z}^N] := \frac{1}{N}\sum_{i=1}^N \delta_{X_i^N}, \quad j^N[\mathcal{Z}^N] := \frac{1}{N}\sum_{i=1}^N V_i^N \delta_{X_i^N}, \tag{4.8}$$

and we consider the unique solution to the Stokes problem (4.3), which in turn gives the extended velocity-field $U_N[\mathcal{Z}^N]$ defined by (4.4).

Let $f \in L^1(\mathbf{R}^3\times\mathbf{R}^3)$ be a probability measure with support in $\Omega_0 \times \mathbf{R}^3$. We define the measures

$$\varrho(x) = \int_{\mathbf{R}^3} f(x,v)\,\mathrm{d}v$$

and

$$j(x) = \int_{\mathbf{R}^3} v f(x,v)\,\mathrm{d}v.$$

Assume that $\varrho \in L^3(\Omega_0)$ and $j \in L^{6/5}(\Omega_0)$ so that there exists a unique solution $u[\varrho,j] \in \dot{H}^1(\mathbf{R}^3)$ to the Stokes-Brinkman problem (4.5) associated to (ϱ,j).

With the above assumptions, we obtain the following quantitative estimate between the extended solution $U_N[\mathcal{Z}^N]$ to the Stokes equation (4.3) and the solution $u[\varrho,j]$ to the Stokes–Brinkman equation (4.5).

Theorem 4.A. *For any $\alpha \in (\frac{2}{3},1)$ and N large enough, there holds*

$$\begin{aligned}&\mathbb{E}\left[\|U_N[\mathcal{Z}^N] - u\|_{L^2_{\mathrm{loc}}(\mathbf{R}^3)}\right]\\ &\quad\lesssim \mathbb{E}\left[W_1(\varrho^N[\mathcal{Z}^N],\varrho)\right]^{\frac{1}{57}} + \mathbb{E}\left[\|j^N[\mathcal{Z}^N] - j\|_{(\mathscr{C}_b^{0,1}(\mathbf{R}^3))'}\right]^{\frac{1}{3}} + N^{-\min\left(\frac{1-\alpha}{95},\frac{3\alpha-2}{2}\right)}\end{aligned} \tag{4.9}$$

where W_1 denotes the Monge-Kantorovich-Wasserstein distance (with cost $|x-y|$) and $\|\cdot\|_{(\mathscr{C}_b^{0,1}(\mathbf{R}^3))'}$ stands for the dual norm of the space of Lipschitz bounded functions on $\mathbf{R}^3$.

As a consequence, if the empirical density $\varrho^N[\mathcal{Z}^N]$ converges to ϱ and the empirical flux $j^N[\mathcal{Z}^N]$ converges to j weakly in the sense of measures, one therefore obtains the convergence

$$\lim_{N\to\infty} \mathbb{E}\left[\|U_N[\mathcal{Z}^N] - u\|_{L^2_{\mathrm{loc}}(\mathbf{R}^3)}\right] = 0.$$

One remarks that, for probability measures, the Monge-Kantorovich-Wasserstein distance W_1 is equivalent to the distance induced by the $(\mathscr{C}_b^{0,1}(\mathbf{R}^3))'$-norm, thus both densities and flux differences estimates appearing in the right-hand side of (4.9) are of the same nature.

Furthermore, one should mention that, under the hypothesis of Theorem 4.A, the sequence $(\mathbb{E}[U_N[\mathcal{Z}^N]])_{N\in\mathbf{N}^*}$ is bounded in $\dot{H}^1(\mathbf{R}^3)$. Therefore, assuming the convergences empirical density $\varrho^N[\mathcal{Z}^N]$ converges to ϱ and the empirical flux $j^N[\mathcal{Z}^N]$ converges to j, as a byproduct of our proof we also obtain that $(\mathbb{E}[U_N[\mathcal{Z}^N]])_{N\in\mathbf{N}^*}$ converges weakly in $\dot{H}^1(\mathbf{R}^3)$ to the unique solution $u[\varrho, j]$ of the Stokes–Brinkman equation.

In order to illustrate one possible application of our result, we shall construct a particular example of a sequence of probability measures on $\mathcal{O}^N$ satisfying the assumptions of Theorem 4.A and for which we obtain a quantitative estimate on the right-hand side of (4.9). As is standard in statistical physics, one way to construct a sequence of probability measures in the phase space of a N-particle system that is asymptotically independent (so that both convergences of the empirical density and the empirical flux above hold) is to take the N-tensor product of a one-particle probability measure and then restrict it to the energy surface of the system. In our framework, considering the above hypothesis on $f \in L^1(\mathbf{R}^3\times\mathbf{R}^3)$, we suppose further that $\varrho \in L^\infty(\Omega_0)$ and $\int_{\mathbf{R}^3\times\mathbf{R}^3} |v|^k f(x,v)\,\mathrm{d}x\,\mathrm{d}v < \infty$ for some $k \geqslant 5$. We then construct a sequence $(\Pi^N[f])_{N\in\mathbf{N}^*}$ of probability measures on $\mathcal{O}^N$ by defining

$$\Pi^N[f](z_1,\ldots,z_N) = \frac{1}{\mathrm{W}_N(f)}\,\mathbf{1}_{\mathcal{O}^N}(z_1,\ldots,z_N)\,f(z_1)\cdots f(z_N)$$

where $\mathrm{W}_N(f)$ is the partition function

$$\mathrm{W}_N(f) := \int_{(\mathbf{R}^3\times\mathbf{R}^3)^N} \mathbf{1}_{\mathcal{O}^N}(z_1,\ldots,z_N)\,f(z_1)\cdots f(z_N)\,\mathrm{d}z_1\ldots\mathrm{d}z_N.$$

One can easily check that $(\Pi^N[f])_{N\in\mathbf{N}^*}$ satisfies assumptions (4.7) and, thanks to the results of Fournier-Guillin [102] on the rate of convergence of empirical measures, one can estimate the first and second terms on right-hand side of (4.9), which finally gives

$$\mathbb{E}\left[\|U_N[\mathcal{Z}^N] - u\|_{L^2_{\mathrm{loc}}(\mathbf{R}^3)}\right] \lesssim N^{-\frac{1}{171}} + N^{-\min\left(\frac{1-\alpha}{95},\frac{3\alpha-2}{2}\right)}.$$

We shall describe in the sequel the main steps in the proof of Theorem 4.A.

4.3.1 Concentrated configurations

As a first step in our proof, we identify a particular set of *concentrated configurations* and prove that they are negligible in the asymptotic limit $N\to\infty$. These configurations correspond to $\mathcal{Z}^N \in \mathcal{O}^N$ such that there exists a couple of particles too close to each other or that there exist too many particles in a same cell of small volume.

More precisely, for $\lambda, \alpha > 0$, and any integer $M \leqslant N$, we define

$$\mathcal{O}^N_\alpha := \{\mathcal{Z}^N \in \mathcal{O}^N \mid \min_{i\neq j} |X_i^N - X_j^N| < N^{-\alpha}\}$$

and

$$\mathcal{O}^N_{\lambda,M} := \{\mathcal{Z}^N \in \mathcal{O}^N \mid \text{there exist at least } M \text{ particles in the same cell of size } \lambda > 0\}.$$

In order to study the weight of the sets $\mathcal{O}^N_{\lambda,M}$ and $\mathcal{O}^N_\alpha$, we define

$$M_N = N^\beta, \qquad \lambda_N := \left(\eta \frac{M_N}{N}\right)^{1/3}, \quad \forall\, N \in \mathbf{N}^*$$

with positive parameters $\alpha, \beta, \eta > 0$ to be fixed later on. Under the assumptions of Theorem 4.A one has:

Proposition 4.1. *Let $\alpha \in (\frac{2}{3}, 1)$ and $\beta \in (0, \frac{1}{2})$. For $\eta > 0$ sufficiently small one has*

$$\mathbb{P}\left(\mathcal{Z}^N \in \mathcal{O}^N_{\lambda_N, M_N} \cup \mathcal{O}^N_\alpha\right) \leqslant \frac{C}{N^{3\alpha-2}},$$

for some positive constant $C > 0$.

4.3.2 Uniform estimates

The second step consists in obtaining uniform estimates satisfied by the extended solution $U_N[\mathcal{Z}^N]$. We obtain the following two properties simultaneously:

— the mean of $U_N[\mathcal{Z}^N]$ is well-defined and uniformly bounded in $\dot{H}^1(\mathbf{R}^3)$;

— the weight of contribution of the concentrated configurations vanishes when $N \to \infty$;

which enables us to get rid of the concentrated configurations in the asymptotic description of U_N. More precisely, under the assumptions of Theorem 4.A, one obtains that:

Proposition 4.2. *For any subset $\mathcal{U}^N$ of $\mathcal{O}^N$ there holds*

$$\mathbb{E}\left[\|\nabla U_N[\mathcal{Z}^N]\|_{L^2(\mathbf{R}^3)} \mathbf{1}_{\mathcal{Z}^N \in \mathcal{U}^N}\right] \leqslant C\left(\mathbb{P}\left(\mathcal{Z}^N \in \mathcal{U}^N\right)^{\frac{1}{2}} + \frac{1}{N^{\frac{3}{2}}}\right),$$

for some positive constant $C > 0$.

4.3.3 Mean-field limit for non-concentrated configurations

In the third step, we prove a mean-field result for *non-concentrated configurations*, corresponding to the situation in which particles are sufficiently distant from each other and they do not concentrate in a box of small size.

This result is the cornerstone of the proof of Theorem 4.A and it is based on a combination of the duality method of Mecherbet-Hillairet [152], performed in the dilute regime (4.6), together with covering arguments of Hillairet [128].

For a given configuration $\mathcal{Z}^N \in \mathcal{O}^N$ one defines the following quantities in order to quantify the regime of non-concentrated configurations:

— $d_{\min}[\mathcal{Z}^N] = \min_{i \neq j} |X^N_i - X^N_j|$ the minimal distance between two different centers X^N_i and X^N_j;

— $\lambda[\mathcal{Z}^N]$ a chosen size for a partition of $\mathbf{R}^3$ in cubes;

— $M[\mathcal{Z}^N]$ the maximum number of centers X^N_i inside one cell of size $\lambda[\mathcal{Z}^N]$.

Remark that if $d_{\min}[\mathcal{Z}^N]$ is sufficiently large and $M[\mathcal{Z}^N]$ is sufficiently small, the particles are sufficiently distant from each others and they do not concentrate in a small box and hence this situation corresponds the complementary set of concentrated configurations discussed in Section 4.3.1.

We then obtain the following estimate:

Theorem 4.1. *Let $\alpha \in (\frac{2}{3}, 1)$, $\eta \in (0,1)$, $R > 0$ and $\delta > \frac{1}{2}$ be fixed. Then there exists a positive constant $K = K(\alpha, R, \Omega_0) > 0$ such that for all $N \in \mathbf{N}^*$ and any configuration $\mathcal{Z}^N \in \mathcal{O}^N$ satisfying*

$$d_{\min}[\mathcal{Z}^N] \geqslant \frac{1}{N^\alpha}, \quad M[\mathcal{Z}^N] \leqslant \frac{N^{\frac{3(1-\alpha)}{5}}}{\eta} \quad \text{and} \quad \lambda[\mathcal{Z}^N] = \left(\frac{\eta M[\mathcal{Z}^N]}{N}\right)^{\frac{1}{3}}, \tag{4.10}$$

one has

$$\begin{aligned}
&\left\| U_N[\mathcal{Z}^N] - u[\varrho, j] \right\|_{L^2(B(0,R))} \\
&\quad \leqslant \frac{K}{\eta} \|j[\mathcal{Z}^N] - j\|_{(\mathscr{C}_b^{0,1/2}(\mathbf{R}^3))'} + \frac{K}{\eta}\left(1 + \frac{1}{N}\sum_{i=1}^N |V_i^N|^2\right)^{\frac{5}{4}} \left(\frac{1 + \|\varrho\|_{L^2(\Omega_0)}}{\delta^{\frac{1}{3}}}\right) \\
&\quad + \frac{K\delta^6}{\eta}\left(1 + \frac{1}{N}\sum_{i=1}^N |V_i^N|^2\right)^{\frac{5}{4}} \left(\frac{1}{N^{\frac{1-\alpha}{5}}} + \|\varrho[\mathcal{Z}^N] - \varrho\|_{(\mathscr{C}_b^{0,1/2}(\mathbf{R}^3))'}\right),
\end{aligned}$$

where $B(0,R)$ denotes the open ball of radius R in $\mathbf{R}^3$.

Let us describe the idea behind the proof of Theorem 4.1 in a formal way. For a given configuration $\mathcal{Z}^N \in \mathcal{O}^N$, recall that $U_N[\mathcal{Z}^N]$ denotes the extended field solution to the Stokes equation (4.3) and denote by $P_N[\mathcal{Z}^N]$ the associated pressure.

Thanks to the variational characterization of the Stokes (4.3) and Stokes–Brinkman (4.5) equations, a duality argument implies that the quantity $\|U_N[\mathcal{Z}^N] - u[\varrho, j]\|_{L^2(B(0,R))}$ that we want to estimate can be bounded by

$$\sup_w \left| \int_{\mathbf{R}^3} \nabla(U_N[\mathcal{Z}^N] - u[\varrho, j]) : \nabla w \, \mathrm{d}x + 6\pi \int_{\mathbf{R}^3} \varrho(U_N[\mathcal{Z}^N] - u[\varrho, j]) \cdot w \, \mathrm{d}x \right| \tag{4.11}$$

where the supremum is taken over the set of divergence-free smooth vector-field $w \in \mathscr{C}_c^\infty(\mathbf{R}^3; \mathbf{R}^3)$ with $\|\nabla w\|_{L^2(\mathbf{R}^3)} + \|\nabla^2 w\|_{L^2(\mathbf{R}^3)} \leqslant 1$. Thanks to a integration by parts, we have then

$$\int_{\mathbf{R}^3} \nabla U_N[\mathcal{Z}^N] : \nabla w \, \mathrm{d}x \approx \sum_{i=1}^N \int_{\partial B(X_i^N, \frac{1}{N})} \Sigma\left(U_N[\mathcal{Z}^N], P_N[\mathcal{Z}^N]\right) n \cdot w \, \mathrm{d}\sigma,$$

where $\Sigma(U,P) = (\nabla U + \nabla U^\top) - P\mathbb{I}_3$ is the fluid stress tensor and n is the normal to $\partial B(X_i^N, \frac{1}{N})$ directed inward the obstacle. In the favorable setting of non-concentrated configurations, we are able to replace w in the boundary integrals on the right-hand side of the above approximation by its value in the center of the obstacle $B(X_i^N, \frac{1}{N})$. Therefore, we can compute the integral of the stress tensor on the boundary $\partial B(X_i^N, \frac{1}{N})$ by using Stokes law (see for instance Desvillettes-Golse-Ricci [77]), which yields

$$\begin{aligned}
&\sum_{i=1}^N \int_{\partial B(X_i^N, \frac{1}{N})} \Sigma\left(U_N[\mathcal{Z}^N], P_N[\mathcal{Z}^N]\right) n \cdot w \, \mathrm{d}\sigma \\
&\qquad \approx \sum_{i=1}^N \int_{\partial B(X_i^N, \frac{1}{N})} \Sigma\left(U_N[\mathcal{Z}^N], P_N[\mathcal{Z}^N]\right) n \cdot w(X_i^N) \, \mathrm{d}\sigma \\
&\qquad \approx \frac{6\pi}{N} \sum_{i=1}^N (V_i^N - \bar{U}_i^N) \cdot w(X_i^N),
\end{aligned}$$

where $V_i^N - \bar{U}_i^N$ stands for the difference between the velocity on the obstacle $B(X_i^N, \frac{1}{N})$ and the *velocity at infinity* seen by this same obstacle. Finally we get the identify

$$\int_{\mathbf{R}^3} \nabla U_N[\mathcal{Z}^N] : \nabla w \, \mathrm{d}x + \frac{6\pi}{N} \sum_{i=1}^N \bar{U}_i^N \cdot w(X_i^N) \approx \frac{6\pi}{N} \sum_{i=1}^N V_i^N \cdot w(X_i^N).$$

that can be rewritten as

$$\int_{\mathbf{R}^3} \nabla U_N[\mathcal{Z}^N] : \nabla w \,\mathrm{d}x + 6\pi \left\langle \varrho^N[\mathcal{Z}^N], w\bar{U} \right\rangle \approx 6\pi \left\langle j^N[\mathcal{Z}^N], w \right\rangle.$$

We then plug this approximate estimate into (4.11) and obtain an estimate where appears the difference between densities $\varrho^N[\mathcal{Z}^N] - \varrho$ and fluxes $j^N[\mathcal{Z}^N] - j$.

The core of the proof of Theorem 4.1 is then to justify the above approximations and to quantify the errors appearing in each one of them. This can indeed be achieved by exploiting the fact that we are dealing with non-concentrated configurations satisfying (4.10) and by using some fine covering arguments inspired by Hillairet [128].

Finally, Theorem 4.A can be proven by gathering the mean-field result for non-concentrated configurations together with the properties of the concentrate configurations in Sections 4.3.1 and the uniform estimates in Section 4.3.2. More precisely, let us fix the parameters $\alpha \in (\frac{2}{3}, 1)$, $\eta > 0$ small enough so that Proposition 4.1 holds and $R > 0$. For any $N \in \mathbf{N}^*$ we denote

$$M_N = N^{\frac{3(1-\alpha)}{5}} \quad \text{and} \quad \lambda_N = \left(\frac{\eta M_N}{N}\right)^{1/3},$$

and then we consider the following decomposition into *non-concentrated* and *concentrated* configurations

$$\mathcal{O}^N = \left(\mathcal{O}^N \setminus (\mathcal{O}^N_{\lambda_N, M_N} \cup \mathcal{O}^N_\alpha)\right) \cup \left(\mathcal{O}^N_{\lambda_N, M_N} \cup \mathcal{O}^N_\alpha\right),$$

in such a way that any configuration $\mathcal{Z}^N \in \mathcal{O}^N \setminus (\mathcal{O}^N_{\lambda_N, M_N} \cup \mathcal{O}^N_\alpha)$ satisfies the assumption (4.10) of Theorem 4.1. We hence split the expectation we want to estimate into two parts

$$\begin{aligned}\mathbb{E}\left[\|U_N[\mathcal{Z}^N] - u\|_{L^2(B(0,R))}\right] = \mathbb{E}&\left[\mathbf{1}_{\mathcal{O}^N \setminus (\mathcal{O}^N_{\lambda_N, M_N} \cup \mathcal{O}^N_\alpha)}(\mathcal{Z}^N)\, \|U_N[\mathcal{Z}^N] - u\|_{L^2(B(0,R))}\right] \\ &+ \mathbb{E}\left[\mathbf{1}_{\mathcal{O}^N_{\lambda_N, M_N} \cup \mathcal{O}^N_\alpha}(\mathcal{Z}^N)\, \|U_N[\mathcal{Z}^N] - u\|_{L^2(B(0,R))}\right].\end{aligned}$$

The first term is controlled thanks to Theorem 4.1 by making a suitable choice of the parameter δ in that theorem. The second term is estimated by Proposition 4.1 and Proposition 4.2.

4.4 Some perspectives

4.4.1 Particles with arbitrary shapes and rotation; nonlinear equations

In the Stokes model considered in (4.3), we have considered spherical particles characterized by their position and velocity. An interesting question would be to consider a more general framework of particles with arbitrary shapes and also taking into account their rotation, hence modifying the boundary condition in (4.3b).

In this setting, we expect that the resulting Stokes–Brinkmann problem is related to the distribution of shapes for the particles in the cloud, that is quantified in terms of Stokes resistance matrix associated with these shapes. Furthermore, the particle rotations influence the effective model only by their contribution to the drag force exerted on the particles.

A first result in that direction is obtained in Hillairet-Moussa-Sueur [129] for fixed configurations in a dilute regime (4.6). A possible direction would be then to combine the methods of this chapter with the results of Hillairet-Moussa-Sueur [129].

Another possible extension of the methods of this chapter would be to consider nonlinear models. More precisely, one could investigate a homogenization problem for the stationary Navier–Stokes equation in a perforated domain with random spheres.

4.4.2 First-order evolution equations

As described in the introduction, our main motivation comes from the the formal limit relating the evolution problem (4.1) to the expected effective limit problem (4.2). As already explained, in this limit two different issues appear: a mean-field limit for the cloud of particles; and an homogenization problem for the macroscopic equation of the fluid, which was the issue treated in this chapter.

The mean-field limit from a many-particle interacting system to its effective Vlasov-type equation is a very important problem in kinetic theory and there is extensive literature on the subject. On the one hand, this limit is well-understood in the case of sufficiently smooth interaction, and we only mention classical references of Braun-Hepp [29] and Dobrushin-[87]. On the other hand, the case of singular interaction is not well-understood. An important result is that direction is the work of Hauray-Jabin [121, 122] which proved the mean-field limit for some singular type of interaction. Very recently new important results on mean-fields limits for first-order many-particle systems with singular interaction were obtained by Jabin-Wang [134], Serfaty [171] and Bresch-Jabin-Wang [33] thanks to the introduction of novel methods.

The interaction between particles in the evolution of the coupled particle-fluid system given by (4.1) is very singular and we believe that the limit from (4.1) to (4.2) is out of reach with current techniques. An interesting question however would be to investigate some related and simpler models than (4.1), namely considering first-order equations instead of second-order equation for the dynamics of particle in (4.1a), by combining the techniques for the homogenization problem developed in this chapter together with new methods developed by Jabin-Wang [134], Serfaty [171] and Bresch-Jabin-Wang [33] in order to treat singular mean-field limits for interacting particle systems.

Chapter 5

The parabolic-parabolic Keller–Segel equation

In this chapter we present the work [56] in collaboration with S. Mischler.

5.1 Introduction

This chapter is devoted to a nonlinear aggregation-diffusion type equation known as the Keller–Segel system (or the Patlak–Keller–Segel system), which is a classical model in chemotaxis. This system describes the collective motion of cells that are attracted by a chemical substance that they are able to emit. We shall consider in this chapter the *parabolic-parabolic Keller–Segel* equation in $\mathbf{R}^2$ given by

$$\begin{cases} \partial_t f = \Delta f - \operatorname{div}(f\, \nabla u) & (5.1a) \\ \varepsilon \partial_t u = \Delta u + f & (5.1b) \end{cases}$$

and which is complemented by non-negative initial data $f_{|t=0} = f_{\text{in}}$ and $u_{|t=0} = u_{\text{in}}$.

The function $f = f(t,x) \geqslant 0$ stands for the mass density of cells while $u = u(t,x) \geqslant 0$ stands for the chemo-attractant concentration, $t \in \mathbf{R}^+$ is the time variable, $x \in \mathbf{R}^2$ is the space variable, and $\varepsilon > 0$ is a constant.

When $\varepsilon = 0$ the system (5.1) is called the *parabolic-elliptic* Keller–Segel equation, it was introduced by Patlak [165] and by Keller-Segel [136] and can also be seen as a quasi-static approximation of the parabolic-parabolic case.

The main feature of system (5.1) is the presence of two competing mechanisms. On the one hand there is a diffusive term Δf, modeling the erratic motion of cells, the expected effect of which is to smooth and spread out the density f all over the plane $\mathbf{R}^2$. On the other hand there is a drift term $-\operatorname{div}(f\nabla u)$ corresponding to an aggregation mechanism, modeling the tendency of cells to follow the gradient of the chemo-attractant they are able to emit, and from which we expect a concentration phenomenon. These two mechanisms are almost of the same order, making their analysis difficult and interesting.

The parabolic-elliptic Keller–Segel system $\varepsilon = 0$ has received a lot of attention in recent years. In summary, the most important feature is the fact that there is a mass threshold dictating the behavior of solutions.

If the initial mass $M = \int_{\mathbf{R}^2} f_{\text{in}} \, \mathrm{d}x$ verifies $M < 8\pi$, then weak solutions exist globally in time as shown by Blanchet-Dolbeault-Perthame [26]. Furthermore we can show that the spreading mechanism resulting from the diffusive term prevails on the aggregation phenomenon for large times, more precisely, Campos-Dolbeault [39] and Fernández-Mischler [97] have proved that, preforming self-similar change of variables, any solution converges to the unique self-similar profile with same mass as time goes to infinity. We also mention the work of Bedrossian-Masmuodi [16] which establishes the well-posedness for measure-valued initial data.

In the mass supercritical case $M > 8\pi$ however, it is the aggregation mechanism that prevails on the diffusion so that any solution blows up in finite time. A particular type of blowing up solutions were constructed by Herrero-Velázquez [127] and the stability under small perturbation of these blowing up solutions has been considered by Raphaël-Schweyer [167]. These results were recently extended in Collot-Ghoul-Masmoudi-Nguyen [69, 68].

Finally, for the mass critical case $M = 8\pi$, it was established by Blanchet-Carrillo-Masmoudi [25] that solutions exist globally in time and concentrate in infinite time, namely the density converges to a Dirac measure with mass 8π as time goes to infinity. We also mention the recent works of Ghoul-Masmoudi [107] and Davila-del Pino-Dolbeault-Musso-Wei [71].

Regarding the parabolic-parabolic Keller–Segel $\varepsilon > 0$, there are fewer works available in the literature and we do not have a complete and detailed description of the behavior of solutions as above. We know that in the mass subcritical setting $M < 8\pi$ weak solutions exist globally in time, as shown by Calvez-Corrias [38]. Furthermore, Biler-Guerra-Karch [24] and Corrias-Escobedo-Matos [70] constructed global solutions for any mass provided that $\varepsilon > 0$ is large enough, which corresponds to a regime with a small nonlinearity. Furthermore, unique self-similar solutions were constructed by Biler-Corrias-Dolbeault [23] and Corrias-Escobedo-Matos [70] in the mass subcritical case $M < 8\pi$ and in the case with any mass but $\varepsilon > 0$ large enough. Finally, Herrero-Velázquez [127] constructed blowing up solutions in the case $M > 8\pi$.

In Section 5.2 we gather some fundamental properties of the parabolic-parabolic Keller-Segel equation. We shall then present two results on system (5.1). The first one concerns the regularization and uniqueness of global weak solutions in the mass subcritical case $M < 8\pi$ and it will be presented in Section 5.3. After that, in Section 5.4, we shall present our second result regarding the large-time behavior of solutions, more precisely the asymptotic stability of the self-similar profile in the mass subcritical case and in a *quasi-parabolic-elliptic regime*, that is for $\varepsilon > 0$ small.

5.2 Fundamental properties

At the formal level, solutions to the parabolic-parabolic Keller–Segel system (5.1) satisfy two fundamental identities.

On the one hand the mass is conserved, that is, for any $t \geqslant 0$ one has

$$\int_{\mathbf{R}^2} f(t,x)\,\mathrm{d}x = \int_{\mathbf{R}^2} f_{\mathrm{in}}(x)\,\mathrm{d}x =: M. \tag{5.2}$$

On the other hand, the free-energy functional $\mathscr{F}$ defined by

$$\mathscr{F}(f,u) := \int_{\mathbf{R}^2} f \log f \,\mathrm{d}x - \int_{\mathbf{R}^2} f u \,\mathrm{d}x + \frac{1}{2}\int_{\mathbf{R}^2} |\nabla u|^2\,\mathrm{d}x, \tag{5.3}$$

satisfies the following free-energy identity, for any $t \geqslant 0$,

$$\mathscr{F}(f,u)(t) + \int_0^t \mathscr{D}(f,u)(s)\,\mathrm{d}s = \mathscr{F}_{\mathrm{in}}, \tag{5.4}$$

where $\mathscr{F}_{\mathrm{in}}$ denotes the free-energy of the initial data, and the non-negative free-energy dissipation functional $\mathscr{D}$ is given by

$$\mathscr{D}(f,u) := \int_{\mathbf{R}^2} f\,|\nabla \log f - \nabla u|^2\,\mathrm{d}x + \frac{1}{\varepsilon}\int_{\mathbf{R}^2} |\Delta u + f|^2\,\mathrm{d}x. \tag{5.5}$$

5.2.1 A priori estimates

We shall now present how, from the above fundamental identities, one can obtain the basic a priori estimates for solutions to (5.1), which in turn will lead to the notion of weak solutions stated in Definition 5.1 below.

As a first step we obtain the control of a logarithmic moment of f. More precisely, we define the function $H(x) := \frac{1}{\pi}\frac{1}{\langle x\rangle^4}$ and compute

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbf{R}^2} f\,(-\log H)\,\mathrm{d}x &= \int_{\mathbf{R}^2} f\,\nabla(\log f - u)\cdot\nabla(\log H)\,\mathrm{d}x\\ &\leqslant \frac{1}{2}\mathscr{D}(f,u) + \frac{1}{2}\int_{\mathbf{R}^2} f\,|\nabla\log H|^2\,\mathrm{d}x.\end{aligned}$$

Gathering this estimate together with (5.4) we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{F}_H(f,u)(t) + \frac{1}{2}\mathscr{D}(f,u)(t) \leqslant M. \tag{5.6}$$

where we define the modified free-energy functional $\mathscr{F}_H$ by

$$\mathscr{F}_H(f,u) = \mathscr{F}(f,u) - \int_{\mathbf{R}^2} f\log H\,\mathrm{d}x.$$

Let us now consider $\bar{u}$ the solution to the Laplace equation

$$-\Delta\bar{u} = f \quad \text{in} \quad \mathbf{R}^2$$

so that $\bar{u} := \mathcal{K}\star f$, where $\mathcal{K}(z) := -\frac{1}{2\pi}\log|z|$ denotes the Laplace kernel. We also define the chemical energy

$$\mathscr{U}(f,u) := \frac{1}{2}\int_{\mathbf{R}^2}|\nabla u|^2\,\mathrm{d}x - \int_{\mathbf{R}^2} f\,u\,\mathrm{d}x$$

and the modified entropy

$$\mathscr{H}_H(f) := \int_{\mathbf{R}^2} f\log\left(\frac{f}{H}\right)\mathrm{d}x,$$

that satisfy

$$\mathscr{F}_H(f,u) = \mathscr{H}_H(f) + \frac{1}{2}\int_{\mathbf{R}^2}|\nabla(u-\bar{u})|^2\,\mathrm{d}x - \frac{1}{2}\iint_{\mathbf{R}^2\times\mathbf{R}^2} f(x)\,f(y)\,\mathcal{K}(x-y)\,\mathrm{d}x\mathrm{d}y, \tag{5.7}$$

by using the fact that

$$\mathscr{U}(f,\bar{u}) = -\frac{1}{2}\iint_{\mathbf{R}^2\times\mathbf{R}^2} f(x)\,f(y)\,\mathcal{K}(x-y)\,\mathrm{d}x\mathrm{d}y.$$

At this point we recall two classical inequalities : the logarithmic Hardy-Littlewood-Sobolev inequality (see for instance Beckner [15] and Carlen-Loss [40]) that states

$$\begin{aligned}\int_{\mathbf{R}^2} f\log f\,\mathrm{d}x - \frac{4\pi}{M}\iint_{\mathbf{R}^2\times\mathbf{R}^2} f(x)\,f(y)\,\mathcal{K}(x-y)\,\mathrm{d}x\mathrm{d}y&\\ - \int_{\mathbf{R}^2} f\log H\,\mathrm{d}x \geqslant -C(M)&,\end{aligned} \tag{5.8}$$

where $C(M) > 0$ is a positive constant depending only on M; as well as the functional inequality

$$\mathscr{H}^+(f) := \int_{\mathbf{R}^2} f(\log f)_+\,\mathrm{d}x \leqslant \mathscr{H}_H(f) - \frac{1}{4}\int_{\mathbf{R}^2} f\log\langle x\rangle^2\,\mathrm{d}x + C(M),$$

where $C(M) > 0$ is another positive constant. Finally, using these two inequalities together with (5.7) and recalling that we have assumed $M < 8\pi$, we obtain

$$\begin{aligned}\mathscr{F}_H(f,u) &\geqslant \left(1-\frac{M}{8\pi}\right)\mathscr{H}_H(f) + \frac{M}{8\pi}\left(\mathscr{H}_H(f) - \frac{4\pi}{M}\iint_{\mathbf{R}^2\times\mathbf{R}^2} f(x)\,f(y)\,\mathcal{K}(x-y)\,\mathrm{d}x\mathrm{d}y\right)\\ &\geqslant C(M)\,\mathscr{H}^+(f) + C(M)\int_{\mathbf{R}^2} f\log\langle x\rangle^2\,\mathrm{d}x - C(M),\end{aligned}$$

where as before $C(M) > 0$ denotes a positive constant depending on M. Therefore, assuming that the initial modified free-energy $\mathscr{F}_{H,\mathrm{in}}$ is finite, the mas conservation (5.2) and the inequality (5.6) provide the following a priori estimates

$$\begin{aligned} C(M)\,\mathscr{H}^+(f(t)) + C(M)\int_{\mathbf{R}^2} f(t)\log\langle x\rangle^2\,\mathrm{d}x + \frac{1}{2}\int_0^t \mathscr{D}(f,u)(s)\,\mathrm{d}s \\ \leqslant \mathscr{F}_{H,\mathrm{in}} + C(M) + Mt. \end{aligned} \tag{5.9}$$

5.2.2 Weak solutions

Motivated by the previous discussion, we can now define our notion of weak solutions. We shall always assume that the initial data $(f_{\mathrm{in}}, u_{\mathrm{in}})$ satisfy

$$\begin{cases} f_{\mathrm{in}}\,(1+\log\langle x\rangle^2) \in L^1(\mathbf{R}^2) \quad \text{and} \quad f_{\mathrm{in}}\log f_{\mathrm{in}} \in L^1(\mathbf{R}^2); \\ u_{\mathrm{in}} \in L^1(\mathbf{R}^2)\cap \dot{H}^1(\mathbf{R}^2); \\ f_{\mathrm{in}}\,u_{\mathrm{in}} \in L^1(\mathbf{R}^2), \end{cases} \tag{5.10}$$

where here and below we define the weight function $\langle x\rangle := (1+|x|^2)^{\frac{1}{2}}$, and we also make the restriction to the mass subcritical case

$$M = \int_{\mathbf{R}^2} f_{\mathrm{in}}\,\mathrm{d}x \in (0, 8\pi), \tag{5.11}$$

Definition 5.1. Let $(f_{\mathrm{in}}, u_{\mathrm{in}})$ satisfy (5.10) and (5.11). We say that (f,u) a global weak solution to the parabolic-parabolic Keller–Segel system (5.1) associated to the initial condition $(f_{\mathrm{in}}, u_{\mathrm{in}})$ if they are non-negative functions such that for any $T > 0$:

(i) The following regularities hold

$$\begin{aligned} f &\in L^\infty(0,T;L^1(\mathbf{R}^2))\cap\mathscr{C}([0,T);\mathscr{D}'(\mathbf{R}^2)) \\ u &\in L^\infty(0,T;L^1(\mathbf{R}^2)\cap\dot{H}^1(\mathbf{R}^2)) \\ fu &\in L^\infty(0,T;L^1(\mathbf{R}^2)). \end{aligned}$$

(ii) The mass conservation (5.2) holds.

(iii) Equation (5.1) holds in the distributional sense: for any $\varphi,\psi \in \mathscr{D}([0,T)\times\mathbf{R}^2)$ there holds

$$\int_{\mathbf{R}^2} f_{\mathrm{in}}\,\varphi(0,x)\,\mathrm{d}x = \int_0^T\!\!\int_{\mathbf{R}^2} f\,\Big((\nabla\log f - \nabla u)\cdot\nabla\varphi - \partial_t\varphi\Big)\,\mathrm{d}x\,\mathrm{d}t \tag{5.12a}$$

$$\varepsilon\int_{\mathbf{R}^2} u_{\mathrm{in}}\,\psi(0,x)\,\mathrm{d}x = \int_0^T\!\!\int_{\mathbf{R}^2} \Big(u\,(-\Delta\psi - \varepsilon\partial_t\psi) - f\psi\Big)\,\mathrm{d}x\,\mathrm{d}t. \tag{5.12b}$$

(iv) The free-energy inequality holds: there exists a constant $C_T > 0$ such that

$$\sup_{t\in[0,T]} \mathscr{F}(f,u)(t) + \sup_{t\in[0,T]}\int_{\mathbf{R}^2} f\,\log\langle x\rangle^2\,\mathrm{d}x + \int_0^T \mathscr{D}(f,u)(t)\,\mathrm{d}t \leqslant C_T. \tag{5.13}$$

We notice that the right-hand side of (5.12a) is indeed well-defined thanks to (5.2) and (5.13), for by Cauchy-Schwarz inequality one has

$$\int_{\mathbf{R}^2} f\,|\nabla\log f - \nabla u|\,\mathrm{d}x \leqslant M^{1/2}\,\mathscr{D}^{1/2}.$$

Moreover, remark that we do not assume the free-energy identity (5.4) to hold, but only the estimate (5.13).

5.3 Regularization and uniqueness

For any initial data $(f_{\text{in}}, u_{\text{in}})$ satisfying (5.10) in the subcritical mass case $M < 8\pi$, Calvez-Corrias [38] established the existence of global weak solution in the sense of Definition 5.1, as well as some a priori regularity estimates. Uniqueness has been proved by Carrillo-Lisini-Mainini [58] in the class of bounded solutions, assuming moreover that the initial datum f_{in} is bounded. We also mention that other well-posedness results have been established in the works of Biler-Guerra-Karch [24], Ferreira-Precioso [98] and Corrias-Escobedo-Matos [70] in some particular regimes.

The first result we shall present concerns regularization properties of weak solutions and their uniqueness in the subcritical mass case $M < 8\pi$ in the same setting as the existence result of Calvez-Corrias [38].

Theorem 5.A. *Consider an initial data $(f_{\text{in}}, u_{\text{in}})$ satisfying (5.10) in the mass subcritical case $M < 8\pi$. Then there exists at most one global weak solution (f, u) to the parabolic-parabolic Keller-Segel equation (5.1) in the sense of Definition 5.1.*

Moreover, this solution is classical in the sense $f, u \in \mathscr{C}^2_b((0,\infty) \times \mathbf{R}^2)$, verifies the short time estimate

$$\lim_{t\to 0} t^{1-\frac{1}{q}} \|f(t)\|_{L^q(\mathbf{R}^2)} = 0 \quad \textit{for any} \quad \frac{4}{3} \leqslant q < 2, \tag{5.14}$$

and satisfies the free-energy identity (5.4).

The proof is based on intermediate regularity a posteriori estimates that enable us to use a DiPerna-Lions [85] renormalization process, which in turn makes possible to first obtain the regularization of solutions and then to get the optimal regularity of solutions for short times (5.14). After that, we follow the uniqueness argument introduced by Ben-Artzi [20] (see also Brezis [35]) for the two-dimensional viscous vortex model. Our proof follows a strategy introduced in Fourier-Hauray-Mischler [104] for the two-dimensional viscous vortex model and generalizes a similar result obtained in Fernández-Mischler [97] for the parabolic-elliptic system $\varepsilon = 0$.

Before explaining in more details the proof of Theorem 5.A, we shall make a comment in regards to the critical and supercritical cases $M \geqslant 8\pi$. We first observe that in the case $M \geqslant 8\pi$ the logarithmic Hardy-Littlewood Sobolev inequality (5.8) does not lead to a global estimate as for the subcritical mass case $M \in (0, 8\pi)$. We can however introduce a modified free-energy

$$\widetilde{\mathscr{F}}(f,u) := \int_{\mathbf{R}^2} \left(f \log\left(\frac{f}{\mathscr{M}}\right) - f + \mathscr{M} \right) \mathrm{d}x - \int_{\mathbf{R}^2} f u \, \mathrm{d}x + \frac{1}{2} \int_{\mathbf{R}^2} |\nabla u|^2 \, \mathrm{d}x$$

where we have defined the function $\mathscr{M} := \frac{M}{\pi \langle x \rangle^4}$ of mass M. One can show then that any solution (f,u) to the Keller-Segel equation (5.1) formally satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t} \widetilde{\mathscr{F}}(f,u) + \widetilde{\mathscr{D}}(f,u) \leqslant C \exp(c \widetilde{\mathscr{F}}(f,u)) + C',$$

for some positive constants $c, C, C' > 0$ and where $\widetilde{\mathscr{D}}(f,u) = \mathscr{D}(f,u) + I(f)$ denotes the non-negative dissipation functional associated to $\widetilde{\mathscr{F}}$, and I is the Fisher information functional defined below in (5.15). On the one hand, this differential inequality provides local a priori estimate on the modified free-energy, from which one can prove a local existence result for the critical and supercritical mass cases $M \geqslant 8\pi$. On the other hand, in the proof of Theorem 5.A we show that the resulting estimate is suitable to obtain uniqueness of the solution. Therefore we can obtain the existence and uniqueness of a maximal weak solution $(f,u) \in \mathscr{C}([0,T^\star); \mathscr{D}'(\mathbf{R}^2) \times \mathscr{D}'(\mathbf{R}^2))$ in the sense of Definition 5.1 such that, for any $T \in (0, T^\star)$, one has

$$\sup_{t\in[0,T)} \widetilde{\mathscr{F}}(f,u)(t) + \int_0^T \widetilde{\mathscr{D}}(f,u)(t) \, \mathrm{d}t < \infty,$$

and the alternative $T^\star = +\infty$ or

$$T^\star < \infty \quad \text{and} \quad \lim_{t\to T^\star} \widetilde{\mathscr{F}}(f,u)(t) = \infty.$$

We shall focus our attention hereafter to the mass subcritical case $M < 8\pi$.

5.3.1 A posteriori regularity estimates

In this subsection we present the main arguments that provide some intermediate regularity estimates for any global weak solution. From now on, we hence consider a global weak solution (f,u) to the parabolic-parabolic Keller–Segel system (5.1) in the sense of Definition 5.1 in the mass subcritical case $M \in (0, 8\pi)$.

We first define the Fisher information of f by

$$I(f) := \int_{\mathbf{R}^2} \frac{|\nabla f|^2}{f} \,\mathrm{d}x = 4 \int_{\mathbf{R}^2} |\nabla \sqrt{f}|^2 \,\mathrm{d}x. \tag{5.15}$$

As a consequence of the integral-in-time bound of the dissipation of the free-energy $\mathscr{D}(f,u)$ in (5.13) and the uniform-in-time bound of the non-negative part of the entropy $\mathscr{H}^+(f)$ established in (5.9), we also obtain an integral-in-time bound for $I(f)$, namely for any $T > 0$ there holds

$$I(f) \in L^1(0,T).$$

Thanks to Hölder and Sobolev inequalities, this last estimate on the Fisher information together with the conservation of mass (5.2) already provide us with some integrability and regularity estimates, more precisely, for any $T > 0$ one has

$$\begin{aligned} &f \in L^{\frac{p}{p-1}}(0,T; L^p(\mathbf{R}^2)) \text{ for any } p \in (1,\infty),\\ &\nabla f \in L^{\frac{2p}{3p-2}}(0,T; L^p(\mathbf{R}^2)) \text{ for any } p \in [1,2),\\ &\Delta u \in L^2(0,T; L^2(\mathbf{R}^2)). \end{aligned} \tag{5.16}$$

With these estimates at hand, we are able to apply the DiPerna-Lions [85, 84] renormalization argument to the equation (5.1a). We hence obtain that any weak solution (f,u) satisfies

$$\begin{aligned} &\int_{\mathbf{R}^2} \beta(f(t_1)) \,\mathrm{d}x + \int_{t_0}^{t_1}\int_{\mathbf{R}^2} \beta''(f(s))\, |\nabla f(s)|^2 \,\mathrm{d}x\,\mathrm{d}s \\ &\qquad \leqslant \int_{\mathbf{R}^2} \beta(f(t_0)) \,\mathrm{d}x + \int_{t_0}^{t_1}\int_{\mathbf{R}^2} \{\beta(f(s)) - f(s)\beta'(f(s))\}\Delta u(s) \,\mathrm{d}x\,\mathrm{d}s, \end{aligned} \tag{5.17}$$

for any times $0 \leqslant t_0 \leqslant t_1 < \infty$ and any renormalizing function $\beta : \mathbf{R} \to \mathbf{R}$ which is convex, piecewise of class $\mathscr{C}^1$ and satisfies the growth condition

$$|\beta(\xi)| \leqslant C(1 + \xi(\log \xi)_+), \quad |\beta(\xi) - \xi\beta'(\xi)| \leqslant C\xi, \quad \forall \xi \in \mathbf{R}.$$

By using (5.17) twice with well-chosen sequences of renormalizing functions β together with the estimate (5.16), we are then able to improve the integrability and regularity estimates for f by obtaining some uniform-in-time bounds. More precisely, we obtain that for any $p \geqslant 2$ and any $t_0 \in [0,T)$ such that $f(t_0) \in L^p(\mathbf{R}^2)$, there exists a positive constant $C > 0$, depending $M, \mathscr{H}_{\mathrm{in}}, \mathscr{F}_{\mathrm{in}}, T, p, \|f(t_0)\|_{L^p(\mathbf{R}^2)}$, such that for all $t_0 < t_1 \leqslant T$ there holds

$$\|f(t_1)\|^p_{L^p(\mathbf{R}^2)} + \frac{1}{2}\int_{t_0}^{t_1} \|\nabla f^{\frac{p}{2}}(t)\|^2_{L^2(\mathbf{R}^2)} \,\mathrm{d}t \leqslant C.$$

Finally, gathering the previous estimates and using a bootstrap argument together with the maximal regularity of parabolic equations in L^p-spaces, we are able to deduce that any weak solution is classical $f, u \in \mathscr{C}^2_b((0,\infty) \times \mathbf{R}^2)$. Furthermore, as a consequence of that, we also get that the free-energy identity (5.4) holds.

5.3.2 Uniqueness of weak solutions

As a consequence of estimates provided by (5.17), the uniform-in-time bound of $\mathscr{H}^+(f)$ and an interpolation argument, we obtain the short-time estimate (5.14). Once this is achieved, we are then able to prove the uniqueness of solutions by following the argument of Ben-Artzi [20] (see also Brezis [35]) for the two-dimensional viscous vortex model, that is we estimate the difference between two solutions in mild form, which is possible thanks to the previous a posteriori estimates together with the short-time estimate (5.14).

More precisely, let (f_1, u_1) and (f_2, u_2) be two weak solutions to the parabolic-parabolic Keller–Segel equation (5.1). Assuming that $f_1(0) = f_2(0)$ and $u_1(0) = u_2(0) = u_{\rm in}$, the difference $F := f_2 - f_1$ satisfies in the mild form

$$\begin{aligned} F(t) = &-\int_0^t \nabla \mathrm{e}^{(t-s)\Delta}\left[F(s)\mathrm{e}^{\frac{s}{\varepsilon}\Delta}(\nabla u_{\rm in})\right]\mathrm{d}s - \int_0^t \nabla \mathrm{e}^{(t-s)\Delta}\left[F(s)\,\frac{1}{\varepsilon}\int_0^s \nabla \mathrm{e}^{\frac{(s-\sigma)}{\varepsilon}\Delta} f_2(\sigma)\,\mathrm{d}\sigma\right]\mathrm{d}s \\ &-\int_0^t \nabla \mathrm{e}^{(t-s)\Delta}\left[f_1(s)\,\frac{1}{\varepsilon}\int_0^s \nabla \mathrm{e}^{\frac{(s-\sigma)}{\varepsilon}\Delta} F(\sigma)\,\mathrm{d}\sigma\right]\mathrm{d}s, \end{aligned}$$

where $\mathrm{e}^{t\Delta}$ denotes the heat semigroup. Defining the quantities

$$\delta(t) := \sup_{0<s\leqslant t} s^{\frac14}\,\|F(s)\|_{L^{4/3}(\mathbf{R}^2)}$$

and, for some fixed $p > 2$,

$$Z_p(t) := \sup_{0<s\leqslant t} s^{\frac12 - \frac{1}{2p}}\,\|f_1(s)\|_{L^{\frac{2p}{p+1}}(\mathbf{R}^2)} + \sup_{0<s\leqslant t} s^{\frac12 - \frac{1}{2p}}\,\|f_2(s)\|_{L^{\frac{2p}{p+1}}(\mathbf{R}^2)},$$

we obtain thanks to the a posteriori bounds and the estimate (5.14) that

$$\delta(t) \leqslant C\left(\eta(t) + Z_p(t)\right)\delta(t),$$

for some constant $C > 0$ and some constructive function η verifying $\lim_{t\to 0}\eta(t) = 0$. This implies $\delta(t) \equiv 0$ on $[0, T)$ if $T > 0$ is small enough, and then we repeat the argument for later times to conclude to the uniqueness.

5.4 Stability of the self-similar profile

In this section we present our second result on the parabolic-parabolic Keller–Segel equation (5.1) that concerns the large-time behavior of solutions.

We start our analysis by looking for self-similar solutions to (5.1), that is solutions of the form

$$f(t,x) = \frac{1}{t}G_\varepsilon\left(\frac{x}{\sqrt{t}}\right) \quad \text{and} \quad u(t,x) = V_\varepsilon\left(\frac{x}{\sqrt{t}}\right),$$

with

$$\int_{\mathbf{R}^2} f(t,x)\,\mathrm{d}x = \int_{\mathbf{R}^2} G_\varepsilon(y)\,\mathrm{d}y = M \in (0, 8\pi).$$

Such a couple of functions (f, u) is a solution to (5.1) if and only if the associated self-similar profile $(G_\varepsilon, V_\varepsilon)$ satisfies the following elliptic system in $\mathbf{R}^2$

$$\Delta G_\varepsilon + \operatorname{div}\left(\frac12\, x\, G_\varepsilon - G_\varepsilon\,\nabla V_\varepsilon\right) = 0 \tag{5.18a}$$

$$\Delta V_\varepsilon + \frac{\varepsilon}{2}\, x\cdot\nabla V_\varepsilon + G_\varepsilon = 0. \tag{5.18b}$$

It was proven by Naito-Suzuki-Yoshida [163], Biler-Corrias-Dolbeault [23], and Corrias-Escobedo-Matos [70] that, for any $\varepsilon \in (0, \frac12)$ and any $M \in (0, 8\pi)$, there exists a unique radially symmetric smooth solution $(G_\varepsilon, V_\varepsilon)$ to (5.18) such that the mass of G_ε equals M.

Hereafter we shall work in self-similar variables, and hence we introduce the rescaled functions (g,v) defined by

$$f(t,x) := \frac{1}{1+t}\, g\left(\log(1+t)^{1/2}, \frac{x}{(1+t)^{1/2}}\right),$$
$$u(t,x) := v\left(\log(1+t)^{1/2}, \frac{x}{(1+t)^{1/2}}\right).$$

where (f,u) satisfies the parabolic-parabolic Keller–Segel equation (5.1). For these new unknowns, the rescaled parabolic-parabolic Keller-Segel system in $(0,\infty)\times\mathbf{R}^2$ reads

$$\begin{cases} \partial_t g = \Delta g + \operatorname{div}\left(\dfrac{1}{2}\, x\, g - g\,\nabla v\right) & (5.19a)\\ \varepsilon\partial_t v = \Delta v + g + \dfrac{\varepsilon}{2}\, x\cdot\nabla v & (5.19b)\end{cases}$$

and therefore the solution $(G_\varepsilon, V_\varepsilon)$ to (5.18) corresponds to a stationary solution to (5.19).

In the parabolic-elliptic case $\varepsilon = 0$, we also obtain a self-similar profile (G_0, V_0) solution to the elliptic system (5.18) with $\varepsilon = 0$ and the associated rescaled evolution equation corresponding to the system (5.19) with $\varepsilon = 0$, which can be written as a single equation for g. In that case, it turns out that the rescaled free-energy, i.e. the free-energy defined in (5.3) in terms of the new unknowns (g,v), is a Lyapunov functional for the system (5.19) with $\varepsilon = 0$ for which the self-similar profile (G_0, V_0) is an extremum. One hence can expect that solutions to the system (5.19) with $\varepsilon = 0$ converge in large time to the self-similar profile (G_0, V_0).

In our case of the parabolic-parabolic rescaled Keller-Segel equation (5.19) with $\varepsilon > 0$, we do not know any Lyapunov functional and hence the possible expected large-time behavior of solutions is not clear at all. However, at a formal level, the parabolic-parabolic system (5.19) converges to the corresponding parabolic-elliptic system in the limit $\varepsilon \to 0$. Therefore one could hope that, for $\varepsilon > 0$ small enough, system (5.19) would inherit the nonlinear stability properties of its associated parabolic-elliptic system.

Following this idea, the result we shall present below concerns the large-time behavior of solutions to (5.19). More precisely, we show the exponential nonlinear stability of the self-similar profile $(G_\varepsilon, V_\varepsilon)$ for any subcritical mass $M \in (0, 8\pi)$ under the strong restriction of radial symmetry and a quasi-parabolic-elliptic regime, that is, for $\varepsilon > 0$ small.

Let us define the norm

$$|\!|\!|(g,v)|\!|\!| := \left(\|\langle x\rangle^k g\|_{L^2}^2 + \|\langle x\rangle^k \nabla g\|_{L^2}^2 + \|v\|_{H^2}^2\right)^{\frac{1}{2}},$$

for some $k > 7$, and where we denote $\langle x\rangle := (1+|x|^2)^{\frac{1}{2}}$.

Theorem 5.B. *Let the mass $M \in (0, 8\pi)$ be fixed. There exist $\varepsilon_0 > 0$ and $\delta_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$ and any radially symmetric initial data $(g_{\rm in}, v_{\rm in})$ satisfying*

$$|\!|\!|(g_{\rm in}, v_{\rm in}) - (G_\varepsilon, V_\varepsilon)|\!|\!| \leqslant \delta_0 \quad \text{and} \quad \int_{\mathbf{R}^2} g_{\rm in}\, \mathrm{d}x = \int_{\mathbf{R}^2} G_\varepsilon\, \mathrm{d}x = M,$$

the solution (g,v) to (5.19) satisfies the following: for any $\lambda \in (0, \frac{1}{2})$ there is a positive constant $C > 0$ such that, for all $t \geqslant 0$, there holds

$$|\!|\!|(g(t), v(t)) - (G_\varepsilon, V_\varepsilon)|\!|\!| \leqslant C\mathrm{e}^{-\lambda t}\, |\!|\!|(g_{\rm in}, v_{\rm in}) - (G_\varepsilon, V_\varepsilon)|\!|\!|.$$

Coming back to the original unknowns (f,u), this result asserts that if the initial data $(f_{\rm in}, u_{\rm in})$ is sufficiently close to the self-similar profile $(G_\varepsilon, V_\varepsilon)$, then the solution (f,u) of the parabolic-parabolic Keller–Segel equation (5.1) behaves, when $t\to\infty$, as

$$f(t,x) \sim \frac{1}{t}\, G_\varepsilon\left(\frac{x}{\sqrt{t}}\right) \quad \text{and} \quad u(t,x) \sim V_\varepsilon\left(\frac{x}{\sqrt{t}}\right),$$

with quantitative rates of convergence.

This result is the first exponential stability result for the system (5.19), even with the restriction of radial symmetry and $\varepsilon > 0$ small enough, and it extends to the parabolic-parabolic Keller-Segel equation similar results obtained for the parabolic-elliptic Keller–Segel equation $\varepsilon = 0$ by Fernández-Mischler [97]. We also mention that Corrias-Escobedo-Matos [70] recently established results on the convergence without rate of some solutions to the associated self-similar profile.

Our proof is based on the study of the linearized equation around the self-similar profile $(G_\varepsilon, V_\varepsilon)$. The first part of our study consists in proving that the associated semigroup is exponentially stable in some weighted $H^1 \times H^2$ space in the quasi-parabolic regime, that is $\varepsilon > 0$ small enough, by taking advantage of the exponential stability of the linearized semigroup in the parabolic-elliptic case $\varepsilon = 0$. Finally, we come back to the nonlinear equation (5.19) and prove that it inherits the exponential stability of the linearized equation by treating the nonlinear term as a perturbation. As a consequence of this, we observe that the exponential rate λ above can be taken arbitrarily close to $\frac{1}{2}$, which is the exponential decay rate in the parabolic-elliptic case established in Campos-Dolbeault [39] and Fernández-Mischler [97].

5.4.1 The linearized operator

By writing the solution of (5.19) as

$$(g, v) = (G_\varepsilon, V_\varepsilon) + (h, w)$$

and neglecting the nonlinear terms, one obtains the linearized equation around the self-similar profile $(G_\varepsilon, V_\varepsilon)$ associated to (5.19), namely

$$\begin{cases} \partial_t h = \Lambda^1_\varepsilon(h, w) := \Delta h + \operatorname{div}\left(\frac{1}{2}\, x\, h - h\, \nabla V_\varepsilon - G_\varepsilon\, \nabla w\right) & \text{(5.20a)} \\ \partial_t w = \Lambda^2_\varepsilon(h, w) := \frac{1}{\varepsilon}(\Delta w + h) + \frac{1}{2}\, x \cdot \nabla w & \text{(5.20b)} \end{cases}$$

that we write in the condensed form

$$\partial_t(h, w) = \Lambda_\varepsilon(h, w).$$

The linearized operator in the parabolic-elliptic case $\varepsilon = 0$

In the parabolic-elliptic case $\varepsilon = 0$, the rescaled equation (5.19) becomes

$$\begin{cases} \partial_t g = \Delta g + \operatorname{div}\left(\frac{1}{2}\, x g - g \nabla v\right) & \text{(5.21a)} \\ -\Delta v = g, & \text{(5.21b)} \end{cases}$$

and the stationary solution (G_0, V_0) to (5.21) verifies the following elliptic system in $\mathbf{R}^2$

$$\Delta G_0 + \operatorname{div}\left(\frac{1}{2}\, x G_0 - G_0 \nabla V_0\right) = 0,$$
$$\Delta V_0 + G_0 = 0,$$

with $\int_{\mathbf{R}^2} G_0(x)\, \mathrm{d}x = M \in (0, 8\pi)$. The linearized equation around the self-similar profile (G_0, V_0) is therefore given by

$$\begin{cases} \partial_t h = \Delta h + \operatorname{div}\left(\frac{1}{2}\, x h - h \nabla V_0 - G_0 \nabla w\right) \\ -\Delta w = h \end{cases}$$

which simplifies into a single equation

$$\partial_t h = \Delta h + \operatorname{div}\left(\frac{1}{2}xh - h\nabla V_0 - G_0\nabla(-\Delta)^{-1}h\right) =: \Omega h. \tag{5.22}$$

Thanks to the existence of a Lyapunov functional for the linearized equation (5.22) provided by the linearization of the rescaled free-energy, a variational approach has been successfully employed by Campos-Dolbeault [39], resulting then in an exponential stability in the space

$$L^2(G_0^{-\frac{1}{2}}) := \left\{h \in L^2(\mathbf{R}^2) \;\middle|\; \int h^2 G_0^{-1}\,\mathrm{d}x < +\infty\right\}.$$

One remarks that G_0 behaves as $\mathrm{e}^{-c|x|^2}$ so that functions in $L^2(G_0^{-\frac{1}{2}})$ decay very fast when $|x| \to \infty$. This linear result was later extended to several larger Banach spaces by Fernández-Mischler [97], whom have then used this linear stability result in order to prove the convergence to the self-similar profile for solutions to the nonlinear equation.

Let us summarize part of these results on the linear operator Ω. Define the space

$$X_1 = L^2_{\mathrm{rad}}(\mathbf{R}^2) \cap L^2_{k,0}(\mathbf{R}^2)$$

endowed with the norm

$$\|h\|_{X_1} := \|\langle x\rangle^k h\|_{L^2},$$

where $L^2_{\mathrm{rad}}(\mathbf{R}^2)$ denotes the subspace of $L^2(\mathbf{R}^2)$ composed by radially symmetric functions, $L^2_k(\mathbf{R}^2)$ is the weighted L^2-space defined by

$$L^2_k(\mathbf{R}^2) := \left\{h \in L^2(\mathbf{R}^2) \;\middle|\; \int_{\mathbf{R}^2} \langle x\rangle^{2k} h^2\,\mathrm{d}x < +\infty\right\}$$

and $L^2_{k,0}$ denotes the subspace of L^2_k formed by functions with zero mean, that is, functions $h \in L^2_k$ such that $\int_{\mathbf{R}^2} h\,\mathrm{d}x = 0$.

Gathering the recent results of Campos-Dolbeault [39] and Fernández-Mischler [97], one obtains that for the linear operator Ω acting on the space X_1, there exists a constant $C > 0$ such that, for all $t \geqslant 0$, one has

$$\|S_\Omega(t)\|_{\mathscr{L}(X_1,X_1)} \leqslant C\mathrm{e}^{-\frac{t}{2}}, \tag{5.23a}$$

where $S_\Omega = (S_\Omega(t))_{t\geqslant 0}$ denotes the semigroup generated by Ω. From this we also obtain that

$$\mathrm{R}_\Omega \in \mathscr{O}\left(\left\{z \in \mathbf{C} \mid \Re z > -\tfrac{1}{2}\right\}; \mathscr{L}(X_1, X_1)\right), \tag{5.23b}$$

where $\mathrm{R}_\Omega : z \mapsto (\Omega - z)^{-1}$ denotes the resolvent map associated to Ω and $\mathscr{O}$ denotes the space of holomorphic functions; as well as

$$\mathrm{sp}(\Omega) \cap \left\{z \in \mathbf{C} \mid \Re z > -\tfrac{1}{2}\right\} = \varnothing. \tag{5.23c}$$

where $\mathrm{sp}(\Omega)$ denotes the spectrum of Ω.

5.4.2 A singular perturbation argument

At a very formal level, one observes that the linearized parabolic-parabolic system (5.20) converges as $\varepsilon \to 0$ to the linearized parabolic-elliptic equation (5.22). Therefore, one could hope that for $\varepsilon > 0$ small enough, the parabolic-system (5.20) would inherit the stability properties of the corresponding limit system (5.22).

One of the difficulties of pursuing this idea is that not only this corresponds to a quite singular limit, but also that the operator Λ_ε that interests us can not be seen as perturbation of some fixed operator Λ, which possesses well suited properties, and we cannot apply the perturbation theory developed by Mischler-Mouhot [153] or Tristani [178].

Still formally, one can also observe that the convergence of the parabolic-parabolic system (5.20) to the parabolic-elliptic system (5.22) as $\varepsilon \to 0$ also holds at the level of the resolvent maps, more precisely

$$\mathrm{R}_{\Lambda_\varepsilon}(z) \to \begin{pmatrix} \mathrm{R}_\Omega(z) & 0 \\ (-\Delta)^{-1}\mathrm{R}_\Omega(z) & 0 \end{pmatrix} \quad \text{as} \quad \varepsilon \to 0.$$

Although we are not able to prove this convergence, using the properties of the resolvent R_Ω described in (5.23b), we can prove some similar properties for the resolvent map $\mathrm{R}_{\Lambda_\varepsilon}$ when $\varepsilon > 0$ is small enough. This information on the resolvent together with some localization properties of the spectrum $\mathrm{sp}(\Lambda_\varepsilon)$ of the operator Λ_ε, will finally gives us the exponential stability of the semigroup S_{Λ_ε} generated by Λ_ε in functional spaces that are appropriated in order to handle the nonlinear equation, and with a rate as close as we want to the exponential stability rate of the semigroup S_Ω.

Let us explain in more details how we are able to deduce spectral and semigroup estimates for the operator Λ_ε from the information we have on the operator Ω described in (5.23). Let us fix a real number $k > 7$ and introduce the Hilbert space

$$X = X_1 \times X_2$$

endowed with the norm

$$\|(h,w)\|_X := \left(\|\langle x\rangle^k h\|_{L^2}^2 + \|w\|_{L^2}^2 \right)^{1/2},$$

where $X_1 = L^2_{\mathrm{rad}}(\mathbf{R}^2) \cap L^2_{k,0}(\mathbf{R}^2)$ and $X_2 = L^2_{\mathrm{rad}}(\mathbf{R}^2)$. We also introduce the Hilbert space $Y \subseteq X$ defined by

$$Y = Y_1 \times Y_2$$

and endowed with the norm

$$\|(h,w)\|_Y := \left(\|(h,w)\|_X^2 + \|\langle x\rangle^k \nabla h\|_{L^2}^2 + \|\nabla w\|_{L^2}^2 \right)^{1/2},$$

where $Y_1 = H^1_k(\mathbf{R}^2) \cap L^2_{k,0}(\mathbf{R}^2) \cap L^2_{\mathrm{rad}}(\mathbf{R}^2)$ and $Y_2 = H^1(\mathbf{R}^2) \cap L^2_{\mathrm{rad}}(\mathbf{R}^2)$, with

$$H^1_k(\mathbf{R}^2) := \left\{ h \in L^2(\mathbf{R}^2) \;\middle|\; \int_{\mathbf{R}^2} \langle x\rangle^{2k} \left(h^2 + |\nabla h|^2\right) \mathrm{d}x < +\infty \right\}.$$

We then factorize the linearized operator as $\Lambda_\varepsilon = A + B_\varepsilon$ with

$$A(h,w) := (R'\chi_R\, h - R'\chi_1 \langle \chi_R\, h\rangle, 0) \quad \text{and} \quad B_\varepsilon(h,w) := (\Lambda_\varepsilon - A)(h,w)$$

where $R, R' > 0$ are constants chosen large enough and χ_R is a smooth cutoff function. One can show that the operators A and B_ε then satisfy the following properties:

- A is bounded from X into X and from Y into Y;
- for any $\lambda \in (0, \frac{1}{2})$, $(B_\varepsilon - \lambda)$ is hypodissipative in both spaces X and Y, in the sense that there is some constant $C > 0$ such that, for all $t \geqslant 0$, one has

  $$\|S_{B_\varepsilon}(t)\|_{\mathscr{L}(X,X)} \leqslant C\mathrm{e}^{-\lambda t} \quad \text{and} \quad \|S_{B_\varepsilon}(t)\|_{\mathscr{L}(Y,Y)} \leqslant C\mathrm{e}^{-\lambda t},$$

  where S_{B_ε} denotes the semigroup generated by B_ε.

— for any $\lambda \in (0, \frac{1}{2})$, S_{B_ε} has the following regularization property: there is $C > 0$ such that, for all $t \geqslant 0$, one has

$$\|S_{B_\varepsilon}(t)\|_{\mathscr{L}(X,Y)} \leqslant C \, \frac{\mathrm{e}^{-\lambda t}}{\min(1, \sqrt{t})}.$$

With these estimates, one can apply the partial spectral mapping theorem and the adapted version of Weyl's theorem developed in Mischler-Scher [155] in order to obtain following property of the spectrum $\mathrm{sp}(\Lambda_\varepsilon)$ of the operator Λ_ε when acting on the space X: there exist $\varepsilon_0, r_0 > 0$ such that, for any $\varepsilon \in (0, \varepsilon_0)$, there holds

$$\mathrm{sp}(\Lambda_\varepsilon) \cap \left\{ z \in \mathbf{C} \mid \Re z > -\tfrac{1}{2} \right\} \subseteq \mathrm{sp}_{\mathrm{disc}}(\Lambda_\varepsilon) \cap \{ z \in \mathbf{C} \mid |z| < r_0 \} \,,$$

where $\mathrm{sp}_{\mathrm{disc}}$ denotes the discrete spectrum.

As explained before, we then investigate the resolvent $\mathrm{R}_{\Lambda_\varepsilon}$ by exploiting the information on the resolvent R_Ω in (5.23b) and by making a perturbative argument with $\varepsilon > 0$ is small enough. We are then able to show that for any $r > 0$, there exists $\varepsilon_0(r) > 0$ such that, for any $\varepsilon \in (0, \varepsilon_0(r))$, there holds

$$\mathrm{R}_{\Lambda_\varepsilon} \in \mathscr{O}\Big(\Big\{ z \in \mathbf{C} \mid \Re z > -\tfrac{1}{2} \text{ and } |z| < r \Big\} ; \mathscr{L}(X, X) \Big) \,.$$

As a consequence of the above information on the spectrum $\mathrm{sp}(\Lambda_\varepsilon)$ and on the resolvent map $\mathrm{R}_{\Lambda_\varepsilon}$, and using again the estimates provided by the factorization $\Lambda_\varepsilon = A + B_\varepsilon$ detailed above, one can apply the principal spectral mapping theorem of Mischler-Scher [155] in order to deduce information on the spectrum of Λ_ε and on the semigroup S_{Λ_ε} generated by Λ_ε. More precisely, for $\lambda_0 < \frac{1}{2}$ being fixed, one obtains that for any $\lambda \in (0, \lambda_0)$ there is $\varepsilon_0 > 0$ such that, when Λ_ε acts on the space X, its spectrum verifies, for any $\varepsilon \in (0, \varepsilon_0)$,

$$\mathrm{sp}(\Lambda_\varepsilon) \cap \{ z \in \mathbf{C} \mid \Re z > -\lambda_0 \} = \varnothing. \tag{5.24a}$$

Furthermore, for any $\lambda \in (0, \lambda_0)$ there is a constant $C > 0$ such that, for all $t \geqslant 0$ and any $\varepsilon \in (0, \varepsilon_0)$, one has

$$\|S_{\Lambda_\varepsilon}(t)\|_{\mathscr{L}(X,X)} \leqslant C \mathrm{e}^{-\lambda t}. \tag{5.24b}$$

In order words, the semigroup S_{Λ_ε} inherits the exponential stability of the semigroup S_Ω associated to the linearized equation in the parabolic-elliptic case $\varepsilon = 0$.

5.4.3 Nonlinear stability

We now focus on the nonlinear parabolic-parabolic Keller-Segel system in self-similar variables. Consider a solution (g, v) to (5.19) and define the perturbation $(h, w) := (g - G_\varepsilon, v - V_\varepsilon)$, which hence satisfies

$$\begin{cases} \partial_t h = \Lambda_\varepsilon^1(h, w) - \operatorname{div}(h \nabla w) & (5.25\mathrm{a}) \\ \partial_t w = \Lambda_\varepsilon^2(h, w), & (5.25\mathrm{b}) \end{cases}$$

together with the initial condition $(h_{\mathrm{in}}, w_{\mathrm{in}}) := (g_{\mathrm{in}} - G_\varepsilon, v_{\mathrm{in}} - V_\varepsilon)$. Writing it in a condensed form one has

$$\partial_t(h, w) = \Lambda_\varepsilon(h, w) + (-\operatorname{div}(h \nabla w), 0) \,.$$

It turns out that the previous properties concerning the linearized operator Λ_ε in the space X are not enough to treat the nonlinear equation (5.25) because of the loss of regularity coming from the nonlinear term. We would like therefore to obtain estimates similar of those in (5.24) in a stronger space $Z \subseteq X$ that is suitable for handling the nonlinear equation. This is actually possible to achieve by using a shrinkage theorem of Mischler-Mouhot [154], which enable us to deduce spectral and semigroup estimates for Λ_ε and S_{Λ_ε} in a stronger space $Z \subseteq X$ provided that the operator Λ_ε factorizes as $\Lambda_\varepsilon = A + B_\varepsilon$ in such a way that the operators A and B_ε possess some boundedness, dissipativity and regularization properties similar of those presented in the above factorization.

Let us define the Hilbert space $Z \subseteq X$ by

$$Z = Z_1 \times Z_2$$

endowed with the norm

$$\|(h,w)\|_Z := \left(\|(h,w)\|_Y^2 + \|\nabla^2 w\|_{L^2}^2 \right)^{1/2},$$

where $Z_1 = H^1_k(\mathbf{R}^2) \cap L^2_{k,0}(\mathbf{R}^2) \cap L^2_{\mathrm{rad}}(\mathbf{R}^2)$ and $Z_2 = H^2(\mathbf{R}^2) \cap L^2_{\mathrm{rad}}(\mathbf{R}^2)$. We first obtain, as explained above, that the spectral and semigroup estimates for Λ_ε and S_{Λ_ε} in the space X also holds in the space $Z \subseteq X$. More precisely one has:

Proposition 5.1. *Let $\lambda_0 \in (0, \frac{1}{2})$. There exists $\varepsilon_0 > 0$ such that, in the space Z, for any $\varepsilon \in (0, \varepsilon_0)$ there holds*

$$\mathrm{sp}(\Lambda_\varepsilon) \cap \{ z \in \mathbf{C} \mid \Re z > -\lambda_0 \} = \varnothing.$$

As a consequence, for any $\lambda \in (0, \lambda_0)$ there exists a constant $C > 0$ such that, for all $t \geqslant 0$ and any $\varepsilon \in (0, \varepsilon_0)$, one has

$$\|S_{\Lambda_\varepsilon}(t)\|_{\mathscr{L}(Z,Z)} \leqslant C\, \mathrm{e}^{-\lambda t}.$$

Using the exponential stability of the semigroup S_{Λ_ε} in the space Z, we are then able to construct a new norm in the space Z, equivalent to the usual one, for which the operator Λ_ε possesses good dissipativity and regularization properties. This is a crucial ingredient in order to deal with the nonlinear equation (5.25), since with the stability estimate of Proposition 5.1 alone we are not able to capture the regularization properties of the operator Λ_ε, which in turn is needed in order to control the loss of regularity coming from the nonlinear term.

Let us define the space $Z_\star \subseteq Z$ by

$$Z_\star = Z_{\star,1} \times Z_{\star,2}$$

endowed with the norm

$$\|(h,w)\|_{Z_\star} := \left(\|(h,w)\|_Z^2 + \|\langle x\rangle^k \nabla^2 h\|_{L^2}^2 + \|\nabla^3 w\|_{L^2}^2 \right)^{1/2},$$

where $Z_{\star,1} = H^2_k(\mathbf{R}^2) \cap L^2_{k,0}(\mathbf{R}^2) \cap L^2_{\mathrm{rad}}(\mathbf{R}^2)$ and $Z_{\star,2} = H^3(\mathbf{R}^2) \cap L^2_{\mathrm{rad}}(\mathbf{R}^2)$, with

$$H^2_k(\mathbf{R}^2) := \left\{ h \in L^2(\mathbf{R}^2) \;\middle|\; \int_{\mathbf{R}^2} \langle x\rangle^{2k} \left(h^2 + |\nabla h|^2 + |\nabla^2 h|^2 \right) \mathrm{d}x < +\infty \right\}.$$

Furthermore, we define for any $\eta > 0$ the norm

$$|||(h,w)|||_Z := \left(\eta \|(h,w)\|_Z^2 + \int_0^\infty \|S_{\Lambda_\varepsilon}(s)(h,w)\|_Z^2 \,\mathrm{d}s \right)^{1/2},$$

which is equivalent to $\|\cdot\|_Z$ thanks to Proposition 5.1, and denote by $\langle\!\langle \cdot, \cdot \rangle\!\rangle_Z$ the associated scalar product.

Proposition 5.2. *Then there is $\eta > 0$ small enough such that the operator Λ_ε is hypodissipative in Z, in the sense that there is a constant $K > 0$ such that*

$$\langle\!\langle \Lambda_\varepsilon(h,w), (h,w) \rangle\!\rangle_Z \leqslant -K \|(h,w)\|_{Z_\star}^2 \quad \textit{for any} \quad (h,w) \in D(\Lambda_\varepsilon).$$

With the semigroup estimates of Proposition 5.1 and the energy estimate of Proposition 5.2, we obtain a key a priori a priori stability estimate on solutions to (5.25). The solution (h, w) to (5.25) satisfies, at least formally, the following differential inequality, for some constants $C, K > 0$,

$$\frac{\mathrm{d}}{\mathrm{d}t} \|\!|(h, w)|\!\|_Z^2 \leqslant -K \, \|(h, w)\|_{Z_\star}^2 + C \, \|\!|(h, w)|\!\|_Z \|(h, w)\|_{Z_\star}^2. \tag{5.26}$$

From this a priori estimate, for initial data $(h_{\text{in}}, w_{\text{in}})$ small enough, one can construct through a standard iterative scheme a unique solution to (5.25) satisfying some strong estimates uniformly in time. More precisely, one obtains that there is $\delta > 0$ small enough such that, if $\|\!|(h_{\text{in}}, w_{\text{in}})|\!\|_Z \leqslant \delta$, then there exists a unique solution $(h, w) \in \mathscr{C}(\mathbf{R}_+; Z)$ to (5.25) that verifies

$$\sup_{t \geqslant 0} \|\!|(h(t), w(t))|\!\|_Z^2 + \int_0^\infty \|(h(t), w(t))\|_{Z_\star}^2 \, \mathrm{d}t \leqslant C\delta^2. \tag{5.27}$$

We then come back to inequality (5.26) together with the estimate (5.27), which implies that if $\delta > 0$ is small enough then one first gets the exponential decay, for all $t \geqslant 0$,

$$\|\!|(h, w)(t)|\!\|_Z \leqslant \mathrm{e}^{-K't} \, \|\!|(h_{\text{in}}, w_{\text{in}})|\!\|_Z,$$

for some constant $K' > 0$. We can finally recover the decay rate $O(\mathrm{e}^{-\lambda t})$, for any $\lambda \in (0, \frac{1}{2})$, as stated in Theorem 5.B by performing a bootstrap argument.

5.5 Some perspectives

Our results on the asymptotic behavior of solutions to the parabolic-parabolic Keller–Segel equation were obtained in the mass subcritical case $M < 8\pi$.

5.5.1 Stability of self-similar profile in a non-radial setting

A first extension of our result on the asymptotic behavior in Theorem 5.B we would like to perform is to remove the radial symmetry assumption. This assumptions is made in order to obtain some precise estimates on solutions to an auxiliary elliptic equation and it is only used in the proof of the spectral estimates (5.24). We believe that this assumption is only a technical issue and we would like to remove it.

5.5.2 Large-time behavior for supercritical mass

Our result on the stability of the self-similar profile in Theorem 5.B considered only the mass subcritical case $M < 8\pi$.

In the mass supercritical case $M > 8\pi$, for $\varepsilon > 0$ sufficiently large we know, on the one hand, that the self-similar profile is unique thanks to Biler-Corrias-Dolbeault [23] and Corrias-Escobedo-Matos [70], and, on the other hand, that we can construct some global solutions thanks to Biler-Guerra-Karch [24] and Corrias-Escobedo-Matos [70].

An interesting question would then to apply the new accurate spectral and semigroups estimates for the linearized operator developed in this chapter in order to study the nonlinear stability of the unique self-similar profile in the mass supercritical case $M > 8\pi$ with $\varepsilon > 0$ large enough.

5.5.3 Finite-time blowup solutions

In the case of parabolic-elliptic Keller–Segel equation (5.1) with $\varepsilon = 0$ in a mass supercritical case $M > 8\pi$, we have already mentioned the construction of solutions that blowup in finite time by Raphaël-Schweyer [167] and Collot-Ghoul-Masmoudi-Nguyen [69, 68]. These results rely on a detailed spectral analysis of the linearized operator and provide a precise description of the singularity formation.

A very interesting problem would be to combine the strategies of Raphaël-Schweyer [167] and Collot-Ghoul-Masmoudi-Nguyen [69, 68] in the parabolic-elliptic case, together with our spectral estimates in order to construct finite-time blowup solutions with precise description of the singularity for the parabolic-parabolic Keller–Segel equation in the mass supercritical setting.

5.5.4 Other aggregation-diffusion equations

An interesting problem would be to investigate other aggregation-diffusion equations other than the Keller–Segel model with the techniques developed in this chapter. More precisely, we could consider aggregation-diffusion equations in the form

$$\partial_t f = \mathcal{D}(f) - \operatorname{div}(\mathcal{K}[f]\, f),$$

where $f = f(t,x)$ represents some density, $t \in \mathbf{R}_+$ being the time variable and $x \in \mathbf{R}^d$ the spatial variable. Here the term $\mathcal{D}(f)$ represents the diffusion phenomenon whereas the term $-\operatorname{div}(\mathcal{K}[f]\, f)$ an aggregation phenomenon, which are hence in competition. We would like then to consider models in which the diffusion is nonlinear or nonlocal and the aggregation term is singular. Typical examples are given by the case of nonlinear diffusion model in which $\mathcal{D}(f) = \Delta f^m$ with $m > 1$, or fractional diffusion modeled by the fractional laplacian $\mathcal{D}(f) = -(-\Delta)^{\frac{\alpha}{2}} f$ with $0 < \alpha < 2$; and an aggregation term of the form $\mathcal{K}[f] = K \star f$ where K is a singular kernel. For instance, some recent results on the following fractional Keller–Segel equation

$$\partial_t f = -(-\Delta)^{\frac{\alpha}{2}} f + \operatorname{div}\left(f \left(\frac{x}{|x|^\beta} \star f \right) \right)$$

were obtained in Lafleche-Salem [142] concerning the existence of global solutions and finite-time blowup. The techniques developed in this chapter could be useful to tackle the issues of uniqueness of solutions as well as their large-time behavior.

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