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Théorèmes asymptotiques pour les équations de Boltzmann et de Landau

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Kleber Carrapatoso

À Vanessa,
aos meus pais e à minha irmã

Théorèmes asymptotiques pour les équations de Boltzmann et de Landau

Résumé. Nous nous intéressons dans cette thèse à la théorie cinétique et aux systèmes de particules dans le cadre des équations de Boltzmann et Landau. Premièrement, nous étudions la dérivation des équations cinétiques comme des limites de champ moyen des systèmes de particules, en utilisant le concept de propagation du chaos. Plus précisément, nous étudions les probabilités chaotiques sur l'espace de phase de ces systèmes de particules : la sphère de Boltzmann, qui correspond à l'espace de phase d'un système de particules qui évolue conservant le moment et l'énergie ; et la sphère de Kac, correspondant à un système de particules qui conserve seulement l'énergie. Ensuite, nous nous intéressons à la propagation du chaos, avec des estimations quantitatives et uniforme en temps, pour les équations de Boltzmann et Landau. Deuxièmement, nous étudions le comportement asymptotique en temps grand des solutions de l'équation de Landau.

Mots-clés. théorie cinétique ; systèmes de particules ; équation de Landau ; équation de Boltzmann ; processus à sauts ; limite de champ moyen ; chaos ; chaos entropique ; Fisher chaos ; propagation du chaos ; entropie ; théorème central limite ; retour à l'équilibre ; convergence exponentielle ; trou spectral ; hypodissipativité ; molécules maxwelliennes ; potentiels durs ; collisions rasantes.

Asymptotic theorems for Boltzmann and Landau equations

Abstract. This thesis is concerned with kinetic theory and many-particle systems in the setting of Boltzmann and Landau equations. Firstly, we study the derivation of kinetic equation as mean field limits of many-particle systems, using the concept of propagation of chaos. More precisely, we study chaotic probabilities on the phase space of such particle systems : the Boltzmann's sphere, which corresponds to the phase space of a many-particle system undergoing a dynamics that conserves momentum and energy ; and the Kac's sphere, which corresponds to the energy conservation only. Then we are concerned with the propagation of chaos, with quantitative and uniform in time estimates, for Boltzmann and Landau equations. Secondly, we study the long-time behaviour of solutions to the Landau equation.

Keywords. kinetic theory ; many-particle system ; Landau equation ; Boltzmann equation ; jump process ; mean field limit ; chaos ; entropic chaos ; Fisher chaos ; propagation of chaos ; entropy ; central limit theorem ; relaxation to equilibrium ; exponential convergence ; spectral gap ; hypodissipativity ; maxwellian molecules ; hard potentials ; grazing collisions.

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Introduction générale

Considérons un système composé d'un grand nombre de particules (ou plus généralement d'éléments) que nous supposons toujours identiques. Alors la description d'un tel système consisterait à étudier les trajectoires de toutes ses particules au cours du temps, ce qui est inaccessible dû au grand nombre de particules en jeu. Une autre façon d'étudier ce système serait de remplacer la description microscopique par une description macroscopique, c'est-à-dire, considérer seulement les observables du système (les quantités que l'on peut effectivement mesurer).

La théorie cinétique propose un niveau de description intermédiaire, le niveau mésoscopique, dans lequel nous nous intéressons au comportement typique d'une particule plutôt qu'à l'analyse detaillée de chacune. Dans cette description statistique, l'inconnue est la densité de probabilité d'une particule dans son espace d'état. Ainsi l'objectif de la théorie cinétique est de simplifier la description d'un système de particules, en fournissant un nouveau modèle qui préserve les informations physiques intéressantes du système.

En résumant, nous avons les différents niveaux de description :

Échelle microscopique description complète du système, les inconnues sont l'état de toutes les particules (lois de Newton).

Échelle mésoscopique description statistique, l'inconnue est la fonction de distribution d'une particule (équations de Boltzmann, de Landau, de Vlasov, etc).

Échelle macroscopique description macroscopique, les inconnues sont les observables du système, tel la densité, vitesse moyene, temperature (équations de Navier-Stokes, d'Euler).

Cette thèse s'insère dans le contexte de la théorie cinétique et des systèmes de particules, dans lequel nous nous intéressons à deux types de problèmes :

- obtention des modèles cinétiques à partir de systèmes de particules, ce qui correspond au passage du niveau microscopique au niveau mésoscopique ;
- étude qualitative et quantitative des modèles cinétiques.

Plus précisément, la première partie est consacrée à la dérivation d'équations cinétiques en utilisant les concepts des limite de champ moyen et de la propagation du chaos, pour les modèles de Landau et Boltzmann. La deuxième partie est consacrée à l'étude du retour vers l'équilibre pour l'équation cinétique de Landau spatialement homogène.

1 Limite de champ moyen et propagation du chaos

Nous présentons le problème du passage du niveau microscopique au niveau mésoscopique. Nous présenterons dans un premier temps deux problèmes historiques importants et nous discuterons ensuite des concepts de limite de champ moyen et propagation du chaos.

Problèmes historiques et approche de Kac

Concernant la dérivation des modèles cinétiques, il existe deux problèmes ouverts importants et difficiles à résoudre en théorie cinétique :

— Dériver rigoureusement l'équation de Boltzmann inhomogène en espace à partir d'un système de particules évoluant selon les lois de Newton. La bonne échelle pour ce problème est connue, c'est la limite de Boltzmann-Grad (ou limite à faible densité), voir [41]. Ce problème a été résolue partiellement (pour des temps courts) par Lanford [51].

— Dériver rigoureusement l'equation de Vlasov-Poisson (particules intéragissant à travers le potentiel coulombien) à partir d'un système de particules évoluant selon les lois de Newton. La bonne échelle pour ce problème est une *limite de champ moyen*. Ce problème a été résolue dans le cas des potentiels réguliers par Braun et Hepp [10] et Dobrushin [31], pour des potentiels singuliers par Hauray et Jabin [45, 44], mais le cas du potentiel coulombien reste encore ouvert.

Dans son célèbre article sur les fondations de la théorie cinètique, Kac [49] propose un problème plus simple :

— Dériver rigoureusement l'équation de Boltzmann *homogène en espace* à partir d'un système de particules qui suit une évolution donné par un processus de Markov (plus précisément un processus à sauts). Nous avons encore ici une *limite de champ moyen*. Le probleme a été résolue par Kac [49] pour un modèle simplifié unidimensionelle (modèle de Kac). Cette aproche peut être utilisée pour d'autres équations cinétiques, comme par exemple pour l'équation de Landau homogène qui est etudié dans le Chapitre 3.

Pour étudier ce problème Kac introduit rigoureusement la notion de chaos (voir définition 1), qui apparaît déjà dans les travaux de Boltzmann sou le nom de « stosszahlansatz » (chaos moléculaire), et étudie la propagation du chaos. Ensuite il montre comment on obtient l'équation cinétique limite à partir de la propriété de propagation du chaos. En outre, comme dans ce cadre le système de particules et l'équation cinétique limite sont dissipatifs, Kac pose la question de relier leur comportement asymptotique. Ce programme est connu aujourd'hui sous le nom de « Programme de Kac ».

Le système à N particules

Essayons maintenant de formaliser le problème. On considère un espace polonais E et on note $\mathbf{P}(E)$ l'espace de mesures de probabilité sur E. On note aussi $\mathbf{P}_{\mathrm{sym}}(E^N)$

l'espace de mesures de probabilités symétriques sur E^N, plus précisément, on dit que $F^N \in \mathbf{P}(E^N)$ est symétrique si

$$\forall \varphi \in C_b(E^N), \qquad \int_{E^N} \varphi_\sigma \, dF^N = \int_{E^N} \varphi \, dF^N$$

pour toute permutation σ de $\{1, \dots, N\}$, où l'on note

$$\forall Z = (z_1, \dots, z_N) \in E^N \qquad \varphi_\sigma := \varphi(Z_\sigma) = \varphi(z_{\sigma(1)}, \dots, z_{\sigma(N)}).$$

Pour $F^N \in \mathbf{P}_{\mathrm{sym}}(E^N)$ et un entier $1 \le \ell \le N$, on note F^N_ℓ ou $\Pi_\ell(F^N) \in \mathbf{P}(E^\ell)$ la ℓ-ième marginale de F^N définie par

$$F^N_\ell := \int_{E^{N-\ell}} F^N,$$

ce qui veut dire, plus précisément,

$$\forall \varphi \in C_b(E^\ell), \qquad \int_{E^\ell} \varphi \, dF^N_\ell := \int_{E^N} \varphi \otimes \mathbf{1}^{\otimes(N-\ell)} \, dF^N.$$

Considérons un système de N particules dont chacune est décrite par sa variable d'état $z \in E$, où E représente l'espace de configurations admissibles. Typiquement, z représente la position de la particule $z = x$ et l'espace de configuration est un domaine $E = \Omega \subset \mathbb{R}^d$; ou z représente la vitesse de la particule $z = v$ et $E = \mathbb{R}^d$; ou encore z représente le couple position et vitesse $z = (x, v)$ et $E = \Omega \times \mathbb{R}^d$. Un système de N particules est donc décrit par la variable

$$Z^N = (z_1, z_2, ..., z_N) \in E^N.$$

Supposons de plus que les particules sont indistinguables, ainsi le système est invariant par permutation des indices et donc

$$Z^N = (z_1, z_2, ..., z_N) \in E^N / \mathfrak{S}_N,$$

où $\mathfrak{S}_N$ est l'espace de permutations de $\{1, \dots, N\}$.

Considérons pour la suite un temps d'observation $T > 0$ et supposons que le système de particules soit décrit par une équation d'évolution : pour tout $t \in [0, T]$, nous avons

$$Z^N(t) : \text{ solution d'une équation d'évolution sur } E^N / \mathfrak{S}_N. \tag{1}$$

Dans le cas d'un système stochastique, nous pouvons aussi le décrire en utilisant la densité de probabilité $F^N(t) \in \mathbf{P}_{\mathrm{sym}}(E^N)$ correspondant à la loi de la variable aléatoire $Z^N(t) \in E^N / \mathfrak{S}_N$, ainsi pour tout $t \in [0, T]$,

$$F^N(t) : \text{ solution d'une équation d'évolution sur } \mathbf{P}_{\mathrm{sym}}(E^N). \tag{2}$$

Le modèle cinétique limite

Nous souhaitons établir une description statistique du système dans la limite où le nombre de particules tend vers l'infini. Pour que cette question ait un sens, il faut que le problème soit observé à la bonne échelle et aussi identifier un objet limite. Plus précisément, nous allons nous intéresser à des « limites de champ moyen » qui correspondent au cas où l'action de chaque particule du système est de l'ordre $O(1/N)$, mais que l'action moyenne est de l'ordre $O(1)$. Les objets qui décrivent le système limite sont soit une densité de probabilité $f \in \mathbf{P}(E)$ décrivant la distribution de particules, soit un processus stochastique $\mathcal{Z} \in E$ (de loi f) décrivant le comportement d'une particule typique.

Pour tout $Z^N = (z_1, \dots, z_N) \in E^N$ on peut définir la mesure empirique par

$$\begin{cases} \mu^N : E^N \to \mathbf{P}(E) \\ \qquad Z^N \mapsto \mu^N_{Z^N} := \dfrac{1}{N} \displaystyle\sum_{i=1}^N \delta_{z_i}. \end{cases} \tag{3}$$

Pour obtenir la convergence du système de N particules vers le modèle cinétique limite associé, nous pouvons montrer que, pout tout $t \in [0, T]$, il existe $f(t) \in \mathbf{P}(E)$ tel que

$$\mu^N_{Z^N(t)} \rightharpoonup f(t) \quad \text{faiblement dans } \mathbf{P}(E) \text{ lorsque } N \to \infty, \tag{4}$$

ou encore,

$$F^N_1(t) \rightharpoonup f(t) \quad \text{faiblement dans } \mathbf{P}(E) \text{ lorsque } N \to \infty. \tag{5}$$

Les convergences (4) et (5) montrent que la distribution moyenne des particules converge vers une distribution typique. En revanche, (4) ne marche que dans le cas déterministe et avec seulement l'information (5) on peut ne pas être capable d'identifier la limite.

Une autre façon de procéder est d'utiliser l'approche proposé par Kac [49], en utilisant la notion de chaos et propagation du chaos. L'idée de cette notion est : une variable aléatoire Z^N est chaotique si elle est une variable aléatoire de coordonnées indépendantes dans l'asymptotique $N \to \infty$.

Définition 1 (Chaos)**.** Soient $f \in \mathbf{P}(E)$ et une suite des mesures de probabilités $(F^N)_{N \in \mathbb{N}^*} \in \mathbf{P}_{\text{sym}}(E^N)$. On dit que F^N est f-chaotique (ou f-Kac chaotique) si pour tout entier positive ℓ fixé, on a

$$F^N_\ell \rightharpoonup f^{\otimes \ell} \quad \text{faiblement dans } \mathbf{P}(E^\ell) \text{ lorsque } N \to \infty, \tag{6}$$

où F^N_ℓ est la ℓ-ième marginale de F^N. On rappelle que la convergence faible dans $\mathbf{P}(E^\ell)$ est donnée par la convergence des intégrales contre des fonctions continues bornées $\varphi \in C_b(E^\ell)$.

On remarque que la propiété de chaos (6) implique les convergences (4) et (5). De plus, la convergence (6) pour tout $\ell \in \mathbb{N}^*$ est équivalente à (6) pour $\ell = 2$, le point clé étant l'hypothèse de symétrie (voir [70]).

Avec cette notion, nous pouvons obtenir l'équation cinétique limite associée au système de N particules en démontrant la propagation du chaos, ce qui consiste à prouver

$$F^N(0) \text{ est } f(0)\text{-chaotique} \quad \text{implique} \quad F^N(t) \text{ est } f(t)\text{-chaotique.}$$

Proposons une autre formulation du chaos. Si $F^N \in \mathbf{P}_{\mathrm{sym}}(E^N)$ est la loi de $Z^N \in E^N/\mathfrak{S}_N$, alors la loi de $\mu^N_{Z^N} \in \mathbf{P}(E)$ est $\hat{F}^N \in \mathbf{P}(\mathbf{P}(E))$ donnée par

$$\forall\, \Phi \in C_b(\mathbf{P}(E)) \qquad \int_{\mathbf{P}(E)} \Phi(\rho)\, \hat{F}^N(d\rho) = \int_{E^N} \Phi(\mu^N_{Z^N})\, F^N(dZ^N). \tag{7}$$

Pour tout $f \in \mathbf{P}(E)$ on définit la mesure de probabilité $\delta_f \in \mathbf{P}(\mathbf{P}(E))$ par

$$\forall\, \Phi \in C_b(\mathbf{P}(E)) \qquad \int_{\mathbf{P}(E)} \Phi(\rho)\, \delta_f(d\rho) = \Phi(f). \tag{8}$$

Le chaos (6) est équivalent à

$$\hat{F}^N \rightharpoonup \delta_f \quad \text{faiblement dans } \mathbf{P}(\mathbf{P}(E)) \text{ lorsque } N \to \infty. \tag{9}$$

Nous pouvons aussi nous intéresser à des versions quantitatives du chaos, plus précisément obtenir des estimations du type

$$\left| \left\langle F^N_\ell - f^{\otimes \ell}, \varphi \right\rangle \right| \le C(\ell)\, \varepsilon(N),$$

pour tout $\varphi \in C_b(E^\ell)$ avec $\|\varphi\| \le 1$, pour une constante $C(\ell) > 0$ dépendant possiblement de ℓ et une fonction constructive $\varepsilon(N) \to 0$ quand $N \to \infty$. Bien sûr, une autre possibilité serait de remplacer le membre de gauche de la dernière equation par une distance sur l'espace des probabilités, comme par exemple les distances de Wasserstein.

Dans cette thèse nous nous intéressons aux limites de champ moyen et à la propagation du chaos avec des applications aux modèles de Landau et Boltzmann, présentés ci-dessous. Premièrement, nous étudions les familles de mesures de probabilités chaotiques sur la sphère de Boltzmann, qui est l'espace de phase associé aux systèmes de N particules des modèles de Landau et Boltzmann (plus généralement, c'est l'espace de phase associé à un système de N particules conservant la quantité de mouvement et l'énergie). On applique ensuite ces résultats au modèle de Boltzmann pour démontrer la propagation du chaos entropique. Deuxièmement nous étudions les propriétés de chaos et chaos entropique dans la sphère de Kac, l'espace de phase associé au modèle de Kac (simplification du modèle de Boltzmann) qui conserve seulement l'énergie. Finalement, nous étudions la propagation du chaos pour le modèle de Landau.

2 Retour vers l'équilibre

Une caractéristique physique très importante prédite par Boltzmann, lorsqu'il introduit son équation, est le phénomène de retour vers l'état d'équilibre, qui a pour motivation de donner une base mathématique au second principe de la thermodynamique. Ce problème a été résolu par le celèbre théorème-H de Boltzmann. Nous nous intéressons bien sûr à ce phénomène pour d'autres équations cinétiques.

Le théorème-H comporte deux parties. D'une part, l'existence d'une fonctionnelle de Lyapunov, qui dans le cas de l'équation de Boltzmann est l'entropie ou fonctionelle H (opposé de « l'entropie physique ») définie par, pour une densité de probabilité f,

$$H(f) := \int f \log f.$$

D'autre part, les extrema de cette fonctionnelle sont atteints par les distributions maxwelliennes (gaussiennes). On conjecture ainsi que la solution converge vers l'équilibre quand $t \to \infty$.

L'étude du retour à l'équilibre consiste donc à prouver cette convergence (dans un certain sens à préciser). On s'intéresse aussi à quantifier la vitesse de convergence.

Usuellement, on peut utiliser différentes méthodes :

— On peut prouver qu'il y a vraiment convergence vers l'équilibre à partir des arguments de compacité. Ces arguments sont non constructives et ne donnent donc aucune information sur la vitesse de convergence.

— On peut utiliser des techniques de linéarisation. On étudie l'opérateur linéarisé (autour de l'équilibre) ce qui peut nous permettre d'obtenir une vitesse de convergence exponentielle $e^{-\lambda_0 t}$, où λ_0 est le trou spectral de l'opérateur linéarisé. En revanche, cette méthode ne donne une estimation de la vitesse de convergence que dans un certain voisinage de l'équilibre, quand les termes linéaires dominent les termes non linéaires. Cela nous pose donc le problème d'estimer le temps pour que la solution entre dans ce voisinage.

— On peut utiliser des méthodes d'entropie. Cela consiste à obtenir des inégalités fonctionnelles entre la dissipation d'entropie et l'entropie relative à l'équilibre ce qui implique le retour vers l'équilibre (mésuré par l'entropie relative) et donne une vitesse de convergence qui dépend de l'inégalité obtenue auparavant. Par exemple, si cette inégalité est linéaire (Conjecture de Cercignani) on obtient une vitesse exponentielle (voir [27] pour une revue sur cette conjecture).

Dans cette thèse nous nous intéressons à l'étude du retour vers l'équilibre quantifié pour l'équation de Landau spatialement homogène. Nous obtenons une vitesse de convergence exponentielle en utilisant la deuxième approche décrite ci-dessus initié par [65] : nous obtenons d'abord le trou spectral dans divers espaces de Banach à poids (par une méthode de [42]), ensuite on couple ce résultat avec des estimations du temps nécéssaire pour que la solution entre dans le bon voisinage (en utilisant des résultats quantifiés sur le retour à l'équilibre de [29]).

3 Modèles cinétiques et systèmes de particules

Nous présentons les sytèmes de particules et leurs modèles cinétiques associés étudiés dans cette thèse.

Équation de Boltzmann

L'équation de Boltzmann est une équation cinétique modélisant un gaz raréfié hors équilibre, dérivée par Boltzmann [9] et Maxwell [56]. L'inconnue est une fonction de distribution positive $f = f(t,x,v) \geq 0$ qui représente la densité de particules à l'instant $t \in \mathbb{R}_+$, à la position $x \in \Omega \subset \mathbb{R}^d$, avec vitesse $v \in \mathbb{R}^d$. L'équation est donnée par (c.f. [22, 76])

$$\partial_t f + v \cdot \nabla_x f = Q_B(f,f).$$

L'opérateur de transport $v \cdot \nabla_v$ décrit la trajectoire libre en ligne droite des particules qui n'intéragissent pas. De plus, l'opérateur de collision de Boltzmann Q_B est un opérateur bilinéaire agissant seulement sur la variable de vitesse v, ce qui représente l'hypothèse de collisions localisés en espace, et s'écrit

$$Q_B(g,f) = \int_{\mathbb{R}^d \times \mathbb{S}^{d-1}} B(v - v_*, \sigma)\Big(g'_* f' - g_* f\Big)\, dv_*\, d\sigma, \tag{10}$$

où nous utilisons dorenavant la notation $g'_* = g(v'_*)$, $g_* = g(v_*)$, $f' = f(v')$ et $f = f(v)$. Les vitesses avant (v, v_*) et après (v', v'_*) collision vérifient

$$\begin{cases} v' = \dfrac{v + v_*}{2} + \dfrac{|v - v_*|}{2}\sigma \\ v'_* = \dfrac{v + v_*}{2} - \dfrac{|v - v_*|}{2}\sigma \end{cases} \tag{11}$$

ce qui est une paramétrisation possible d'une collision élastique

$$\begin{cases} v' + v'_* = v + v_* \\ |v'|^2 + |v'_*|^2 = |v|^2 + |v_*|^2. \end{cases}$$

Nous considérons par la suite le cas où la densité des particules ne dépend pas de la position $f = f(t,v)$, obtenant ainsi *l'équation de Boltzmann homogène en espace*

$$\begin{cases} \partial_t f = Q_B(f,f) \\ f_{|t=0} = f_0. \end{cases} \tag{12}$$

Les caractéristiques de l'intéraction entre particules sont prises en compte dans le noyau de collision positif $B(v - v_*, \sigma) \geq 0$. Grâce à des considérations physiques, nous supposons que le noyau dépend seulement de la vitesse relative $|v - v_*|$ et $\cos\theta = \sigma \cdot (v - v_*)/|v - v_*|$ (nous renvoyons à [76] pour plus de détails). Nous considérons des noyaux de collision de la forme

$$B(|v - v_*|, \cos\theta) = \Gamma(|v - v_*|) b(\cos\theta), \tag{13}$$

où Γ et b sont des fonctions positives. Nous présentons ici deux types de noyau :

— Les intéractions à courte portée sont modélisées par un noyau de collision de sphères dures
$$B(|v-v_*|,\cos\theta) = C|v-v_*|, \quad C>0. \tag{14}$$

— Les intéractions à longue portée sont modélisées par des noyaux de collision dérivés des potentiels d'interactions
$$U(r) = Cr^{-s}, \quad s>-2,$$
où r représente la distance entre particules. Ils vérifient
$$\Gamma(|v-v_*|) = |v-v_*|^{\gamma}, \quad \gamma = \frac{s-2d+2}{s} \tag{15}$$
et
$$\sin^{d-2}\theta\, b(\cos\theta) \underset{\sim 0}{\sim} C_b\,\theta^{-1-\nu}, \quad C_b>0,\ \nu\in(0,2). \tag{16}$$
Notons avec ces hypothèses la singularité de b proche de 0, ce qui implique que la fonction b n'est pas intégrable. Une hypothèse simplificatrice, connue sous le nom de troncature angulaire de Grad, est de supposer b intégrable.
Usuellement nous appelons potentiels durs pour $\gamma>0$, molécules maxwelliennes quand $\gamma=0$ et potentiels mous lorsque $\gamma<0$. Pour chaque cas nous parlons aussi de noyau de collision sans troncature pour b satisfaisant (16) ou avec troncature de Grad pour $b\in L^1$.

En considérant une fonctions test $\varphi=\varphi(v)$, on obtient les formulations faibles
$$\begin{aligned}\int_{\mathbb{R}^d} Q_B(f,f)\varphi\,dv \\ = \frac{1}{2}\int_{\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{S}^{d-1}} \Gamma(|v-v_*|)\, b(\cos\theta)(\varphi'_*+\varphi'-\varphi_*-\varphi)\, f_*f\,d\sigma\,dv_*\,dv\end{aligned} \tag{17}$$
ou encore
$$\begin{aligned}\int_{\mathbb{R}^d} Q_B(f,f)\varphi\,dv \\ = -\frac{1}{4}\int_{\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{S}^{d-1}} \Gamma(|v-v_*|)\, b(\cos\theta)(\varphi'_*+\varphi'-\varphi_*-\varphi)\,(f'_*f'-f_*f)\,d\sigma\,dv_*\,dv.\end{aligned} \tag{18}$$
On déduit de (17), formellement, que l'équation de Boltzmann homogène (12) conserve la masse, la quantité de mouvement et l'énergie, en particulier on a
$$\frac{d}{dt}\int_{\mathbb{R}^d} f\varphi = \int_{\mathbb{R}^d} Q_B(f,f)\varphi = 0 \qquad \text{pour} \qquad \varphi(v) = 1, v, |v|^2, \tag{19}$$
et les fonctions $\varphi(v)=1,v,|v|^2$ sont appelées invariants de collision. D'autre part, on obtient que l'entropie définie par
$$H(f) := \int_{\mathbb{R}^d} f\log f$$

est décroissante. En effet, formellement, en prenant $\varphi = \log f$ dans (18), la dissipation de l'entropie $D_B(f) := -\frac{d}{dt}H(f)$ vérifie

$$D_B(f) = \frac{1}{4}\int_{\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{S}^{d-1}} \Gamma(|v-v_*|)\, b(\cos\theta)\,(f'_*f' - f_*f)\log\frac{f'_*f'}{f_*f}\,d\sigma\,dv_*\,dv \geq 0, \quad (20)$$

Si une distribution f vérifie $D_B(f) = 0$, alors $\log f$ est une combinaison linéaire des invariants de collision, ce qui implique que tout équilibre est une distribution maxwellienne

$$\mu_{\rho_f,u_f,T_f}(v) := \frac{\rho}{(2\pi T)^{d/2}}e^{-\frac{|v-u|^2}{2T}},$$

où $\rho_f > 0$ est la densité, $u_f \in \mathbb{R}^d$ est la vitesse moyenne et $T_f > 0$ la température associées à f, définies par

$$\rho_f = \int_{\mathbb{R}^d} f, \qquad u_f = \frac{1}{\rho_f}\int_{\mathbb{R}^d} vf, \qquad T_f = \frac{1}{d\rho_f}\int_{\mathbb{R}^d}|v-u|^2 f,$$

Ceci est le celèbre théorème-H de Boltzmann. On espère donc qu'une solution $f(t,\cdot)$ de l'équation de Boltzmann converge vers l'équilibre maxwellien μ_{ρ_f,u_f,T_f} associé lorsque $t\to+\infty$.

Équation de Landau

L'équation de Landau est un modèle cinétique utilisé pour la modélisation de plasmas. Elle décrit l'évolution de la densité de particules $f = f(t,x,v)$ dans l'espace de phase des positions et vitesses. Ici la variable $t \in \mathbb{R}_+$ représente le temps, $x \in \Omega \subset \mathbb{R}^d$ la position et $v \in \mathbb{R}^d$ la vitesse. L'équation est donnée par (c.f. [76])

$$\partial_t f + v\cdot\nabla_x f = Q_L(f,f).$$

Comme dit avant, l'opérateur de transport $v\cdot\nabla_x$ décrit la trajectoire libre des particules qui ne collisionent pas. L'opérateur de collision de Landau Q_L est bilinéaire et n'agit que sur la variable de vitesse v (comme pour l'opérateur de Boltzmann décrit ci-dessus) et est donné par

$$Q_L(g,f) = \partial_i\int_{\mathbb{R}^d} a_{ij}(v-v_*)\,(g_*\partial_j f - \partial_j g_* f)\,dv_*, \quad (21)$$

où par la suite nous utilisons la convention de sommation des indices et les notations $g_* = g(v_*)$, $\partial_j g_* = \partial_{v_{*j}} g(v_*)$, $\partial_j f = \partial_{v_j} f(v)$ et $f = f(v)$.

Dans cette thèse, nous considérons seulement le cas où la densité de particules vérifie $f = f(t,v)$, c'est-à-dire est indépendante de la position x, ce qui nous donne *l'équation de Landau homogène en espace*

$$\begin{cases} \partial_t f = Q_L(f,f) \\ f_{|t=0} = f_0. \end{cases} \quad (22)$$

La matrice a est positive, symétrique et dépend de l'intéraction entre les particules. Si les particules intéragissent selon un potentiel (voir [76] pour plus d'informations)

$$U(r) = Cr^{-s}, \quad C > 0,\ s \geq 2,$$

alors nous avons

$$a_{ij}(z) = \Lambda |z|^{\gamma+2} \Pi_{ij}(z), \qquad \Pi_{ij}(z) = \delta_{ij} - \frac{z_i z_j}{|z|^2}, \tag{23}$$

avec $\gamma = (s - 2d + 2)/s$ et une constante $\Lambda > 0$. Par analogie avec l'équation de Boltzmann, nous appelons potentiels durs le cas $\gamma > 0$, molécules maxwelliennes quand $\gamma = 0$, potentiels mous pour $-d < \gamma < 0$ et potentiel coulombien lorsque $\gamma = -d$.

L'équation de Landau a été derivée par Landau (voir [52, 23]) par une approche phénoménologique. Plus tard, l'opérateur de Landau a été justifié comme un cas limite de l'opérateur de Boltzmann, voir [26, 1, 25, 73], dans la limite de collisions rasantes (voir ci-dessous).

Limite de collisions rasantes. Nous appelons collision rasante une collision dans laquelle l'angle de déviation est proche de zéro. La limite de collisions rasantes consiste à rendre toutes les collision rasantes. Plus précisément, en considérant une suite de noyaux de collision B_ε avec $\varepsilon \to 0$, on dit que la partie angulaire b_ε (13) se concentre sur les collisions rasantes si :

$$\begin{cases} \forall\, \theta_0 > 0, \qquad \sup_{\theta \in [\theta_0, \pi]} b_\varepsilon(\cos\theta) \to 0 \qquad \text{quand } \varepsilon \to 0, \\ \\ \Lambda_\varepsilon = \displaystyle\int_{\mathbb{S}^{d-1}} b_\varepsilon(\cos\theta)(1 - \cos\theta)\, d\sigma \to \Lambda > 0 \qquad \text{quand } \varepsilon \to 0. \end{cases} \tag{24}$$

Dans cette limite l'opérateur de Boltzmann associé à B_ε tend vers l'opérateur de Landau.

En définissant les nouvelles quantités

$$b_i(z) = \partial_j a_{ij} = -\Lambda(d-1) z_i |z|^\gamma, \qquad c(z) = \partial_{ij} a_{ij} = -\Lambda(d-1)(\gamma + d)|z|^\gamma, \tag{25}$$

nous pouvons réécrire l'opérateur de Landau (21) de la façon suivante

$$Q_L(g, f) = (a_{ij} * g)\partial_{ij} f - (c * g) f. \tag{26}$$

On considère maintenant une fonction test $\varphi = \varphi(v)$ et en intégrant cette fonction contre l'opérateur de Landau, on obtient les formulations faibles

$$\int_{\mathbb{R}^d} Q_L(f, f)\varphi\, dv = -\frac{1}{2} \iint_{\mathbb{R}^d \times \mathbb{R}^d} a_{ij}(v - v_*) \left(\frac{\partial_i f}{f} - \frac{\partial_i f_*}{f_*} \right) (\partial_j \varphi - \partial_j \varphi_*)\, f f_*\, dv_*\, dv \tag{27}$$

ou encore

$$\begin{aligned} \int_{\mathbb{R}^d} Q_L(f, f)\varphi\, dv = \frac{1}{2} &\iint_{\mathbb{R}^d \times \mathbb{R}^d} a_{ij}(v - v_*)(\partial_{ij}\varphi + \partial_{ij}\varphi_*)\, f f_*\, dv_*\, dv \\ &+ \iint_{\mathbb{R}^d \times \mathbb{R}^d} b_i(v - v_*)(\partial_i \varphi - \partial_i \varphi_*)\, f f_*\, dv_*\, dv. \end{aligned} \tag{28}$$

On voit aisément que l'équation de Landau homogène (22) conserve, au moins formellement, la masse, la quantité de mouvement et l'énergie. En effet, de (27)-(28) et de $a_{ij}(z)z_i = 0$ grâce à (23), on déduit

$$\frac{d}{dt}\int_{\mathbb{R}^d} f\varphi = \int_{\mathbb{R}^d} Q_L(f,f)\varphi = 0 \qquad \text{pour} \qquad \varphi(v) = 1, v, |v|^2. \tag{29}$$

En outre, on obtient que l'entropie définie par

$$H(f) := \int_{\mathbb{R}^d} f \log f$$

est décroissante. En effet, formellement, en prenant $\varphi = \log f$ dans (27) on a que la dissipation de l'entropie $D(f) := -\frac{d}{dt}H(f)$ vérifie

$$D_L(f) = \frac{1}{2}\iint_{\mathbb{R}^3\times\mathbb{R}^3} a_{ij}(v-v_*)\left(\frac{\partial_i f}{f} - \frac{\partial_{i*} f_*}{f_*}\right)\left(\frac{\partial_j f}{f} - \frac{\partial_{j*} f_*}{f_*}\right) f f_*\, dv_*\, dv \geq 0, \tag{30}$$

car la matrice a est positive. Il s'en suit que tout équilibre est une distribution maxwellienne

$$\mu_{\rho,u,T}(v) := \frac{\rho}{(2\pi T)^{d/2}} e^{-\frac{|v-u|^2}{2T}},$$

pour $\rho > 0$, $u \in \mathbb{R}^d$ et $T > 0$. Ceci est la version Landau du celèbre théorème-H de Boltzmann présenté ci-dessus (pour plus d'informations voir [29, 74]). Ainsi on s'attend à ce qu'une solution $f(t,\cdot)$ de l'équation de Landau converge vers l'équilibre maxwellien μ_{ρ_f,u_f,T_f} lorsque $t \to +\infty$, où ρ_f représente la densité, u_f la vitesse moyenne et T_f la température associées à f, données par

$$\rho_f = \int_{\mathbb{R}^d} f, \qquad u_f = \frac{1}{\rho_f}\int_{\mathbb{R}^d} vf, \qquad T_f = \frac{1}{d\rho_f}\int_{\mathbb{R}^d} |v-u|^2 f,$$

et ces quantités sont définies par la donnée initiale f_0 grâce aux propriétés conservées (29).

Modèle de Kac-Boltzmann : systèmes de particules

On présente un modèle de système de particules décrit par un processus de Markov et pour lequel la limite de champ moyen attendue est l'équation de Boltzmann spatialement homogène.

Considérons un système de N particules en dimension $d \geq 2$ décrit par les vitesses $V = (v_1, \dots, v_N) \in \mathbb{R}^{dN}$ et un noyau de collision $B(|z|, \cos\theta) = \Gamma(|z|)b(\cos\theta)$ comme défini dans (13). Alors le processus est donné par [57, 49, 62, 12, 63] :

— pour tout $i' \neq j'$, on choisit un temps aléatoire de collision $T(\Gamma(|v_{i'} - v_{j'}|))$ grâce à une loi exponentielle de paramètre $\Gamma(|v_{i'} - v_{j'}|)$; ensuite on prend le temps minimal T_1 et le couple (v_i, v_j) qui subira une collision tel que

$$T_1 = T(\Gamma(|v_i - v_j|)) = \min_{i',j'} T(\Gamma(|v_{i'} - v_{j'}|)),$$

— on choisit $\sigma \in \mathbb{S}^{d-1} \subset \mathbb{R}^d$ avec la loi $b(\cos\theta_{ij})$, où

$$\cos\theta_{ij} = \sigma \cdot \frac{(v_i - v_j)}{|v_i - v_j|},$$

— après la collision, le nouvel état du système est

$$V'_{ij} = (v_1, \ldots, v'_i, \ldots, v'_j, \ldots, v_N)$$

où les vitesses post-colisionnelles v'_i et v'_j sont données par

$$v'_i = \frac{v_i + v_j}{2} + \frac{|v_i - v_j|}{2}\sigma, \qquad v'_j = \frac{v_i + v_j}{2} - \frac{|v_i - v_j|}{2}\sigma. \tag{31}$$

On obtient par itération le processus de Markov associé $(\mathcal{V}_t^N)_{t\geq 0}$ sur $\mathbb{R}^{dN}$. Par contre, en considérant les quantités conservées dans les collisions, ce processus admet des sous-varietés de $\mathbb{R}^{dN}$ qui sont invariantes, et donc le processus peut être restreint à ces sous-varietés (voir ci-dessous pour plus de détails).

On effectue un changement d'échelle $t \to t/N$ pour que le nombre d'intéractions dans un intervalle de temps fini soit d'ordre $O(1)$ et on note f_t^N la loi de $\mathcal{V}_t^N$. L'équation d'évolution pour f_t^N est appelée *l'équation maîtresse de Kac-Boltzmann* et donnée par, dans sa formulation duale,

$$\partial_t \left\langle f_t^N, \varphi \right\rangle = \left\langle f_t^N, \Lambda_B^N \varphi \right\rangle \tag{32}$$

pour toute fonction test $\varphi \in C_b(\mathbb{R}^{dN})$, où pour tout $V = (v_1, \ldots, v_N) \in \mathbb{R}^{dN}$ on a

$$\Lambda_B^N \varphi(V) = \frac{1}{2N} \sum_{i,j=1}^{N} \Gamma(|v_i - v_j|) \int_{\mathbb{S}^{d-1}} b(\cos\theta_{ij}) \left(\varphi'_{ij} - \varphi\right) d\sigma \tag{33}$$

avec la notation $\varphi'_{ij} = \varphi(V'_{ij})$ et $\varphi = \varphi(V) \in C_b(\mathbb{R}^{dN})$.

Ce processus de collision est invariant par permutation de vitesses et satisfait la conservation microscopique de la quantité de mouvement et de l'énergie

$$v'_i + v'_j = v_i + v_j, \qquad |v'_i|^2 + |v'_j|^2 = |v_i|^2 + |v_j|^2.$$

Par conséquent, si la loi initiale est une probabilité symétrique $f_0^N \in \mathbf{P}_{\mathrm{sym}}(\mathbb{R}^{dN})$ la solution de l'équation maîtresse f_t^N l'est aussi pour tout temps. De plus, l'évolution conserve la quantité de mouvement

$$\int_{\mathbb{R}^{dN}} \varphi\left(\sum_{k=1}^{N} v_k\right) df_t^N = \int_{\mathbb{R}^{dN}} \varphi\left(\sum_{k=1}^{N} v_k\right) df_0^N$$

et l'énergie

$$\int_{\mathbb{R}^{dN}} \varphi\left(\sum_{k=1}^{N} |v_k|^2\right) df_t^N = \int_{\mathbb{R}^{dN}} \varphi\left(\sum_{k=1}^{N} |v_k|^2\right) df_0^N.$$

Grâce aux quantités conservées lors d'une collision élastique (quantité de mouvement et énergie), le processus peut être restreint à la sous-varieté suivante de $\mathbb{R}^{dN}$

$$\mathcal{S}^N(\mathcal{M},\mathcal{E}) := \left\{ V = (v_1, \dots, v_N) \in \mathbb{R}^{dN} \;\middle|\; \frac{1}{N}\sum_{k=1}^N v_k = \mathcal{M},\ \frac{1}{N}\sum_{k=1}^N |v_k - \mathcal{M}|^2 = \mathcal{E} \right\} \quad (34)$$

pour $\mathcal{M} \in \mathbb{R}^d$ et $\mathcal{E} \in \mathbb{R}_+$. Sans perte de généralité, nous considérons le cas $\mathcal{M} = 0$, $\mathcal{E} = d$ et nous notons $\mathcal{S}^N_{\mathcal{B}} := \mathcal{S}^N(0, d)$. Ces espaces sont appelés *sphères de Boltzmann.*

Remarque 2. Le système de particules introduit par Kac [49], appelé modèle de Kac, est une simplification unidimensionelle du processus décrit ci-dessus. Dans ce cas les vitesses scalaires changent de la façon suivante après une collision

$$\begin{aligned} v_i' &= v_i \cos\theta + v_j \sin\theta, \\ v_j' &= -v_i \sin\theta + v_j \cos\theta, \end{aligned} \quad (35)$$

et nous avons seulement la conservation de l'énergie $|v_i'|^2 + |v_j'|^2 = |v_i|^2 + |v_j|^2$. La sous-variété invariante associée à ce processus est donc la sphère $\mathcal{S}^N_{\mathcal{K}} := \mathbb{S}^{N-1}(\sqrt{N})$ (en supposant encore sans perte de généralité $\mathcal{E} = d = 1$), aussi appelé *sphère de Kac.*

La différence entre considérer le processus sur $\mathbb{R}^{dN}$ ou sur le sous-espace invariant $\mathcal{S}^N_{\mathcal{B}}$ réside dans l'étude du comportement asymptotique des solutions en temps grand. En effet, l'évolution du processus est decouplée dans les differents sous espaces $\mathcal{S}^N(\mathcal{M},\mathcal{E})$ pour des valeurs différentes de $\mathcal{M}$ et $\mathcal{E}$. Dans chacun de ces sous-espaces, le processus de Markov de N particules admet une mesure invariante γ^N unique et constante. Par contre, lorsque le processus est considéré dans $\mathbb{R}^{dN}$ on a un nombre infini de mesures invariantes.

Modèle de Kac-Landau : systèmes de particules

Maintenant, on présente un modèle de système de particules dont l'état est décrit par ses vitesses, et pour lequel la limite de champ moyen attendue est l'équation de Landau spatialement homogène.

Pour tout $i = 1, \dots, N$, on considère les variables aléatoires $(X_t^i)_{t\geq 0}$ à valeurs dans $\mathbb{R}^d$ vérifiant l'équation suivante

$$dX_t^i = \frac{\sqrt{2}}{\sqrt{N}} \sum_{\substack{k=1\\k\neq i}}^N \sigma(X_t^i - X_t^k)\, dZ_t^{i,k} + \frac{2}{N}\sum_{\substack{k=1\\k\neq i}}^N b(X_t^i - X_t^k)\, dt \quad (36)$$

où, pour tout $1 \leq i \leq N$ et $i < k$, $Z_t^{i,k} = B_t^{i,k}$ sont $N(N-1)/2$ mouvements browniens indépendants à valeurs dans $\mathbb{R}^d$, et les autres termes sont anti-symétrique, i.e. $Z_t^{k,i} = -B_t^{i,k}$. La fonction symétrique σ est donnée par $a(z) = \sigma(z)\sigma^*(z)$ où la matrice a est définie dans (23) et b dans (25).

En considérant f_t^N la loi du processus $\mathcal{X}_t^N = (X_t^1, \dots, X_t^N)$ définit ci-dessus, on obtient *l'équation maîtresse de Kac-Landau* donnée par, dans sa formulation duale,

$$\partial_t \left\langle f_t^N, \varphi \right\rangle = \left\langle f_t^N, \Lambda_L^N \varphi \right\rangle \tag{37}$$

pour toute fonction test $\varphi = \varphi(V) \in C_b^2(\mathbb{R}^{dN})$, où pour tout $V = (v_1, \dots, v_N) \in \mathbb{R}^{dN}$ avec $v_k = (v_{k,\alpha})_{1 \le \alpha \le d} \in \mathbb{R}^d$, on a

$$\begin{aligned} \Lambda_L^N \varphi(V) = {} & \frac{1}{N} \sum_{i,j=1}^N b(v_i - v_j) \cdot (\nabla_i \varphi - \nabla_j \varphi) \\ & + \frac{1}{2N} \sum_{i,j=1}^N a(v_i - v_j) : (\nabla_{ii}^2 \varphi + \nabla_{jj}^2 \varphi - \nabla_{ij}^2 \varphi - \nabla_{ji}^2 \varphi). \end{aligned} \tag{38}$$

Ici on utilise la notation, pour deux matrices de dimension $d \times d$,

$$A : B = \sum_{\alpha,\beta=1}^d A_{\alpha\beta} B_{\alpha\beta},$$

et

$$\nabla_i \varphi = \left(\partial_{v_{i,\alpha}} \varphi(V)\right)_{1 \le \alpha \le d}, \qquad \nabla_{ij} \varphi = \left(\partial_{v_{i,\alpha}} \partial_{v_{j,\beta}} \varphi(V)\right)_{1 \le \alpha,\beta \le d}.$$

Ce processus est aussi invariant par permutation des vitesses, conserve la quantité de mouvement et l'énergie

$$\begin{aligned} \int_{\mathbb{R}^{dN}} \varphi\left(\sum_{k=1}^N v_k\right) df_t^N &= \int_{\mathbb{R}^{dN}} \varphi\left(\sum_{k=1}^N v_k\right) df_0^N, \\ \int_{\mathbb{R}^{dN}} \varphi\left(\sum_{k=1}^N |v_k|^2\right) df_t^N &= \int_{\mathbb{R}^{dN}} \varphi\left(\sum_{k=1}^N |v_k|^2\right) df_0^N. \end{aligned}$$

On en déduit que, comme dans le cas du modèle de Kac-Boltzmann, ce processus peut être considéré dans $\mathbb{R}^{dN}$ ou dans la sphère de Boltzmann $\mathcal{S}_{\mathcal{B}}^N$.

L'équation maîtresse (37) est établie dans le chapitre 3, en appliquant la limite des collisions rasantes à l'équation maîtresse Kac-Boltzmann (32). Cette équation a été aussi introduite par Balescu et Prigorine dans les années 1950 (voir [50]), et est aussi étudiée dans [50, 58].

4 Chapitre 1 : Chaos quantitative et qualitative sur la sphère de Boltzmann

Dans ce chapitre nous nous intéressons aux suites de probabilités chaotiques sur la sphère de Boltzmann $\mathcal{S}_{\mathcal{B}}^N$ (34). Nous rappelons que cet espace est l'espace de phase d'un système de N particules décrit par ses vitesses et qui suit un processus conservant la quantité de mouvement et l'énergie.

Une première question est savoir s'il existe des suites chaotiques sur la sphère de Boltzmann et comment les caractériser. Dans le contexte du modèle de Kac (simplification en dimension $d = 1$ du modèle de Kac-Boltzmann, voir remarque 2) et de la sphère de Kac $\mathcal{S}^N_{\mathcal{K}} = \mathbb{S}^{N-1}(\sqrt{N})$, Kac [49] a l'idée de construire une suite des probabilités chaotiques à partir de la tensorisation d'une probabilité $f \in \mathbf{P}(\mathbb{R})$, centrée et de variance unitaire, qui est ensuite restreinte à $\mathcal{S}^N_{\mathcal{K}}$. Cette idée vient du fait que, si l'on considère la mesure produite $f^{\otimes N}$, alors par la loi de grands nombres $\sum_{k=1}^N v_k^2 \approx N$ presque sûrement par rapport à $f^{\otimes N}$, donc cette mesure est concentrée sur $\mathcal{S}^N_{\mathcal{K}}$ et la restriction à $\mathcal{S}^N_{\mathcal{K}}$ ne devrait pas changer beaucoup. Cela a été demontré par Kac [49] pour des probabilités $f \in \mathbf{P}(\mathbb{R})$ régulières. Ce résultat a été étendu pour une classe plus générale de probabilités $f \in \mathbf{P}(\mathbb{R})$ par Carlen, Carvalho, Le Roux, Loss et Villani dans [12], de plus ils introduisent la notion de chaos entropique (définition 3) et montre que cette mesure tensorisée et restreinte à la sphère de Kac est aussi entropie chaotique. Plus récemment, Hauray et Mischler [46] démontrent des versions quantitatives du chaos, introduisent la notion de Fisher-chaos (en utilisant l'information de Fisher, voir définition 3) et montrent la relation entre ces différents types de chaos.

En utilisant cette idée dans notre cadre, pour $f \in \mathbf{P}(\mathbb{R}^d)$ telle que

$$\int_{\mathbb{R}^d} v\, f(dv) = 0 \quad \text{et} \quad \int_{\mathbb{R}^d} |v|^2\, f(dv) = d,$$

on définit la mesure de probabilité $F^N \in \mathbf{P}_{\mathrm{sym}}(\mathcal{S}^N_{\mathcal{B}})$ comme la tensorisation de f renstreinte à la sphère de Boltzmann, donnée par

$$F^N := [f^{\otimes N}]_{\mathcal{S}^N_{\mathcal{B}}} = \frac{f^{\otimes N}}{\int_{\mathcal{S}^N_{\mathcal{B}}} f^{\otimes N}\, d\gamma^N}\, \gamma^N \tag{39}$$

où γ^N est la probabilité uniforme sur $\mathcal{S}^N_{\mathcal{B}}$.

Nous considérons aussi deux notions de chaos plus forte que la définition 1, le chaos entropique introduit dans [12] et le Fisher chaos introduit dans [46]. Étant donné $f \in \mathbf{P}(\mathbb{R}^d)$ on définit l'entropie relative de f par rapport à γ, la mesure gaussienne $\gamma(v) = (2\pi)^{-d/2} \exp(-|v|^2/2)$ de même quantité de mouvement et énergie que f, par

$$H(f|\gamma) := \int_{\mathbb{R}^d} h \log h\, d\gamma, \qquad h = \frac{df}{d\gamma} \tag{40}$$

si f est absolument continue par rapport à γ et $H(f|\gamma) = +\infty$ sinon. De façon analogue on définit l'information de Fisher relative de f par rapport à γ par

$$I(f|\gamma) := \int_{\mathbb{R}^d} |\nabla \log h|^2\, d\gamma, \qquad h = \frac{df}{d\gamma} \tag{41}$$

si f est absolument continue par rapport à γ et $I(f|\gamma) = +\infty$ sinon. Pour $F^N \in \mathbf{P}_{\mathrm{sym}}(\mathcal{S}^N_{\mathcal{B}})$ on définit l'entropie relative de F^N par rapport à γ^N par

$$H(F^N|\gamma^N) := \int_{\mathcal{S}^N_{\mathcal{B}}} h^N \log h^N\, d\gamma^N, \qquad h^N = \frac{dF^N}{d\gamma^N} \tag{42}$$

et l'information de Fisher relative de F^N par rapport à γ^N par

$$I(F^N|\gamma^N) := \int_{\mathcal{S}_{\mathcal{B}}^N} |\nabla_{\mathcal{S}^N} \log h^N|^2 \, d\gamma^N, \qquad h^N = \frac{dF^N}{d\gamma^N}, \tag{43}$$

où $\nabla_{\mathcal{S}^N}$ est le gradient par rapport à la sphère de Boltzmann $\mathcal{S}_{\mathcal{B}}^N$, i.e. la partie du gradient usuel dans $\mathbb{R}^{dN}$ qui est tangente à $\mathcal{S}_{\mathcal{B}}^N$. Dans le cas où F^N n'est pas absolument continue par rapport à γ^N, on pose $H(F^N|\gamma^N) = +\infty$ et $I(F^N|\gamma^N) = +\infty$.

On définit

Définition 3. Soit une suite de mesures de probabilités $(F^N)_{N\in\mathbb{N}^*} \in \mathbf{P}_{\rm sym}(\mathcal{S}_{\mathcal{B}}^N)$ qui est f-chaotique pour un $f \in \mathbf{P}(\mathbb{R}^d)$.

(1) On dit que F^N est f-entropie chaotique si

$$\lim_{N\to\infty} \frac{1}{N} H(F^N|\gamma^N) = H(f|\gamma).$$

(2) On dit que F^N est f-Fisher chaotique si

$$\lim_{N\to\infty} \frac{1}{N} I(F^N|\gamma^N) = I(f|\gamma).$$

Nous démontrons ainsi le résultat suivant (voir Theorem 1.18 et Theorem 1.19) :

Théorème 4. *Soit $f \in \mathbf{P}_6 \cap L^p(\mathbb{R}^d)$ avec $p > 1$. Alors, on peut construire une suite de mesures de probabilité $(F^N)_{N\in\mathbb{N}^*} \in \mathbf{P}_{\rm sym}(\mathcal{S}_{\mathcal{B}}^N)$, où $F^N := [f^{\otimes N}]_{\mathcal{S}_{\mathcal{B}}^N}$ est contruite par la tensorisation de f et la restriction à la sphère de Boltzmann, tel que :*

(i) F^N est f-chaotique, de plus on a une estimation quantitative.

(ii) F^N est f-entropie chaotique, de plus on a une estimation quantitative.

Un autre résultat est le lien entre les différents types de chaos (voir Theorem 1.24) :

Théorème 5. *Soient $f \in \mathbf{P}(\mathbb{R}^d)$ et $(G^N)_{N\in\mathbb{N}^*} \in \mathbf{P}_{\rm sym}(\mathcal{S}_{\mathcal{B}}^N)$ tel que $\Pi_1(G^N) \rightharpoonup f$ faiblement dans $\mathbf{P}(\mathbb{R}^d)$ lorsque $N \to \infty$. Alors*

(i) Si $H(f|\gamma) < \infty$ et $\lim_{N\to\infty} \frac{1}{N} H(G^N|\gamma^N) = H(f|\gamma)$, alors F^N est f-chaotique.

(ii) Si $I(f|\gamma) < \infty$ et $\lim_{N\to\infty} \frac{1}{N} I(G^N|\gamma^N) = I(f|\gamma)$, alors F^N est f-chaotique.

Le théorème 4 montre qu'il existe des suites de probabilités chaotiques sur la sphère de Boltzmann et, en outre, donne une façon de les construire. Nous pouvons donc nous demander s'il existe d'autres suites chaotiques différentes des F^N construits dans le Théorème 4 et comment les caractériser. Le théorème suivant donne une réponse (voir Theorem 1.31) :

Théorème 6. *Soit $(G^N)_{N\in\mathbb{N}^*} \in \mathbf{P}_{\rm sym}(\mathcal{S}_{\mathcal{B}}^N)$. Supposons que G^N est f-chaotique, pour $f \in \mathbf{P}(\mathbb{R}^d)$, et aussi que*

$$M_k(\Pi_1(G^N)) \le C, \; k \ge 6, \quad \frac{1}{N} H(G^N|\sigma^N) \le C, \quad \frac{1}{N} I(G^N|\sigma^N) \le C,$$

où M_k denote le moment d'ordre k.

Alors G^N est f-entropie chaotique et, de plus, on a une estimation quantitative.

Ensuite nous nous intéressons à l'application de ces résultats au modèle de Boltzmann présenté ci-dessus (équation cinétique limite (12) et équation maîtresse du système de particules (32)). Nous démontrons la propagation quantitative du chaos entropique pour l'équation de Boltzmann homogène (12) avec molécules maxwelliennes sans troncature (16) (voir Theorem 1.8) :

Théorème 7. *Soient $f_0 \in \mathbf{P}_6 \cap L^p(\mathbb{R}^d)$, pour $p > 1$, et $F_0^N := [f_0^{\otimes N}]_{\mathcal{S}_\mathcal{B}^N} \in \mathbf{P}_{\mathrm{sym}}(\mathcal{S}_\mathcal{B}^N)$ construite comme dans le théorème 4. Considérons $(f_t)_{t\geq 0}$ la solution de l'équation de Boltzmann homogène* (12) *avec molécules maxwelliennes sans troncature* (16) *associée à la donnée initiale f_0, et $(F_t^N)_{t\geq 0}$ la solution de l'équation maîtresse de Kac-Boltzmann* (32) *avec molécules maxwelliennes sans troncature* (16) *associée à la donnée initiale F_0^N.*

Alors, F_t^N est f_t-entropie chaotique uniformément en temps. De plus on a l'estimation

$$\sup_{t\geq 0} \left| \frac{1}{N} H(F_t^N|\sigma^N) - H(f_t|\gamma) \right| \leq C N^{-\theta}$$

pour une constante $C > 0$ et un $\theta > 0$ constructifs.

Ce résultat se base sur les théorèmes 4 et 6, ainsi que sur le résultat de propagation du chaos quantitative pour l'équation de Boltzmann avec molécules maxweliennes sans troncature [62].

5 Chapitre 2 : Chaos et chaos entropique dans le modèle de Kac sans moments de grand ordre

Dans ce chapitre nous nous intéressons aux propriétés de chaos et chaos entropique (définition 3) dans le cadre du modèle de Kac (voir remarque 2).

Comme mentionné avant, on montre dans Kac [49] que la suite de probabilités définie par la tensorisation et restriction à la sphère de Kac d'une probabilité $f \in \mathbf{P}(\mathbb{R})$ (assez régulière) centrée et de variance égal à 1, plus précisément

$$F^N := [f^{\otimes N}]_{\mathcal{S}_\mathcal{K}^N} = \frac{f^{\otimes N}}{\int_{\mathcal{S}_\mathcal{K}^N} f^{\otimes N}\, d\sigma^N}\, \sigma^N \in \mathbf{P}_{\mathrm{sym}}(\mathcal{S}_\mathcal{K}^N), \tag{44}$$

est f-chaotique, où σ^N dénote la mesure de probabilité uniforme sur $\mathcal{S}_\mathcal{K}^N$. Plus tard, Carlen, Carvalho, Le Roux, Loss et Villani [12] étendent ce résultat pour une classe plus large de probabilités : pour $f \in \mathbf{P}(\mathbb{R})$ (centrée et de variance 1) avec moment d'ordre 4 fini et telle que $f \in L^p(\mathbb{R})$ pour un $p > 1$. Ensuite, encore sous l'hypothèse du moment d'ordre 4 fini, [12] montre ce résultat pour d'autres familles de probabilités en utilisant des résultats de stabilité. Finalement, tous ces résultat sont aussi démontrés au sens entropique (chaos entropique).

Ce résultat est basé sur des estimations sur la quantité

$$Z^N(f;r) := \int_{\mathbb{S}^{N-1}(r)} f^{\otimes N}\, d\sigma_r^N, \quad \sigma_r^N = \text{probabilité uniforme sur } \mathbb{S}^{N-1}(r), \tag{45}$$

qui sont obtenues comme consequence du Théorème Central Limite, d'où la condition du moment d'ordre 4 fini. Par contre, le TCL usuel n'est pas suffisant pour ces estimations et [12] établit une version locale du TCL en norme L^∞ ce qui permet d'obtenir des estimations asymptotiques précises sur $Z^N(f;r)$, point clé pour montrer que F^N (44) est f-chaotique.

L'objectif de ce chapitre est de relaxer l'hypothèse du moment d'ordre 4 fini et d'étendre les résultats présentés ci-dessus.

Tout d'abord, on s'attend à ce que dans le cadre de cette généralisation on doive travailler avec la convergence stable (au lieu de la convergence vers la gaussienne du TCL classique). Ainsi, on établit une version locale du TCL stable en norme L^∞ avec des estimations fines pour le terme résiduel (voir Theorem 2.12) :

Théorème 8. *Soit $g \in \mathbf{P}(\mathbb{R}) \cap L^p(\mathbb{R})$ pour un $p > 1$. On suppose g dans le domaine naturel d'attraction (NDA) de $\gamma_{\sigma,\alpha,\beta}$, pour un $\sigma > 0$, β et $1 < \alpha < 2$, et que g a un moment fini d'ordre $k > 0$. On définit*

$$g_N(x) = N^{\frac{1}{\alpha}} g^{*N}\left(N^{\frac{1}{\alpha}} x\right) \qquad et \qquad \gamma_{\sigma,\alpha,\beta}(x) = \frac{1}{2\pi}\int_{\mathbb{R}} \widehat{\gamma}_{\sigma,\alpha,\beta}(\xi) e^{i\xi x} d\xi.,$$

où $\gamma_{\sigma,\alpha,\beta}$ est la densité de probabilité d'un processus stable, i.e.

$$\widehat{\gamma}_{\sigma,\alpha,\beta}(\xi) = \exp\big(-\sigma|\xi|^\alpha \{1 + i\beta \mathrm{sign}(\xi)\tan(\alpha\pi/2)\}\big).$$

Alors, pour N assez grand, on a

$$\|g_N - \gamma_{\sigma,\alpha,\beta}\|_{L^\infty} \le \epsilon(N), \tag{46}$$

pour une fonction constructive $\epsilon(N) \to 0$ lorsque $N \to \infty$ qui dépend des données de l'énoncé.

Ce résultat est basé sur des techniques de Fourier developpées par Goudon, Junca et Toscani [40].

Avec ce théorème nous pouvons construire des suites de probabilités chaotiques, comme demontré dans le résultat suivant (voir Theorem 2.13) :

Théorème 9. *Soit $f \in \mathbf{P}(\mathbb{R}) \cap L^p(\mathbb{R})$ pour un $p > 1$, et $\int_{\mathbb{R}} x^2 f(x)\,dx = 1$. Soit*

$$\nu_f(x) = \int_{-\sqrt{x}}^{\sqrt{x}} y^4 f(y) dy \tag{47}$$

et supposons que $\nu_f(x) \underset{x\to\infty}{\sim} C_S\, x^{2-\alpha}$ pour une constante $C_S > 0$ et $1 < \alpha < 2$.

Alors la famille de tensorisation et restriction de f, c'est-à-dire, $F^N = [f^{\otimes N}]_{\mathcal{S}^N_{\mathcal{K}}} \in \mathbf{P}_{\mathrm{sym}}(\mathcal{S}^N_{\mathcal{K}})$, définie dans (44), est f-chaotique. En outre, F^N est f-entropie chaotique.

Ensuite, on montre la propriété de stabilité suivante (voir Theorem 2.16) :

Théorème 10. *Considérons f satisfaisant les mêmes hypothèses que dans le théorème 9, $F^N = [f^{\otimes N}]_{\mathcal{S}^N_{\mathcal{K}}}$ et supposons de plus $f \in L^\infty(\mathbb{R})$. Soit $(G^N)_{N\in\mathbb{N}^*} \in \mathbf{P}_{\mathrm{sym}}(\mathcal{S}^N_{\mathcal{K}})$ tel que*

$$\lim_{N\to\infty} \frac{H(G^N|F^N)}{N} = 0.$$

Alors G^N est f-chaotique et, de plus, est f-entropie chaotique.

On clôt ce chapitre avec une approche différente au résultat de stabilité, en utilisant l'information de Fisher. Ce résultat caractérise une classe de famille de probabilités entropie chaotique (voir Theorem 2.18) :

Théorème 11. *Soit $(G^N)_{N\in\mathbb{N}^*} \in \mathbf{P}_{\mathrm{sym}}(\mathcal{S}^N_{\mathcal{K}})$ tel que G^N soit f-chaotique, pour un $f \in \mathbf{P}(\mathbb{R})$. Supposons qu'il existe une constante $C_S > 0$ et $1 < \alpha < 2$ tel que*

$$\int_{-\sqrt{x}}^{\sqrt{x}} y^4\, \Pi_1(G^N)(dy) \underset{x\to\infty}{\sim} C_S\, x^{2-\alpha} \tag{48}$$

uniformément en N, et que

$$\frac{H(G^N|\sigma^N)}{N} \le C, \quad \frac{I(G^N|\sigma^N)}{N} \le C \tag{49}$$

pour tout N.

Alors G^N est f-entropie chaotique.

6 Chapitre 3 : Propagation du chaos pour l'équation de Landau homogène avec molécules maxwelliennes

Comme expliqué précédemment, un problème important en théorie cinétique est de dériver rigoureusement l'équation de Boltzmann inhomogène en espace à partir d'un système de particules suivant les loi de Newton. De la même façon, un autre problème important est de dériver l'équation de Landau inhomogène. Ceci est une question ouverte mais l'échelle correcte est connue : c'est la limite à couplage faible, voir [8] pour plus d'informations.

Nous proposons dans ce chapitre d'utiliser l'approche de Kac, c'est à dire suivre le « Programme de Kac », et d'obtenir rigoureusement l'équation de Landau homogène en espace avec molécules maxwelliennes (22) comme la limite de champ moyen du modèle de Kac-Landau avec molécules maxwelliennes (37). Pour cela nous étudions la propagation du chaos et du chaos entropique pour ce modèle. De plus, nous démontrons le retour à l'équilibre pour l'équation maîtresse de Kac-Landau (37). Nous considérons le processus à N particules (36)-(37) dans le sous-espace invariant, c'est-à-dire la sphère de Boltzmann $\mathcal{S}^N_{\mathcal{B}}$ (34).

Avant de présenter notre contribution, nous introduisons quelques résultats connus. Le travail de Fontbona, Guérin et Méléard [37] considère un processus de diffusion non linéaire avec un bruit blanc qui a une interprétation en termes d'equation aux derivées partielles correspondant à l'équation de Landau homogène. Ensuite, ils construisent un système de N particules qui converge dans la limite $N \to \infty$ vers ce processus de diffusion non linéaire, cette preuve est en temps fini et ils obtiennent un taux de convergence quantitative pour la distance de Wasserstein W_2. Plus tard, Fournier [38], avec le même type d'approche, obtient un meilleur taux de convergence, encore en temps fini. Il faut remarquer que le système de particules utilisé dans ces deux travaux est différent de (36)-(37), en particulier leur système ne satisfait pas la conservation microscopique de l'énergie et donc le processus ne peut pas être considéré sur la sphère de Boltzmann (seulement sur $\mathbb{R}^{dN}$).

La stratégie est d'utiliser la méthode de consistence-stabilité devellopée par Mischler, Mouhot et Wennberg [62, 63]. En considérant les semi-groupes associés à l'équation maîtresse et à l'équation limite (de champ moyen), cette méthode réduit le problème de propagation du chaos à des estimations de consistence et de stabilité des ces semi-groupes et leurs générateurs associés.

Nous démontrons la propagation du chaos et du chaos entropique uniformément en temps ainsi que la convergence vers l'équilibre des solutions du systèmes de particules (voir Theorem 3.14, Theorem 3.15 et Theorem 3.31) :

Théorème 12. *Soient $f_0 \in \mathbf{P}(\mathbb{R}^d)$ assez regulière et $F_0^N \in \mathbf{P}_{\mathrm{sym}}(\mathcal{S}_{\mathcal{B}}^N)$ telle que F_0^N soit f_0-chaotique. On considère $(f_t)_{t\geq 0}$ solution de l'équation de Landau homogène avec molécules maxwelliennes* (22) *et $(F_t^N)_{t\geq 0}$ solution de l'équation maîtresse de Kac-Landau avec molécules maxwelliennes* (37)*. Alors on a :*

(i) F_t^N est f_t-chaotique uniformément en temps et on a une estimation quantitative.

(ii) F_t^N est f_t-entropie chaotique uniformément en temps et on a une estimation quantitative.

(iii) F_t^N converge vers γ^N lorsque $t \to \infty$ et on a une estimation quantitative indépendante du nombre N de particules.

7 Chapitre 4 : Convergence exponentielle vers l'équilibre pour l'équation de Landau homogène

Dans ce chapitre nous nous intéressons à la convergence des solutions de l'équation de Landau homogène avec potentiels durs (22) vers l'équilibre. Comme dèjà presenté dans (29), cette équation satisfait la conservation de la masse, de la quantité de mouvement et de l'énergie. En outre, on a le Thèorème-H qui nous dit que l'équilibre est une distribution maxwelienne caracterisée par la donnée initiale f_0. On rappelle l'équation de Landau homogène (en dimension $d = 3$)

$$\begin{cases} \partial_t f = Q_L(f,f) \\ f_{|t=0} = f_0, \end{cases} \tag{50}$$

avec

$$Q_L(g,f) = \partial_i \int_{\mathbb{R}^d} a_{ij}(v-v_*) \left(g_* \partial_j f - \partial_j g_* f\right) \, dv_*. \tag{51}$$

On considère, sans perte de généralité, seulement le cas d'une donnée initiale $f_0 = f_0(v)$, $v \in \mathbb{R}^3$, vérifiant

$$\int_{\mathbb{R}^3} f_0 = 1, \quad \int_{\mathbb{R}^3} v f_0 = 0, \quad \int_{\mathbb{R}^3} |v|^2 f_0 = 3,$$

et l'équilibre associé (de même masse, quantité de mouvement et énergie),

$$\mu(v) = \frac{e^{-\frac{|v|^2}{2}}}{(2\pi)^{3/2}}.$$

On linéarise l'équation autour de l'équilibre μ, avec une perturbation

$$f = \mu + h,$$

ainsi $h = h(t,v)$ vérifie

$$\begin{cases} \partial_t h = \mathcal{L}h + Q(h,h), \\ h_0 = f_0 - \mu \end{cases} \tag{52}$$

où l'opérateur linéarisé de Landau $\mathcal{L}$ est donné par

$$\begin{aligned} \mathcal{L}h &= Q(\mu,h) + Q(h,\mu) \\ &= (a_{ij} * \mu)\partial_{ij} f - (c * \mu) f + (a_{ij} * f)\partial_{ij}\mu - (c * f)\mu. \end{aligned} \tag{53}$$

Grâce aux quantités conservéés (29), le noyau de l'opérateur $\mathcal{L}$ est de dimension 5 et donné par (voir [24, 43, 2, 64, 66])

$$\mathcal{N}(\mathcal{L}) = \mathrm{Vect}\{\mu, v_1\mu, v_2\mu, v_3\mu, |v|^2\mu\}. \tag{54}$$

Mentionnons quelques résultats connus concernant le retour à l'équilibre. Premièrement, considérons l'équation linéarisée

$$\begin{cases} \partial_t h = \mathcal{L}h, \\ h_{|t=0} = h_0. \end{cases} \tag{55}$$

Notons $\mathcal{D}(h)$ la forme de Dirichlet associé à $-\mathcal{L}$, plus précisément

$$\mathcal{D}(h) := \langle -\mathcal{L}h, h\rangle_{L^2(\mu^{-1/2})} := \int_{\mathbb{R}^3} (-\mathcal{L}h) h \mu^{-1}.$$

En calculant cette quantité on obtient

$$\mathcal{D}(h) = \frac{1}{2}\iint_{\mathbb{R}^3\times\mathbb{R}^3} a_{ij}(v-v_*)\{\partial_i(\mu^{-1}h) - \partial_i(\mu_*^{-1}h_*)\}\{\partial_j(\mu^{-1}h) - \partial_j(\mu_*^{-1}h_*)\}\,\mu_*\mu\,dv_*\,dv \geq 0,$$

ce qui implique que $\mathcal{L}$ est négatif et son spectre est inclus dans $\mathbb{R}_-$. Plus précisément, d'après les travaux de Degond et Lemou [24], Guo [43], Mouhot et Strain [66], Baranger et Mouhot [2], dans le cas de l'équation de Landau homogène avec potentiels durs $\gamma \in (0,1]$

et molécules maxwelliennes $\gamma = 0$ (voir (23)), il existe une constante constructive $\lambda_0 > 0$ (appelée trou spectral) telle que

$$\mathcal{D}(h) \geq \lambda_0 \|h\|^2_{L^2(\mu^{-1/2})}, \qquad \forall\, f \in \mathcal{N}(\mathcal{L})^\perp. \tag{56}$$

Cela implique un retour exponentiel vers l'équilibre pour l'équation linéarisée (55), plus précisément

$$\forall\, t \geq 0,\ \forall\, h_0 \in L^2(\mu^{-1/2}), \qquad \|h_t - \Pi h_0\|_{L^2(\mu^{-1/2})} \leq e^{-\lambda_0 t} \|h_0 - \Pi h_0\|_{L^2(\mu^{-1/2})}, \tag{57}$$

où Π représente la projection sur $\mathcal{N}(\mathcal{L})$ (54). Cette approche nous donne une vitesse de convergence exponentielle mais seulement dans un certain voisinage de l'équilibre.

Une autre approche possible est d'étudier directement l'équation non linéaire (50), en établissant des inégalités fonctionnelles entre l'entropie relative définie par

$$H(f|\mu) := \int_{\mathbb{R}^3} f \log(f/\mu)$$

et la production d'entropie $D(f)$ définie dans (30). En effet, Desvillettes et Villani [29] montrent qu'il existe des constantes $\delta_1, \delta_2 > 0$ telles que

$$D(f) \geq \min\left\{\delta_1 H(f|\mu), \delta_2 H(f|\mu)^{1+\gamma/2}\right\} \tag{58}$$

ce qui implique, pour l'équation (50), le retour à l'équilibre avec taux de convergence polynomial

$$\forall t \geq 0, \quad \|f_t - \mu\|_{L^1} \leq C(1+t)^{-1/\gamma}. \tag{59}$$

Remarque 13. Dans le cas de molécules maxwelliennes $\gamma = 0$, Desvilletes et Villani [29] montre l'inégalité linéaire suivante

$$D(f) \geq \delta_0\, H(f|\mu), \tag{60}$$

pour une constante constructive $\delta_0 > 0$. Cela implique un retour exponentiel en entropie relative et en distance L^1

$$\forall t \geq 0, \quad H(f_t|\mu) \leq\, e^{-\delta_0 t} H(f_0|\mu) \quad \Rightarrow \quad \|f_t - \mu\|_{L^1} \leq C\, e^{-\delta_0 t/2}. \tag{61}$$

L'inégalité (60) est une réponse positive à la Conjecture de Cercignani (dans le cadre de l'équation de Landau), pour plus d'informations voir [27].

Notre objectif dans ce chapitre est de démontrer la convergence exponentielle des solutions de l'équation de Landau homogène (50) vers l'équilibre, dans le cas de potentiels durs $\gamma \in (0,1]$ (23). Nous démontrons (voir Theorem 4.1) :

Théorème 14. *Soit $f_0 \in L^1_{2+\delta}(\mathbb{R}^3)$ pour un $\delta > 0$. Alors, pour toute solution faible $(f_t)_{t\geq 0}$ de l'équation de Landau homogène avec potentiels durs* (50)*, il existe une constante $C > 0$ telle que*

$$\|f_t - \mu\|_{L^1(\mathbb{R}^3)} \leq Ce^{-\lambda_0 t},$$

où λ_0 est le trou spectral de l'opérateur linéarisé $\mathcal{L}$ dans $L^2(\mu^{-1/2})$ (57)*.*

La stratégie pour montrer ce théorème est :

(1) Des nouvelles estimations de trou spectral pour l'opérateur linéarisé de Landau $\mathcal{L}$ dans des espaces de Banach L^p avec poids polynomial et exponentiel. Ces nouvelles estimations sont basées sur la *méthode d'extension de l'espace fonctionnelle* pour lequel on a un trou spectral, méthode developpée par Gualdani, Mischler et Mouhot [42]. Plus précisément, à partir du trou spectral de $\mathcal{L}$ dans un « petit espace » $L^2(\mu^{-1/2})$ (ce qui correspond à un espace L^2 avec un poids exponentiel), nous démontrons en utilisant leur méthode le trou spectral dans des espaces « plus larges » L^p avec poids (voir théorème ci-dessous).

(2) La théorie de Cauchy pour l'équation (non linéaire) de Landau homogène dans le cas des potentiels durs developpée par Desvillettes et Villani [29].

(3) Le couplage entre la théorie linéaire et non linéaire : pour des temps petits, on utilise le retour polynomial donné par (59) ; ensuite, quand la solution entre dans un certain voisinage de l'équilibre, nous utilisons le retour exponentiel donné par les nouvelles estimation de la théorie linéaire (1).

Cette méthode a été introduite par Mouhot [65] pour prouver le retour exponentiel pour l'équation de Boltzmann homogène avec potentiels durs. Plus récemment, la même approche a été utilisée par Gualdani, Mischler et Mouhot [42] pour prouver la convergence exponentielle vers l'équilibre pour l'équation de Boltzmann inhomogène avec sphères dures dans le tore.

Le point (1) ci-dessus est donné par le résultat suivant (voir Theorem 4.3) :

Théorème 15. *On considère l'opérateur de Landau linéarisé* $\mathcal{L}$ (53) *avec potentiels durs* ($\gamma \in (0,1]$), $p \in [1,2]$ *et un poid* m *polynomial ou exponentiel. Alors, il existe* $C > 0$ *telle que*

$$\forall\, t > 0,\ \forall\, h \in L^p(m), \quad \|e^{t\mathcal{L}} h - \Pi h\|_{L^p(m)} \leq C e^{-\lambda_0 t} \|h - \Pi h\|_{L^p(m)},$$

où λ_0 *est le trou spectral de l'opérateur linéarisé* $\mathcal{L}$ *dans* $L^2(\mu^{-1/2})$ (57) *et* Π *est la projection sur* $\mathcal{N}(\mathcal{L})$ (54)*.*

8 Liste de travaux rassemblés dans la thèse

Les chapitres de ce manuscrit sont composés des travaux suivants :

— Chapitre 1 : article [19], soumis aux *Annales de l'Institut Henri Poincaré - Probabilités et Statistique.*

— Chapitre 2 : article [20], écrit en collaboration avec Amit Einav et paru à *Eletronic Journal of Probability.*

— Chapitre 3 : article [18], prépublication.

— Chapitre 4 : article [17], prépublication.

Première partie

Limite de champ moyen et propagation du chaos

Chapitre 1

Quantitative and qualitative Kac's chaos on the Boltzmann's sphere

Abstract. We investigate the construction of chaotic probability measures on the Boltzmann's sphere, which is the state space of the stochastic process of a many-particle system undergoing a dynamics preserving energy and momentum.

Firstly, based on a version of the local Central Limit Theorem (or Berry-Esseen theorem), we construct a sequence of probabilities that is Kac chaotic and we prove a quantitative rate of convergence. Then, we investigate a stronger notion of chaos, namely entropic chaos introduced in [12], and we prove, with quantitative rate, that this same sequence is also entropically chaotic.

Furthermore, we investigate more general class of probability measures on the Boltzmann's sphere. Using the HWI inequality we prove that a Kac chaotic probability with bounded Fisher's information is entropically chaotic and we give a quantitative rate. We also link different notions of chaos, proving that Fisher's information chaos, introduced in [46], is stronger than entropic chaos, which is stronger than Kac's chaos. We give a possible answer to [12, Open Problem 11] in the Boltzmann's sphere's framework.

Finally, applying our previous results to the recent results on propagation of chaos for the Boltzmann equation [62], we prove a quantitative rate for the propagation of entropic chaos for the Boltzmann equation with Maxwellian molecules.

1.1 Introduction

1.1.1 Motivation

In his celebrated paper [49], Kac introduced the notion of propagation of chaos in order to connect a stochastic process of a system of N identical particles undergoing binary collisions to its mean field equation.

Our interest in this paper is to investigate chaotic distributions supported by the phase space of the stochastic process of the N-particle system as we shall explain. We refer to [12] for a detailed introduction on this topic and on Kac's paper [49].

Consider a system of N identical particles of mass $\rho > 0$ such that its evolution is described by a jump process with binary collisions that preserves energy and momentum. Let us denote by i, j the particles undergoing the collision, with pre-collisional velocities $v_i, v_j \in \mathbb{R}^d$ and post-collisional velocities $v_i^*, v_j^* \in \mathbb{R}^d$. We have then the conservation of momentum

$$\rho v_i^* + \rho v_j^* = \rho v_i + \rho v_j,$$

and the conservation of energy

$$\frac{\rho}{2}|v_i^*|^2 + \frac{\rho}{2}|v_j^*|^2 = \frac{\rho}{2}|v_i|^2 + \frac{\rho}{2}|v_j|^2.$$

If the system has initial energy $\mathcal{E} = \frac{1}{2}\sum_{i=1}^N \rho |v_i|^2 \in \mathbb{R}_+$ and initial momentum $M = \rho m = \sum_{i=1}^N \rho v_i \in \mathbb{R}^d$, then both energy and momentum will be unchanged under the dynamics. The phase space of this process is then the manifold $\mathcal{S}^N(\sqrt{\mathcal{E}}, m) \subset \mathbb{R}^{dN}$ defined by

$$\mathcal{S}^N(\sqrt{\mathcal{E}}, m) := \left\{ V = (v_1, \ldots, v_N) \in \mathbb{R}^{dN} \;\Big|\; \frac{1}{2}\sum_{i=1}^N \rho |v_i|^2 = \mathcal{E},\ \sum_{i=1}^N \rho v_i = \rho m \right\},$$

which is the intersection of a sphere of radius $\sqrt{2\mathcal{E}/\rho}$ and a hyperplane. This space $\mathcal{S}^N(\sqrt{\mathcal{E}}, m)$ is in fact a sphere in $\mathbb{R}^{dN}$ of dimension $d(N-1)-1$ with radius $\sqrt{2\mathcal{E}/\rho - |m|^2/N}$ and center $(m, \ldots, m)/\sqrt{N}$. We remark that we need $|m|^2 \leq 2N\mathcal{E}/\rho$ in order to $\mathcal{S}^N(\sqrt{\mathcal{E}}, m)$ be non empty.

Now choosing units such that the mass ρ of each particle is equal to 2, the total value of kinetic energy is dN and, without loss of generality, choosing $m = 0$, the state space of this dynamics is

$$\mathcal{S}^N_{\mathcal{B}} := \mathcal{S}^N(\sqrt{dN}, 0) = \left\{ V = (v_1, \ldots, v_N) \in \mathbb{R}^{dN} \;\Big|\; \sum_{i=1}^N |v_i|^2 = dN,\ \sum_{i=1}^N v_i = 0 \right\} \tag{1.1}$$

and we shall call the manifold $\mathcal{S}^N_{\mathcal{B}}$ the Boltzmann's sphere.

An example of this kind of dynamics is the space homogeneous Boltzmann model that we shall explain. Given a pre-collisional system of velocities $V = (v_1, \ldots, v_N) \in \mathbb{R}^{dN}$ and a collision kernel (for more information on the collision kernel we refer to [76, 62])

$$B(z, \cos\theta) = \Gamma(|z|) b(\cos\theta), \tag{1.2}$$

for some nonnegative functions Γ and b, the process is:

— for any $i' \neq j'$, pick a random time $T(\Gamma(|v_{i'} - v_{j'}|))$ of collision accordingly to an exponential law of parameter $\Gamma(|v_{i'} - v_{j'}|)$ and choose the minimum time T_1 and the colliding pair (v_i, v_j) such that

$$T_1 = T(\Gamma(|v_i - v_j|)) = \min_{i', j'} T(\Gamma(|v_{i'} - v_{j'}|)),$$

— draw $\sigma \in \mathbb{S}^{d-1} \subset \mathbb{R}^d$ according to the law $b(\cos\theta_{ij})$, with

$$\cos\theta_{ij} = \sigma \cdot \frac{(v_i - v_j)}{|v_i - v_j|},$$

— after collision the new velocities become

$$V_{ij}^* = (v_1, \dots, v_i^*, \dots, v_j^*, \dots, v_N)$$

where the post-collisional velocities v_i^* and v_j^* are given by

$$v_i^* = \frac{v_i + v_j}{2} + \frac{|v_i - v_j|}{2}\sigma, \qquad v_j^* = \frac{v_i + v_j}{2} - \frac{|v_i - v_j|}{2}\sigma. \tag{1.3}$$

Iterating this construction we built then the associated Markov process $(\mathcal{V}_t)_{t\geq 0}$ on $\mathbb{R}^{dN}$. The equation of the associated law is given by, after a rescaling of time, (see [62])

$$\partial_t G_t^N = L_N G_t^N = \frac{1}{N} \sum_{i<j} \int_{\mathbb{S}^{d-1}} \Big[G_t^N(V_{ij}^*) - G_t^N(V) \Big] \, B(|v_i - v_j|, \cos\theta)\, d\sigma \tag{1.4}$$

with initial data G_0^N and where $V_{ij}^* = (v_1, \dots, v_i^*, \dots, v_j^*, \dots, v_N)$. This equation is known as the *master equation*.

Associated to this process, we have the (limit) spatially homogeneous Boltzmann equation [62, 63, 76]

$$\partial_t f(t,v) = \int_{\mathbb{R}^d \times \mathbb{S}^{d-1}} B(|v-w|, \cos\theta) \Big(f(w^*) f(v^*) - f(w) f(v) \Big)\, dw\, d\sigma \tag{1.5}$$

with initial data $f(0,\cdot) = f_0$ and where the post-collisional velocities v^* and w^* are obtained by (1.3).

We shall highlight here the models we consider in the last part of this work (see Theorem 1.8 below), and we refer to [76] for more details concerning the collision kernel. Assuming a collision kernel B derived from inverse-power law interaction potentials

$$\phi(r) = r^{-(s-1)}, \quad s > 2,$$

we have that the collision kernel has the form

$$B(z, \cos\theta) = |z|^\gamma\, b(\cos\theta), \quad \gamma = \frac{s - (2d-1)}{s-1}, \tag{1.6}$$

where the function b is locally smooth and has a nonintegrable singularity

$$\sin^{d-2}\theta\, b(\cos\theta) \sim_{\theta\sim 0} C_b\, \theta^{-1-\nu}, \quad \nu \in (0,2),\ C_b > 0. \tag{1.7}$$

In the particular case of three dimensions $d = 3$, we have $\gamma = (s-5)/(s-1)$ and $\nu = 2/(s-1)$. If we replace the angular collision kernel b by a locally integrable one, we speak of cutoff collision kernels (or Grad's cutoff).

We shall consider in this work the case of Maxwellian molecules, in which the collision kernel does not depend on the relative velocity, i.e. $\gamma = 0$ in (1.6). We consider the general assumption

$$\begin{cases} B(|v-w|, \cos\theta) = b(\cos\theta), \\ \forall\, \alpha > 0, \quad \displaystyle\int_0^\pi b(\cos\theta)\,(1-\cos\theta)^{\alpha+1/4}\,\sin^{d-2}\theta\,d\theta < +\infty. \end{cases} \tag{1.8}$$

This is the same assumption made in [62], since in Theorem 1.8 we use their results. Remark that (1.8) includes the true Maxwellian molecules (or Maxwellian molecules without cutoff) in dimension $d = 3$, when $\gamma = 0$, $\nu = 1/2$ and

$$B(z, \cos\theta) = b(\cos\theta), \qquad b(\cos\theta) \sim_{\theta\sim 0} C_b\,\theta^{-5/2}, \quad (d = 3). \tag{1.9}$$

Also, it includes the Grad's cutoff Maxwellian molecules, when the singularity is removed,

$$B(z, \cos\theta) = b(\cos\theta), \qquad \int_0^\pi b(\cos\theta)\,\sin^{d-2}\theta\,d\theta < +\infty. \tag{1.10}$$

Some results in Theorem 1.8 will consider the general assumption (1.8) and others the cutoff Maxwellian molecules (1.10).

The program set by Kac in [49] was to investigate the behavior of solutions of the mean field equation (1.5) in terms of the behaviour of the solutions of the master equation (1.4). Moreover, the notion of propagation of chaos introduced by Kac means that if the initial distribution G_0^N is f_0-chaotic (Definition 1.1 below) then, for all $t > 0$, the solution G_t^N of (1.4) is f_t-chaotic, where f_t is the solution of (1.5). For more information on this topic we refer to the recent results of Mischler, Mouhot and Wennberg [62, 63].

This paper is inspired by the works of Carlen, Carvalho, Le Roux, Loss and Villani [12] and also of Hauray and Mischler [46], which investigate chaotic probabilities on the usual sphere in $\mathbb{R}^N$ with radius $\sqrt{N}$ (also called Kac's sphere). This sphere is the phase space of Kac's model, which is a one-dimensional simplification, introduced in [49], of the model presented above, with energy conservation only.

The novelty here is that we investigate chaotic probability sequences in the Boltzmann's sphere $\mathcal{S}_{\mathcal{B}}^N \subset \mathbb{R}^{dN}$ and, furthermore, we prove quantitative rates of chaos convergence. Moreover, we apply our results to the Boltzmann equation with true Maxwellian molecules to prove quantitative propagation of entropic chaos.

1.1.2 Definitions and main results

Let E be a Polish space, then we shall denote by $\mathbf{P}(E)$ the space of Borel probability measures on E. Furthermore, through this paper, on the space E^N we will only consider symmetric measures, more precisely, we say that $G^N \in \mathbf{P}(E^N)$ is symmetric if for all $\varphi \in C_b(E^N)$ we have

$$\int_{E^N} \varphi\,dG^N = \int_{E^N} \varphi_\sigma\,dG^N,$$

for any permutation σ of $\{1,\dots,N\}$, and where

$$\varphi_\sigma := \varphi(V_\sigma) = \varphi(v_{\sigma(1)},\dots,v_{\sigma(N)}),$$

for $V=(v_1,\dots,v_N)\in E^N$.

For $G^N \in \mathbf{P}(E^N)$ and a integer $\ell \in [1,N]$ we denote by G^N_ℓ (or $\Pi_\ell(G^N)$) the ℓ-marginal of G^N, defined by

$$\forall \varphi \in C_b(E^\ell), \qquad \int_{E^\ell} \varphi\, dG^N_\ell = \int_{E^N} \varphi \otimes \mathbf{1}^{\otimes(N-\ell)}\, dG^N.$$

We shall use through the paper the same notation to represent a probability measure and its density with respect to the Lebesgue measure.

We can now give the notion of chaos formalized by Kac in [49], we also refer to [70] for an introduction on this topic with a probabilistic approach and to [60] for a short survey.

Definition 1.1 (Kac's chaos)**.** Consider $f \in \mathbf{P}(E)$. We say that $G^N \in \mathbf{P}(E^N)$ is f-chaotic (or f-Kac chaotic), if for each fixed positive integer ℓ, G^N_ℓ converges to $f^{\otimes \ell}$ in the sense of measures in $\mathbf{P}(E^\ell)$ when N goes to infinity, i.e. if for all $\varphi \in C_b(E^\ell)$,

$$\lim_{N\to\infty} \int_{E^\ell} \varphi\, dG^N_\ell = \int_{E^\ell} \varphi\, df^{\otimes \ell}. \tag{1.11}$$

In fact, it is well known that we need condition (1.11) to hold for only one $\ell \ge 2$ (see for instance [70]).

We also introduce the Monge-Kantorovich-Wasserstein (MKW) distance and for more information about it we refer to [78]. Consider an integer ℓ and $p \in [1,\infty)$, we define then the space

$$\mathbf{P}_p(E^\ell) := \left\{ F^\ell \in \mathbf{P}(E^\ell);\ M_p(F^\ell) := \int_{E^\ell} |X|^p\, dF^\ell(X) < \infty \right\}.$$

Then, for $F^\ell, G^\ell \in \mathbf{P}_p(E^\ell)$ we define the MKW distance between F^ℓ and G^ℓ by

$$W_p(F^\ell,G^\ell) := \inf_{\pi \in \Pi(F^\ell,G^\ell)} \left(\int_{E^\ell\times E^\ell} d_{E^\ell}(X,Y)^p\, d\pi(X,Y) \right)^{1/p}, \tag{1.12}$$

where $\Pi(F^\ell,G^\ell)$ is the set of transfer plan between F^ℓ and G^ℓ, which is the set of probabilty measures on $E^\ell \times E^\ell$ with marginals F^ℓ and G^ℓ respectively, and where we define the distace d_{E^ℓ} as

$$\forall\, X=(x_1,\dots,x_\ell), Y=(y_1,\dots,y_\ell) \in E^\ell, \qquad d_{E^\ell}(X,Y) := \sum_{i=1}^{\ell} d_E(x_i,y_i).$$

In the paper we will use the Euclidean distance in $E=\mathbb{R}^d$, i.e. $d_E(x_i,y_i)=|x_i-y_i|$ for all $x_i,y_i \in E$. More precisely, we shall use

$$\forall\, f,g \in \mathbf{P}_1(\mathbb{R}^d), \qquad W_1(f,g) = \inf_{\pi\in\Pi(f,g)} \int_{\mathbb{R}^d\times\mathbb{R}^d} |x-y|\, d\pi(x,y)$$

and

$$\forall\, f, g \in \mathbf{P}_2(\mathbb{R}^d), \qquad W_2(f,g) = \inf_{\pi \in \Pi(f,g)} \left(\int_{\mathbb{R}^d \times \mathbb{R}^d} |x-y|^2 \, d\pi(x,y) \right)^{1/2}.$$

Moreover, for $F^N, G^N \in \mathbf{P}(\mathcal{S}^N_{\mathcal{B}})$ we shall use in the definition of $W_p(F^N, G^N)$ the Euclidean distance inherited from $\mathbb{R}^{dN}$, which means that for $X, Y \in \mathcal{S}^N_{\mathcal{B}}$ we shall use $d_{\mathcal{S}^N_{\mathcal{B}}}(X,Y) = |X - Y|$.

Let γ be the Gaussian probability measure on $\mathbb{R}^d$, $\gamma(v) = (2\pi)^{-d/2}\, e^{-|v|^2/2}$, and $\mu \in \mathbf{P}(\mathbb{R}^d)$. We define the relative entropy of μ with respect to γ by

$$H(\mu|\gamma) := \int_{\mathbb{R}^d} \log \frac{d\mu}{d\gamma}\, d\mu, \tag{1.13}$$

if μ is absolutely continuous with respect to γ, otherwise $H(\mu|\gamma) \; := \; +\infty$.

Moreover, for $G^N \in \mathbf{P}(\mathcal{S}^N_{\mathcal{B}})$ we define the relative entropy with respect to γ^N, the uniform probability measure on $\mathcal{S}^N_{\mathcal{B}}$, by

$$H(G^N|\gamma^N) := \int_{\mathcal{S}^N_{\mathcal{B}}} \left(\log \frac{dG^N}{d\gamma^N} \right) dG^N. \tag{1.14}$$

We shall now define a stronger notion of chaos, namely the entropic chaos introduced in [12].

Definition 1.2 (Entropic chaos)**.** We say that the sequence $G^N \in \mathbf{P}(\mathcal{S}^N_{\mathcal{B}})$ is entropically f-chaotic, for some $f \in \mathbf{P}(\mathbb{R}^d)$, if G^N is f-chaotic in Kac's sense (Definition 1.1) and

$$\lim_{N\to\infty} \frac{1}{N}\, H(G^N|\gamma^N) = H(f|\gamma) \tag{1.15}$$

with $H(f|\gamma) < \infty$.

Finally, with these definitions at hand we can state the main results of the paper.

Theorem 1.3. *For any $f \in \mathbf{P}_6(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$ with $1 < p \leq \infty$, there exists a sequence of probability measures $F^N := [f^{\otimes N}]_{\mathcal{S}^N_{\mathcal{B}}} \in \mathbf{P}(\mathcal{S}^N_{\mathcal{B}})$, contructed by conditioning the N-fold tensorization of f to the Boltzmann's sphere, such that*

(i) F^N is f-chaotic. More precisely, for any $\ell \geq 1$ fixed there exists a constant $C = C(\ell) > 0$ such that for $N \geq \ell + 1$ we have

$$W_1(F^N_\ell, f^{\otimes \ell}) \leq \frac{C}{\sqrt{N}};$$

(ii) F^N is entropically f-chaotic. More precisely, there exists a constant $C > 0$ such that

$$\left| \frac{1}{N}\, H(F^N|\gamma^N) - H(f|\gamma) \right| \leq \frac{C}{\sqrt{N}}.$$

Let us now define the relative Fisher's information of a probability measure $\mu \in \mathbf{P}(\mathbb{R}^d)$ with respect to γ by

$$I(\mu|\gamma) := \int_{\mathbb{R}^d} \left|\nabla \log \frac{d\mu}{d\gamma}\right|^2 \, d\mu, \tag{1.16}$$

and, as we did for entropy, we also define for $G^N \in \mathbf{P}(\mathcal{S}_{\mathcal{B}}^N)$ the relative Fisher's information with respect to γ^N by

$$I(G^N|\gamma^N) := \int_{\mathcal{S}_{\mathcal{B}}^N} \left|\nabla_{\mathcal{S}} \log \frac{dG^N}{d\gamma^N}\right|^2 \, dG^N, \tag{1.17}$$

where $\nabla_{\mathcal{S}}$ stands for the gradient on the Boltzmann's sphere, i.e. the component of the usual gradient in $\mathbb{R}^{dN}$ that is tangent to the sphere $\mathcal{S}_{\mathcal{B}}^N$.

We define then another stronger notion of chaos, the Fisher's information chaos, in an analogous way of Definition 1.2.

Definition 1.4 (Fisher's information chaos)**.** We say that the sequence $G^N \in \mathbf{P}(\mathcal{S}_{\mathcal{B}}^N)$ is Fisher's information f-chaotic, for some $f \in \mathbf{P}(\mathbb{R}^d)$, if G^N is f-chaotic in Kac's sense (Definition 1.1) and

$$\lim_{N\to\infty} \frac{1}{N} \, I(G^N|\gamma^N) = I(f|\gamma)$$

with $I(f|\gamma) < \infty$.

Remark 1.5. The Fisher's information chaos is introduced in [46] in a weaker way, which is in fact equivalent to Definition 1.4 thanks to Theorem 1.6.

Next, we may compare as follows the several notions of chaos:

Theorem 1.6. *Consider $G^N \in \mathbf{P}(\mathcal{S}_{\mathcal{B}}^N)$, with k-th order moment $M_k(G_1^N)$ bounded, for some $k \geq 6$, and suppose that $G_1^N \rightharpoonup f$ in $\mathbf{P}(\mathbb{R}^d)$.*

Then, each assertion listed below implies the further one:

(i) $N^{-1} I(G^N|\gamma^N) \to I(f|\gamma)$, with $I(f|\gamma) < \infty$.

(ii) $N^{-1} I(G^N|\gamma^N)$ is bounded and G^N is f-chaotic in Kac's sense.

(iii) $N^{-1} H(G^N|\gamma^N) \to H(f|\gamma)$, with $H(f|\gamma) < \infty$.

(iv) G^N is f-chaotic in Kac's sense.

As a consequence, in Definition 1.2 of the entropic chaos and in Definition 1.4 of Fisher's information chaos, we only need the convergence of the first marginal, i.e. $G_1^N \rightharpoonup f$, instead of the convergence of all marginals. Hence, this theorem asserts that Fisher's information chaos implies entropic chaos, which in turns implies chaos (or Kac's chaos). Furthermore, we prove a quantitative rate for the implication $(ii) \Rightarrow (iii)$.

Another main result of the paper is a possible answer to [12, Open Problem 11] in the setting of Boltzmann's sphere given in Theorem 1.7. First of all, let us state the problem. For $G^N \in \mathbf{P}(\mathcal{S}_{\mathcal{B}}^N)$ and $f \in \mathbf{P}_6(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$ with $p > 1$, consider the following two conditions:

$$\lim_{N\to\infty} \frac{1}{N} \, H(G^N|[f^{\otimes N}]_{\mathcal{S}_{\mathcal{B}}^N}) = 0, \tag{1.18}$$

and

$$\forall\, \ell \in \mathbb{N}, \qquad \lim_{N\to\infty} H(G_\ell^N | f^{\otimes \ell}) = 0, \tag{1.19}$$

where $[f^{\otimes N}]_{\mathcal{S}_\mathcal{B}^N}$ is the probability measure constructed in Theorem 1.3. In the Kac's sphere setting (i.e. $\mathbb{S}^{N-1}(\sqrt{N})$ instead of $\mathcal{S}_\mathcal{B}^N$), [12] proved that condition (1.19) holds when G^N is the conditioned tensor product $G^N = [f^{\otimes N}]_{\mathcal{S}_\mathcal{B}^N}$. As discussed in [12], conditions (1.15), (1.18) and (1.19) really mean that G^N is "strongly" close to $f^{\otimes N}$, not only in the weak measure sense for marginals as in Kac's chaos. In view of this, they formulated the following problem.

Problem 1 ([12, Open Problem 11])**.** Does condition (1.18) imply condition (1.19) ? More generally, does condition (1.19) hold for a larger and easily recognized class of chaotic sequences, larger than those contructed by means of conditioning tensor products ?

We give a partial answer to Problem 1 in the following theorem.

Theorem 1.7. *Consider $G^N \in \mathbf{P}(\mathcal{S}_\mathcal{B}^N)$ such that G^N is f-chaotic, for some $f \in \mathbf{P}(\mathbb{R}^d)$, and suppose that*

$$M_k(G_1^N) \le C,\ k > 2, \qquad \frac{1}{N}\, I(G^N|\gamma^N) \le C.$$

Suppose further that $f \in L^\infty(\mathbb{R}^d)$ and $f(v_1) \ge \exp(-a|v_1|^2)$ for some constant $a > 0$. Then for any fixed ℓ, there exists a constant $C = C(d, \ell, \|f\|_{L^\infty}, M_k(G_1^N, f)) > 0$ such that for all $N \ge \ell + 1$ we have

$$H(G_\ell^N | f^{\otimes \ell}) \le C\, W_1(G_\ell^N, f^{\otimes \ell})^{\theta(\ell,d,k)},$$

where $\theta(\ell, d, k)$ is constructive and depends on ℓ, d and k. As a consequence, $H(G_\ell^N | f^{\otimes \ell}) \to 0$ as $N \to \infty$ and condition (1.19) *holds.*

This theorem exhibes a class of chaotic sequences in the Boltzmann's sphere that satisfy condition (1.19). At a first sight, the hypotheses needed on G^N and f to (1.19) be true may seen stronger than the conditioned tensor product, in which case [12] proved that (1.19) holds (as said above). However, as remarked in [12, 46], the conditioned tensor product assumption is not propagated along time by the Boltzmann equation but the assumptions needed in Theorem 1.7 may be. It is indeed true for the Boltzmann equation with Maxwellian molecules (see point (*iv*) of Theorem 1.8 below for a precise statement), hence, in this setting, the assumptions in Theorem 1.7 are natural, which gives a satisfying answer to the second question on Problem 1 in the Maxwellian case.

The interest here is that, as already remarked in [12, 62, 46], a natural step on Kac's program would be to study the propagation of conditions (1.15) or (1.18) or (1.19) (which are stronger than Kac's chaos) under the master equation (1.4). As explained above, as a consequence of Theorem 1.7, the propagation of (1.19) holds true for Maxwellian molecules. We continue the investigation of these issues in Theorem 1.8 below, proving also the propagation of entropic chaos (1.15) and (1.18).

We can apply our previous results to the Boltzmann equation for Maxwellian molecules. Some of the results concern assumption (1.8), i.e. Maxwellian molecules with and without cutoff, others concern only the Grad's cutoff Maxwellian molecules (1.10). Thanks to the work on propagation of chaos of [62], we can establish the following theorem.

Theorem 1.8. *Let $f_0 \in \mathbf{P}(\mathbb{R}^d)$ and $G_0^N \in \mathbf{P}(\mathcal{S}_\mathcal{B}^N)$. Consider then, for all $t > 0$, the solution G_t^N of the Boltzmann master equation* (1.4) *with Maxellian molecules (*(1.8) *or* (1.10)*) associated to the initial condition G_0^N, and the solution f_t of the limiting Boltzmann equation* (1.5) *with Maxellian molecules (*(1.8) *or* (1.10)*) associated to the initial data f_0.*

Then we have

(i) Let (1.10) *be in force. Consider $f_0 \in \mathbf{P}_6 \cap L^p(\mathbb{R}^d)$ for $p > 1$. If G_0^N is entropically f_0-chaotic, then for all $t > 0$, G_t^N is entropically f_t-chaotic, more precisely*

$$\lim_{N\to\infty} \frac{1}{N} H(G_t^N|\gamma^N) = H(f_t|\gamma).$$

(ii) Let (1.8) *be in force. Consider $f_0 \in \mathbf{P}_6(\mathbb{R}^d)$ with $I(f_0|\gamma) < \infty$. If $G_0^N = [f_0^{\otimes N}]_{\mathcal{S}_\mathcal{B}^N} \in \mathbf{P}(\mathcal{S}_\mathcal{B}^N)$ as in Theorem 1.3, then, for all $t > 0$, G_t^N is entropically f_t-chaotic. More precisely, for any*

$$\epsilon < \frac{48}{(7d+6)^2(5d+24)}$$

there exists a constant $C := C(\epsilon) > 0$ such that

$$\sup_{t\geq 0} \left| \frac{1}{N} H(G_t^N|\gamma^N) - H(f_t|\gamma) \right| \leq C N^{-\epsilon}.$$

(iii) Let (1.10) *be in force. Consider $f_0 \in \mathbf{P}_6 \cap L^\infty(\mathbb{R}^d)$ and $f_0(v_1) \geq \exp(-\alpha|v_1|^2 + \beta)$ for $\alpha > 0$ and $\beta \in \mathbb{R}$. If G_0^N satisfies condition* (1.18)

$$\lim_{N\to\infty} \frac{1}{N} H(G_0^N|[f_0^{\otimes N}]_{\mathcal{S}_\mathcal{B}^N}) = 0,$$

then, for all $t > 0$, G_t^N also satisfies condition (1.18)

$$\lim_{N\to\infty} \frac{1}{N} H(G_t^N|[f_t^{\otimes N}]_{\mathcal{S}_\mathcal{B}^N}) = 0.$$

(iv) Let (1.10) *be in force. Consider $f_0 \in \mathbf{P}_6 \cap L^\infty(\mathbb{R}^d)$ and $f_0(v_1) \geq \exp(-\alpha|v_1|^2 + \beta)$ for $\alpha > 0$, $\beta \in \mathbb{R}$. Consider also G_0^N that is f_0-chaotic and has $M_k(\Pi_1(G_0^N))$ and $N^{-1}I(G_0^N|\gamma^N)$ finite, for some $k > 2$.*
Then, for all $t \geq 0$, G_t^N satisfies condition (1.19)

$$\forall\, \ell \in \mathbb{N}, \qquad \lim_{N\to\infty} H(\Pi_\ell(G_t^N)|f_t^{\otimes \ell}) = 0.$$

Theorem 1.8 improves the results of [62] where Kac's chaos is established with a rate but entropic chaos is proved without any rate. Indeed, point (i) here is proved in [62] and point (ii) gives a quantitative propagation of entropic chaos. Moreover, point (iii) answers a question of [62, Remark 7.11] and point (iv) is a consequence of Theorem 1.7 as said above.

It is worth mentioning that point (i) was proved in [62] for both the Maxwellian molecules with cutoff (1.10) and the hard spheres case (which corresponds to the collision kernel $B(z, \cos\theta) = |z|$). The proof of point (iii) also shows that (iii) is valid for hard spheres, indeed the proof is based on the fact that (1.15) and (1.18) are equivalent under some hypotheses on f (see Theorem 1.25) and these properties are also propagated along time in the hard spheres case (propagation of L^∞, moments and lower Maxwellian bounds, see e.g. [76] and the references therein). However, the results (ii) and (iv) are valid only for the Maxwellian case, the reason behind this is that a key ingredient of the proof is the propagation of the Fisher's information bound, and such property is only know to hold for Maxwellian molecules.

1.1.3 Strategy

We construct a probability on $\mathcal{S}^N_\mathcal{B}$ based on tensorization and conditioning of some probabilty measure on $\mathbb{R}^d$. To this purpose, we use an explicit formula for the marginals of the uniform probablity on $\mathcal{S}^N_\mathcal{B}$ and a version of the local Central Limit Theorem (also known as Berry-Esseen), which is the cornerstone of the proof.

In order to study more general probabilities on the Boltzmann's sphere, we use an interpolation-type inequality, relating entropy, Fisher's information and the 2-MKW distance, called HWI inequality from [67, 53, 78], to show that Kac chaotic probabilities with finite Fisher's information are entropically chaotic.

Finally, the application of our results to the Boltzmann equation is based on recent results of propagation of chaos from [62] and on the relations of different notions of measuring chaos from the work [46].

1.1.4 Previous works

In [49] it is proved that the N-fold tensorization of a smooth probability on $\mathbb{R}$ conditioned to the Kac's sphere, i.e. the usual sphere $\mathbb{S}^{N-1}(\sqrt{N})$, is Kac chaotic. Then, the work [12] extends this result to a more general class of probabilities on $\mathbb{R}$, introduces the notion of entropic chaos and also proves that the N-fold tensorization conditioned to the Kac's sphere is entropically chaotic. Furthermore, the recent work [46] gives quantitative rates of the results before, introduces the notion of Fisher's information chaos and links these three notions of chaos.

1.1.5 Organization of the paper

In Section 1.2 we shall study the uniform probability measure on $\mathcal{S}^N_\mathcal{B}$. In Section 1.3 we construct a chaotic distribution on Boltzmann's sphere based on a probability measure on $\mathbb{R}^d$. Furthermore we prove a quantitative chaos convergence rate and we prove point (i) of Theorem 1.3. Then, in Section 1.4 we investigate the entropic and Fisher's information chaos. First, we study the entropic chaos for the probability distribution built before in Section 1.3 and we prove point (ii) of Theorem 1.3. Then, we link these three notions of chaos and investigate a more general class of probability measures on $\mathcal{S}^N_{\mathcal{B}}$, proving Theorem 1.6 and Theorem 1.7. Finally, in Section 1.5 we use our previous results to prove Theorem 1.8.

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1.2 Uniform probability measure

Consider $V = (v_1, \dots, v_N) \in \mathbb{R}^{dN}$, $r \in \mathbb{R}_+$ and $z \in \mathbb{R}^d$. We define the sphere

$$\mathcal{S}^N(r,z) := \left\{ V = (v_1, \dots, v_N) \in \mathbb{R}^{dN} \;\Big|\; \sum_{i=1}^N v_i^2 = r^2,\ \sum_{i=1}^N v_i = z \right\}.$$

We denote by $\gamma^N_{r,z}$ the uniform probability measure on $\mathcal{S}^N(r,z)$. We recall that $\mathcal{S}^N_{\mathcal{B}} := \mathcal{S}^N(\sqrt{dN}, 0)$ is the Boltzmann sphere and we denote by $\gamma^N := \gamma^N_{\sqrt{dN},0}$ its uniform probability measure. Moreover, we also denote by $\mathbb{S}^{n-1}(r) \subset \mathbb{R}^n$ the usual sphere of dimension $n-1$ and radius r, $\mathbb{S}^{n-1} := \mathbb{S}^{n-1}(1)$ and by $\left|\mathbb{S}^{n-1}\right|$ its measure. We can easily compute the measure of $\mathcal{S}^N(r,z)$ by

$$\left|\mathcal{S}^N(r,z)\right| = \left|\mathbb{S}^{d(N-1)-1}\right| \left(r^2 - \frac{|z|^2}{N} \right)_+^{\frac{d(N-1)-1}{2}}, \tag{1.20}$$

For $V = (v_1, \dots, v_N) \in \mathbb{R}^{dN}$, we shall use through the paper the notation $V_\ell = (v_1, \dots, v_\ell) \in \mathbb{R}^{d\ell}$, $V_{\ell,N} = (v_{\ell+1}, \dots, v_N) \in \mathbb{R}^{d(N-\ell)}$ and $\bar{V}_\ell = \sum_{i=1}^{\ell} v_i \in \mathbb{R}^d$.

We begin with the following result of a change of variables, proved in Appendix 1.A.1.

Lemma 1.9. *Consider $V \in \mathcal{S}^N(r,z)$. We can make a change of coordinates $(v_1, \dots, v_N) \to (u_1, \dots, u_N)$ in the following way*

$$\begin{aligned} u_N &= \frac{1}{\sqrt{N}}(v_1 + \cdots + v_N) \\ u_k &= \frac{1}{\sqrt{k(k+1)}}(v_1 + \cdots + v_k - k\, v_{k+1}), \quad 1 \le k \le N-1, \end{aligned} \tag{1.21}$$

such that the Jacobian is equal to one, $|u_1|^2 + \cdots + |u_N|^2 = |v_1|^2 + \cdots + |v_N|^2$ *and*

$$\begin{cases} |v_1|^2 + \cdots + |v_N|^2 = r^2 \\ v_{1,\alpha} + \cdots + v_{N,\alpha} = z_\alpha \end{cases} \quad \to \quad \begin{cases} |u_1|^2 + \cdots + |u_{N-1}|^2 = r^2 - \dfrac{|z|^2}{N} \\ u_{N,\alpha} = \dfrac{z_\alpha}{\sqrt{N}}, \quad 1 \le \alpha \le d. \end{cases} \tag{1.22}$$

With these definitions and notations at hand we can study some properties of the uniform probability measure γ^N on $\mathcal{S}^N_{\mathcal{B}}$. We remark that these estimates can also be obtained using correlation operators on the Boltzmann's sphere as in Carlen, Carvalho and Loss [14].

Lemma 1.10. *We have the following properties*

(i) for any $\ell \le N-1$ the ℓ-marginal of γ^N is given by $\gamma^N_\ell(dV_\ell) = \gamma^N_\ell(V_\ell)\, dV_\ell$ with

$$\gamma^N_\ell(V_\ell) = \frac{\left|\mathbb{S}^{d(N-\ell-1)-1}\right|}{\left|\mathbb{S}^{d(N-1)-1}\right|} \frac{N^{\frac{d}{2}}}{(N-\ell)^{\frac{d}{2}}} \frac{\left(dN - |V_\ell|^2 - \frac{|\bar V_\ell|^2}{N-\ell}\right)_+^{\frac{d(N-\ell-1)-2}{2}}}{(dN)^{\frac{d(N-1)-2}{2}}}, \tag{1.23}$$

where $dV_\ell = dv_1 \dots dv_\ell$ is the Lebesgue measure on $\mathbb{R}^{d\ell}$.

(ii) the moments of γ^N_ℓ are uniformly bounded in N, more precisely, for $k \ge 1$ we have $M_k(\gamma^N_\ell) \le C_{d,k,\ell}$, where $C_{d,k,\ell}$ depends on d, k and ℓ.

Before the proof, we refer to [35] where a Fubini-like theorem on $\mathcal{S}^N(r,z)$ is proved, which yields a generalization of (1.23) for the ℓ-marginal of $\gamma^N_{r,z}$.

Proof. Let us split the proof.

(i) We can define $\gamma^N_{r,z}$ by

$$\gamma^N_{r,z} := \frac{1}{Z^N_{r,z}} \lim_{h\to 0} \frac{1}{h}\left(\mathbf{1}_{B^N_z(r+h)} - \mathbf{1}_{B^N_z(r)}\right), \quad B^N_z(r) := \{V \in \mathbb{R}^{dN}; |V| \le r, \sum_{i=1}^N v_i = z\},$$

where $Z^N_{r,z}$ is the normalization constant so that the integral of $\gamma^N_{r,z}$ is one.

Consider $\varphi \in C(\mathbb{R}^{d\ell})$, for $\ell \le N-1$, then

$$\begin{aligned} &\left\langle \mathbf{1}_{B^N_z(r)}, \varphi \otimes \mathbf{1}^{N-\ell} \right\rangle \\ &= \int_{\mathbb{R}^{dN}} \mathbf{1}_{|V_\ell|^2 + |V_{\ell,N}|^2 \le r^2}\, \mathbf{1}_{\bar V_\ell + v_{\ell+1} + \cdots + v_N = z}\, \varphi(V_\ell)\, dV_\ell\, dV_{\ell,N} \\ &= \int_{\mathbb{R}^{d\ell}} \varphi(V_\ell) \left(\int_{\mathbb{R}^{d(N-\ell)}} \mathbf{1}_{|V_{\ell,N}|^2 \le r^2 - |V_\ell|^2} \mathbf{1}_{v_{\ell+1} + \cdots + v_N = z - \bar V_\ell}\, dV_{\ell,N} \right) dV_\ell \\ &= \int_{\mathbb{R}^{d\ell}} \varphi(V_\ell) \left|\mathbb{B}^{d(N-\ell-1)}\right| \left(r^2 - |V_\ell|^2 - \frac{|z - \bar V_\ell|^2}{N-\ell} \right)_+^{\frac{d(N-\ell-1)}{2}} dV_\ell, \end{aligned}$$

where $|\mathbb{B}^{d(N-\ell-1)}|$ is the measure of the unit ball in dimension $d(N-\ell-1)$. We deduce then that the ℓ-marginal of $\gamma^N_{r,z}$, denoted by $\Pi_\ell(\gamma^N_{r,z})$, is given by

$$\begin{aligned}
\Pi_\ell(\gamma^N_{r,z}) &= \frac{1}{Z^N_{r,z}}\frac{d}{dr}\left[\left|\mathbb{B}^{d(N-\ell-1)}\right|\left(r^2-|V_\ell|^2-\frac{|z-\bar{V}_\ell|^2}{N-\ell}\right)_+^{\frac{d(N-\ell-1)}{2}}\right] \\
&= \frac{\left|\mathbb{B}^{d(N-\ell-1)}\right|}{Z^N_{r,z}} d(N-\ell-1)\, r\left(r^2-|V_\ell|^2-\frac{|z-\bar{V}_\ell|^2}{N-\ell}\right)_+^{\frac{d(N-\ell-1)-2}{2}} \\
&= \frac{\left|\mathbb{S}^{d(N-\ell-1)-1}\right|}{Z^N_{r,z}} r\left(r^2-|V_\ell|^2-\frac{|z-\bar{V}_\ell|^2}{N-\ell}\right)_+^{\frac{d(N-\ell-1)-2}{2}}
\end{aligned}$$

and in the particular case $r^2 = dN$, $z = 0$

$$\Pi_\ell(\gamma^N) = \gamma^N_\ell = \frac{\left|\mathbb{S}^{d(N-\ell-1)-1}\right|}{Z^N_{\sqrt{dN},0}}(dN)^{1/2}\left(dN-|V_\ell|^2-\frac{|\bar{V}_\ell|^2}{N-\ell}\right)_+^{\frac{d(N-\ell-1)-2}{2}}. \tag{1.24}$$

Now we shall compute $Z^N := Z^N_{\sqrt{dN},0}$, with

$$Z^N = \left|\mathbb{S}^{d(N-\ell-1)-1}\right|(dN)^{1/2}\int_{\mathbb{R}^{d\ell}}\left(dN-|V_\ell|^2-\frac{|\bar{V}_\ell|^2}{N-\ell}\right)_+^{\frac{d(N-\ell-1)-2}{2}} dV_\ell. \tag{1.25}$$

We start by the integral

$$A = \int_{\mathbb{R}^{d\ell}}\left(dN-|V_\ell|^2-\frac{|\bar{V}_\ell|^2}{N-\ell}\right)_+^{\frac{d(N-\ell-1)-2}{2}} dV_\ell,$$

with the changement of variable (1.21)-(1.22) (replacing N by ℓ), with the notation $U = U_{\ell-1} = (u_1,\dots,u_{\ell-1})$ and $x = u_\ell$ to simplify, we obtain

$$A = \int_{\mathbb{R}^{d\ell}}\left(dN-|U|^2-\frac{N}{N-\ell}|x|^2\right)_+^{\frac{d(N-\ell-1)-2}{2}} dU\,dx.$$

Changing U to spherical coordinates in dimension $d(\ell-1)$, we have

$$\begin{aligned}
A &= \int_{\mathbb{R}^d}\int_0^\infty |\mathbb{S}^{d(\ell-1)-1}|\left(dN-\rho^2-\frac{N}{N-\ell}|x|^2\right)_+^{\frac{d(N-\ell-1)-2}{2}}\rho^{d(\ell-1)-1}\,d\rho\,dx \\
&= |\mathbb{S}^{d(\ell-1)-1}|\int_0^\infty\left(\int_{\mathbb{R}^d}\left(dN-\rho^2-\frac{N}{N-\ell}|x|^2\right)_+^{\frac{d(N-\ell-1)-2}{2}} dx\right)\rho^{d(\ell-1)-1}\,d\rho.
\end{aligned} \tag{1.26}$$

Looking first to the integral over $\mathbb{R}^d$ we obtain, changing x to spherical coordinates in dimension d,

$$\begin{aligned} B &= \int_{\mathbb{R}^d} \left(dN - \rho^2 - \frac{N}{N-\ell}|x|^2 \right)_+^{\frac{d(N-\ell-1)-2}{2}} dx \\ &= |\mathbb{S}^{d-1}| \int_0^\infty \left(dN - \rho^2 - \frac{N}{N-\ell} y^2 \right)_+^{\frac{d(N-\ell-1)-2}{2}} y^{d-1} dy, \end{aligned}$$

and after some computations we get

$$\begin{aligned} B &= \frac{|\mathbb{S}^{d-1}|}{2} \left(\frac{N-\ell}{N} \right)^{d/2} (dN - \rho^2)_+^{\frac{d(N-\ell)-2}{2}} \int_0^1 (1-y)^{\frac{d(N-\ell-1)-2}{2}} y^{\frac{d-2}{2}} \, dy \\ &= \frac{|\mathbb{S}^{d-1}|}{2} \left(\frac{N-\ell}{N} \right)^{d/2} (dN - \rho^2)_+^{\frac{d(N-\ell)-2}{2}} \frac{\Gamma\left(\frac{d(N-\ell-1)-2}{2}+1\right)\Gamma\left(\frac{d-2}{2}+1\right)}{\Gamma\left(\frac{d(N-\ell-1)-2}{2}+\frac{d-2}{2}+2\right)}. \end{aligned}$$

Plugging this expression in (1.26) we get

$$\begin{aligned} A = |\mathbb{S}^{d(\ell-1)-1}| \frac{|\mathbb{S}^{d-1}|}{2} \left(\frac{N-\ell}{N} \right)^{d/2} & \frac{\Gamma\left(\frac{d(N-\ell-1)-2}{2}+1\right)\Gamma\left(\frac{d-2}{2}+1\right)}{\Gamma\left(\frac{d(N-\ell-1)-2}{2}+\frac{d-2}{2}+2\right)} \\ \times \int_0^\infty (dN-\rho^2)_+^{\frac{d(N-\ell)-2}{2}} \rho^{d(\ell-1)-1} \, d\rho, & \end{aligned}$$

and we can compute the last integral

$$\begin{aligned} C &:= \int_0^\infty (dN-\rho^2)_+^{\frac{d(N-\ell)-2}{2}} \rho^{d(\ell-1)-1} \, d\rho \\ &= \frac{1}{2}(dN)^{\frac{d(N-1)-2}{2}} \frac{\Gamma\left(\frac{d(N-\ell)-2}{2}+1\right)\Gamma\left(\frac{d(\ell-1)-2}{2}+1\right)}{\Gamma\left(\frac{d(N-\ell)-2}{2}+\frac{d(\ell-1)-2}{2}+2\right)}. \end{aligned}$$

Finally, plugging this in (1.25), we obtain

$$\begin{aligned} Z^N = \left|\mathbb{S}^{d(N-\ell-1)-1}\right| \left|\mathbb{S}^{d(\ell-1)-1}\right| \frac{\left|\mathbb{S}^{d-1}\right|}{2} \left(\frac{N-\ell}{N} \right)^{d/2} \frac{1}{2}(dN)^{\frac{d(N-1)-1}{2}} \\ \times \frac{\Gamma\left(\frac{d(N-\ell-1)}{2}\right)\Gamma\left(\frac{d}{2}\right)}{\Gamma\left(\frac{d(N-\ell)}{2}\right)} \frac{\Gamma\left(\frac{d(N-\ell)}{2}\right)\Gamma\left(\frac{d(\ell-1)}{2}\right)}{\Gamma\left(\frac{d(N-1)}{2}\right)} \end{aligned}$$

and using the fact that

$$|\mathbb{S}^{n-1}| = \frac{2\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)} \tag{1.27}$$

we have

$$Z^N = \left|\mathbb{S}^{d(N-1)-1}\right| (dN)^{\frac{d(N-1)-1}{2}} \left(\frac{N-\ell}{N} \right)^{d/2}, \tag{1.28}$$

then we conclude by plugging (1.28) in (1.24).

(ii) Let $k \geq 1$ be a even integer. We have then to compute $M_k(\gamma_\ell^N)$

$$\begin{aligned}\int_{\mathbb{R}^{d\ell}} |V_\ell|^k \, \gamma_\ell^N(V_\ell)\, dV_\ell = \frac{\left|\mathbb{S}^{d(N-\ell-1)-1}\right|}{\left|\mathbb{S}^{d(N-1)-1}\right|} \frac{\left(\frac{N}{N-\ell}\right)^{\frac{d}{2}}}{(dN)^{\frac{d(N-1)-2}{2}}} \\ \times \int_{\mathbb{R}^{d\ell}} |V_\ell|^k \left(dN - |V_\ell|^2 - \frac{|\bar{V}_\ell|^2}{N-\ell}\right)_+^{\frac{d(N-\ell-1)-2}{2}} dV_\ell.\end{aligned} \tag{1.29}$$

As in the proof of (i), we use the change of coordinates (1.21)-(1.22), then to simplify we denote $U = U_{\ell-1} = (u_1, \ldots, u_{\ell-1})$ and $x = u_\ell$. Hence we can compute the integral

$$\begin{aligned}A_k &= \int_{\mathbb{R}^{d\ell}} |V_\ell|^k \left(dN - |V_\ell|^2 - \frac{|\bar{V}_\ell|^2}{N-\ell}\right)_+^{\frac{d(N-\ell-1)-2}{2}} dV_\ell \\ &= \int_{\mathbb{R}^{d\ell}} \left(|U|^2 + |x|^2\right)^{\frac{k}{2}} \left(dN - |U|^2 - \frac{N}{N-\ell}|x|^2\right)_+^{\frac{d(N-\ell-1)-2}{2}} dU\, dx.\end{aligned}$$

With another change of coordinates, U to spherical coordinates in dimension $d(\ell-1)$, x also to spherical coordinates in dimension d we have

$$\begin{aligned}A_k &= \left|\mathbb{S}^{d(\ell-1)-1}\right| \left|\mathbb{S}^{d-1}\right| \int_0^\infty\!\!\int_0^\infty \left(\rho^2 + y^2\right)^{\frac{k}{2}} \left(dN - \rho^2 - \frac{N}{N-\ell}y^2\right)_+^{\frac{d(N-\ell-1)-2}{2}} \rho^{d(\ell-1)-1} y^{d-1} d\rho\, dy \\ &\leq C \left|\mathbb{S}^{d(\ell-1)-1}\right| \left|\mathbb{S}^{d-1}\right| \int_0^\infty \rho^k \left\{\int_0^\infty \left(dN - \rho^2 - \frac{N}{N-\ell}y^2\right)_+^{\frac{d(N-\ell-1)-2}{2}} y^{d-1}\, dy\right\} \rho^{d(\ell-1)-1}\, d\rho \\ &+ C \left|\mathbb{S}^{d(\ell-1)-1}\right| \left|\mathbb{S}^{d-1}\right| \int_0^\infty \left\{\int_0^\infty y^k \left(dN - \rho^2 - \frac{N}{N-\ell}y^2\right)_+^{\frac{d(N-\ell-1)-2}{2}} y^{d-1}\, dy\right\} \rho^{d(\ell-1)-1}\, d\rho \\ &=: I_1 + I_2.\end{aligned}$$

For the first term we have (already computed in (i))

$$\begin{aligned}I_1 &= \frac{1}{2} \left|\mathbb{S}^{d(\ell-1)-1}\right| \left|\mathbb{S}^{d-1}\right| \left(\frac{N-\ell}{N}\right)^{\frac{d}{2}} \frac{\Gamma\left(\frac{d(N-\ell-1)}{2}\right)\Gamma\left(\frac{d}{2}\right)}{\Gamma\left(\frac{d(N-\ell)}{2}\right)} \\ &\quad\times \int_0^\infty (dN - \rho^2)^{\frac{d(N-\ell)-2}{2}} \rho^{d(\ell-1)-1+k}\, d\rho \\ &= \frac{1}{2} \left|\mathbb{S}^{d(\ell-1)-1}\right| \left|\mathbb{S}^{d-1}\right| \left(\frac{N-\ell}{N}\right)^{\frac{d}{2}} \frac{\Gamma\left(\frac{d(N-\ell-1)}{2}\right)\Gamma\left(\frac{d}{2}\right)}{\Gamma\left(\frac{d(N-\ell)}{2}\right)} \\ &\quad\times \frac{1}{2}(dN)^{\frac{d(N-1)-2+k}{2}} \frac{\Gamma\left(\frac{d(N-\ell)}{2}\right)\Gamma\left(\frac{d(\ell-1)+k}{2}\right)}{\Gamma\left(\frac{d(N-1)+k}{2}\right)}.\end{aligned}$$

In the same way, we can compute the second term to get

$$
\begin{aligned}
I_2 &= \frac{1}{2}\left|\mathbb{S}^{d(\ell-1)-1}\right|\left|\mathbb{S}^{d-1}\right|\left(\frac{N-\ell}{N}\right)^{\frac{d+k}{2}} \frac{\Gamma\left(\frac{d(N-\ell-1)}{2}\right)\Gamma\left(\frac{d+k}{2}\right)}{\Gamma\left(\frac{d(N-\ell)+k}{2}\right)} \\
&\quad \times \int_0^\infty (dN-\rho^2)^{\frac{d(N-\ell)-2+k}{2}} \rho^{d(\ell-1)-1}\, d\rho \\
&= \frac{1}{2}\left|\mathbb{S}^{d(\ell-1)-1}\right|\left|\mathbb{S}^{d-1}\right|\left(\frac{N-\ell}{N}\right)^{\frac{d+k}{2}} \frac{\Gamma\left(\frac{d(N-\ell-1)}{2}\right)\Gamma\left(\frac{d+k}{2}\right)}{\Gamma\left(\frac{d(N-\ell)+k}{2}\right)} \\
&\quad \times \frac{1}{2}(dN)^{\frac{d(N-1)-2+k}{2}} \frac{\Gamma\left(\frac{d(N-\ell)+k}{2}\right)\Gamma\left(\frac{d(\ell-1)}{2}\right)}{\Gamma\left(\frac{d(N-1)+k}{2}\right)}.
\end{aligned}
$$

Plugging this two estimates in (1.29) we obtain after some simplifications

$$
\begin{aligned}
M_k(\gamma_\ell^N) &\leq \frac{\left|\mathbb{S}^{d(N-\ell-1)-1}\right|}{\left|\mathbb{S}^{d(N-1)-1}\right|} \frac{\left(\frac{N}{N-\ell}\right)^{\frac{d}{2}}}{(dN)^{\frac{d(N-1)-2}{2}}} (I_1+I_2) \\
&\leq (dN)^{\frac{k}{2}} \frac{\Gamma\left(\frac{d(N-1)}{2}\right)}{\Gamma\left(\frac{d(N-1)+k}{2}\right)} \frac{\Gamma\left(\frac{d(\ell-1)+k}{2}\right)}{\Gamma\left(\frac{d(\ell-1)}{2}\right)} + (dN)^{\frac{k}{2}} \frac{\Gamma\left(\frac{d(N-1)}{2}\right)}{\Gamma\left(\frac{d(N-1)+k}{2}\right)} \frac{\Gamma\left(\frac{d+k}{2}\right)}{\Gamma\left(\frac{d}{2}\right)}.
\end{aligned}
$$

Using the fact that for k even we have

$$
\begin{aligned}
\Gamma\left(\frac{n}{2}+\frac{k}{2}\right) &= \frac{(n+k-2)}{2}\frac{(n+k-4)}{2}\cdots\frac{n}{2}\,\Gamma\left(\frac{n}{2}\right) \\
&= \frac{1}{2^{\frac{k}{2}}} \underbrace{(n+k-2)(n+k-4)\cdots n}_{k/2 \text{ terms}}\,\Gamma\left(\frac{n}{2}\right),
\end{aligned}
$$

we conclude that

$$
\begin{aligned}
M_k(\gamma_\ell^N) &\leq \frac{(dN)^{\frac{k}{2}}}{[d(N-1)+k-2][d(N-1)+k-4]\cdots[d(N-1)]} \\
&\quad \times \Big([d(\ell-1)+k-2][d(\ell-1)+k-4]\cdots[d(\ell-1)] \\
&\qquad + (d+k-2)(d+k-4)\cdots d\Big) \\
&\leq \frac{(dN)^{\frac{k}{2}}}{[d(N-1)]^{\frac{k}{2}}}\Big([d(\ell-1)+k-2][d(\ell-1)+k-4]\cdots[d(\ell-1)] \\
&\qquad + (d+k-2)(d+k-4)\cdots d\Big) \\
&\leq 2^{\frac{k}{2}}\Big([d(\ell-1)+k-2][d(\ell-1)+k-4]\cdots[d(\ell-1)] \\
&\qquad + (d+k-2)(d+k-4)\cdots d\Big) \\
&\leq C_{d,k,\ell},
\end{aligned} \tag{1.30}
$$

where $C_{d,k,\ell}$ depends only on d, k and ℓ.

We proved then a uniform bound in N for k even. If k is odd we use $|v|^k \leq |v|^{k-1} + |v|^{k+1}$ with the last estimate to conclude.

$\square$

Now, using this explicit formula for γ_ℓ^N computed above, we prove that γ^N is γ-chaotic, where γ is the Gaussian probability measure in $\mathbb{R}^d$, i.e. $\gamma(v) = (2\pi)^{-d/2}\, e^{-|v|^2/2}$, for $v \in \mathbb{R}^d$. The proof presented here is an adaptation of [30], where it is proved that the uniform probability measure on the sphere $\mathbb{S}^{n-1}(\sqrt{n}) \subset \mathbb{R}^n$ is γ_1-chaotic, with $\gamma_1(x) = (2\pi)^{-1/2}\, e^{-x^2/2}$ the one-dimensional Gaussian measure.

Lemma 1.11. *The sequence of probability measures $\gamma^N \in \mathbf{P}(\mathcal{S}_\mathcal{B}^N)$ is γ-chaotic, more precisely, for any integer ℓ such that $d\ell \leq d(N-2) - 3$ we have*

$$\|\gamma_\ell^N - \gamma^{\otimes \ell}\|_{L^1} \leq 2\, \frac{d(\ell+2)+2}{dN - d(\ell+2) - 2}.$$

Proof. Let ℓ be an even integer. Then we have

$$\begin{aligned}\frac{\left|\mathbb{S}^{d(N-\ell-1)-1}\right|}{\left|\mathbb{S}^{d(N-1)-1}\right|} &= \frac{1}{\pi^{\frac{d\ell}{2}}}\, \frac{\Gamma\left(\frac{d(N-1)}{2}\right)}{\Gamma\left(\frac{d(N-\ell-1)}{2}\right)} \\ &= \frac{(dN)^{\frac{d\ell}{2}}}{(2\pi)^{\frac{d\ell}{2}}} \left(1 - \frac{d+2}{dN}\right)\left(1 - \frac{d+4}{dN}\right)\cdots\left(1 - \frac{d(\ell+1)}{dN}\right).\end{aligned}$$

By the explicit formula of γ_ℓ^N in Lemma 1.10 we obtain

$$\gamma_\ell^N = \frac{\left(\frac{N}{N-\ell}\right)^{\frac{d}{2}}}{(2\pi)^{\frac{d\ell}{2}}} \left(1 - \frac{d+2}{dN}\right)\cdots\left(1 - \frac{d(\ell+1)}{dN}\right) \left(1 - \frac{|V_\ell|^2}{dN} - \frac{|\bar{V}_\ell|^2}{dN(N-\ell)}\right)_+^{\frac{d(N-\ell-1)-2}{2}}.$$

Since γ_ℓ^N and $\gamma^{\otimes \ell}$ are probability densities, the L^1 norm of their difference can be computed in the following way

$$\|\gamma_\ell^N - \gamma^{\otimes \ell}\|_{L^1} = 2 \int_{\mathbb{R}^{d\ell}} \left(\frac{\gamma_\ell^N}{\gamma^{\otimes \ell}} - 1\right)_+ \gamma^{\otimes \ell}\, dV_\ell, \tag{1.31}$$

and we shall denote

$$\frac{\gamma_\ell^N}{\gamma^{\otimes \ell}} = \left(\frac{N}{N-\ell}\right)^{\frac{d}{2}} h(V_\ell)\, A$$

with

$$h(V_\ell) := e^{\frac{|V_\ell|^2}{2}} \left(1 - \frac{|V_\ell|^2}{dN} - \frac{|\bar{V}_\ell|^2}{dN(N-\ell)}\right)_+^{\frac{d(N-\ell-1)-2}{2}}$$

and

$$A := \left(1 - \frac{d+2}{dN}\right)\cdots\left(1 - \frac{d(\ell+1)}{dN}\right).$$

We obtain that

$$\log h(V_\ell) = \frac{|V_\ell|^2}{2} + \frac{d(N-\ell-1)-2}{2} \log\left(1 - \frac{|V_\ell|^2}{dN} - \frac{|\bar{V}_\ell|^2}{dN(N-\ell)}\right)$$
$$\leq \frac{|V_\ell|^2}{2} + \frac{d(N-\ell-1)-2}{2} \log\left(1 - \frac{|V_\ell|^2}{dN}\right),$$

and since the function $\alpha(z) = z/2 + [(d(N-\ell-1)-2)/2]\log(1-z/dN)$ has a maximum for $z = d(\ell+1)+2$, we deduce

$$\log h(V_\ell) \leq \frac{d(\ell+1)+2}{2} + \frac{d(N-\ell-1)-2}{2} \log\left(1 - \frac{d(\ell+1)+2}{dN}\right), \tag{1.32}$$

for $d\ell \leq d(N-1)-3$.

On the other hand, for the quantity A, we have

$$\log\left[\left(1 - \frac{d(\ell+1)+2}{dN}\right) A\right] = \sum_{j=1}^{(d(\ell+1)+2)/2} \log\left(1 - \frac{2j}{dN}\right)$$
$$\leq \int_0^{(d(\ell+1)+2)/2} \log\left(1 - \frac{2x}{dN}\right) dx$$
$$= -\frac{d(N-\ell-1)-2}{2} \log\left(1 - \frac{d(\ell+1)+2}{dN}\right) - \frac{d(\ell+1)+2}{2}, \tag{1.33}$$

again for $d\ell \leq d(N-1)-3$.

Combining (1.32) and (1.33) we obtain

$$\log\left[h(V_\ell)\left(1 - \frac{d(\ell+1)+2}{dN}\right) A\right] \leq 0$$

and then

$$\left(1 - \frac{d(\ell+1)+2}{dN}\right) \frac{\gamma_\ell^N}{\gamma^{\otimes \ell}} \leq \frac{(N-\ell)^{\frac{d}{2}}}{N^{\frac{d}{2}}},$$

which implies

$$\frac{\gamma_\ell^N}{\gamma^{\otimes \ell}} - 1 \leq \frac{d(\ell+1)+2}{dN - d(\ell+1) - 2}.$$

Plugging this expression in (1.31) we deduce

$$\|\gamma_\ell^N - \gamma^{\otimes \ell}\|_{L^1} \leq \frac{2d(\ell+1)+4}{dN - d(\ell+1) - 2},$$

which is valid if ℓ is even.

Finally, if ℓ is odd, then $\ell+1$ is even and we shall write

$$\|\gamma_\ell^N - \gamma^{\otimes \ell}\|_{L^1} \leq \|\gamma_{\ell+1}^N - \gamma^{\otimes \ell+1}\|_{L^1} \leq 2\, \frac{d(\ell+2)+2}{dN - d(\ell+2) - 2}$$

for $d\ell \leq d(N-2)-3$, which concludes the proof.

□

1.3 Chaotic sequences in Kac's sense

In this section, inpired by the work [12], we shall construct a chaotic sequence of probability measures on the Boltzmann's sphere based on the tensorization of some suitable probability f on $\mathbb{R}^d$ and conditioning to $\mathcal{S}^N_{\mathcal{B}}$. We shall give a quantitative rate of the chaos convergence, proving a precise version of point (i) in Theorem 1.3.

First of all, we define

$$Z_N(f;r,z) = \int_{\mathcal{S}^N(r,z)} f^{\otimes N}\, d\gamma^N_{r,z}, \quad \text{and} \quad Z'_N(f;r,z) = \int_{\mathcal{S}^N(r,z)} \frac{f^{\otimes N}}{\gamma^{\otimes N}}\, d\gamma^N_{r,z}, \tag{1.34}$$

for $r \in \mathbb{R}_+$ and $z \in \mathbb{R}^d$, and we shall investigate their asymptotic behaviour. We remark that, since $\gamma^{\otimes N}$ is constant on $\mathcal{S}^N(r,z)$, we have

$$Z'_N(f;r,z) = \frac{Z_N(f;r,z)}{\gamma^{\otimes N}}$$

and we shall study in the sequel only the behaviour of $Z'_N(f;r,z)$.

Define the space $\mathbf{P}_k(\mathbb{R}^d) := \{f \in \mathbf{P}(\mathbb{R}^d);\ M_k(f) := \int |v|^k f\, dv < \infty\}$, for some $k \geq 1$. Let us consider $f \in \mathbf{P}_6(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$, for some $p > 1$, a probability measure that verifies

$$\begin{aligned} &\int_{\mathbb{R}^d} v f(v)\, dv = 0\,, && \int_{\mathbb{R}^d} v \otimes v\, f(v)\, dv = \mathcal{E} I_d, \\ &\int_{\mathbb{R}^d} |v|^2\, f(v)\, dv = d\mathcal{E} = E\,, && \int_{\mathbb{R}^d} (|v|^2 - E)^2 f(v)\, dv = \Sigma^2, \end{aligned} \tag{1.35}$$

where I_d is the d-dimensional identity matrix.

1.3.1 Preliminary results

Before study the asymptotic behaviour of Z'_N, we shall state some preliminary results that will be useful in the sequel.

Consider $(\mathcal{V}_j)_{j\in\mathbb{N}^*}$ a sequence of random variables i.i.d. in $\mathbb{R}^d$ with same law f, then the law of the couple $(\mathcal{V}_1, \mathcal{V}_1^2)$ is

$$h(v,u) = f(v)\, \delta_{u=|v|^2} \in \mathbf{P}(\mathbb{R}^d \times \mathbb{R}_+). \tag{1.36}$$

Moreover, we have the following lemma.

Lemma 1.12. *The random variable* $S_N := \sum_{j=1}^N (\mathcal{V}_j, |\mathcal{V}_j|^2)$ *has law* $s^N(z,u)\, dzdu$ *with*

$$s^N(z,u) := \frac{\left|\mathcal{S}^N(\sqrt{u}, z)\right|}{2\left(u - \frac{|z|}{N^2}\right)^{1/2} N^{d/2}}\, Z_N(f;\sqrt{u}, z),$$

where $z \in \mathbb{R}^d$ *and* $u \in \mathbb{R}_+$.

Proof. Let $\varphi \in C_b(\mathbb{R}^d \times \mathbb{R}_+)$, with the change of coordinates (1.21)-(1.22) $v \to u$, we have

$$\begin{aligned}\mathbb{E}\left[\varphi\left(\sum_{j=1}^N \mathcal{V}_j, \sum_{j=1}^N |\mathcal{V}_j|^2\right)\right] &= \int_{\mathbb{R}^{dN}} \varphi\left(\sum_{j=1}^N v_j, \sum_{j=1}^N |v_j|^2\right) f^{\otimes N}\, dV \\ &= \int_{\mathbb{R}^{dN}} \varphi\left(\sqrt{N} u_N, \sum_{j=1}^N |u_j|^2\right) f^{\otimes N}\, dU.\end{aligned}$$

Denoting $r^2 = \sum_{j=1}^{N-1} |u_j|^2$ and splitting the integral, the last equation is equal to

$$\int_0^\infty\!\!\int_{\mathbb{R}^d} \varphi(\sqrt{N}u_N, r^2 + |u_N|^2) \left\{\left|\mathbb{S}^{d(N-1)-1}(r)\right| \int_{\mathbb{S}^{d(N-1)-1}(r)} f^{\otimes N}\, d\sigma_r^{d(N-1)-1}\right\} du_N\, dr$$

where σ_R^{n-1} is the uniform probability measure on $\mathbb{S}^{n-1}(R)$. Making the change of coordinates $w = r^2 + |u_N|^2$ and $z = \sqrt{N}u_N$, we obtain

$$\begin{aligned}&\int_0^\infty\!\!\int_{\mathbb{R}^d} \varphi(z,w) \left\{\frac{\left|\mathbb{S}^{d(N-1)-1}\left(\sqrt{w - \frac{|z|^2}{N}}\right)\right|}{2\left(w - \frac{|z|}{N^2}\right)^{1/2} N^{d/2}} \int_{\mathbb{S}^{d(N-1)-1}\left(\sqrt{w - \frac{|z|^2}{N}}\right)} f^{\otimes N}\, d\sigma^{d(N-1)-1}_{\sqrt{w-|z|^2/N}}\right\} dz\, dw \\ &= \int_0^\infty\!\!\int_{\mathbb{R}^d} \varphi(z,w)\left\{\frac{\left|\mathcal{S}^N(\sqrt{w}, z)\right|}{2\left(w - \frac{|z|}{N^2}\right)^{1/2} N^{d/2}} Z_N(f; \sqrt{w}, z)\right\} dz\, dw,\end{aligned}$$

from which we conclude. $\square$

Since S_N is the summation of independent random variables, its law's density is also given by

$$s^N(z,u) = h^{*N}(z,u), \tag{1.37}$$

and we deduce from the lemma above

$$Z_N(f; \sqrt{u}, z) = \frac{2\left(u - \frac{|z|}{N^2}\right)^{1/2} N^{d/2}\, h^{(*N)}(z,u)}{|\mathcal{S}^N(\sqrt{u}, z)|}. \tag{1.38}$$

Lemma 1.13. *If $f \in \mathbf{P}_{2k}(\mathbb{R}^d)$ then $h \in \mathbf{P}_k(\mathbb{R}^{d+1})$.*

Proof. Let $y = (v,u) \in \mathbb{R}^{d+1}$ with $v \in \mathbb{R}^d$ and $u \in \mathbb{R}$. Then we have

$$\begin{aligned}\int_{\mathbb{R}^{d+1}} |y|^k\, h(y)\, dy &= \int_{\mathbb{R}^{d+1}} \left(|v|^2 + |u|^2\right)^{k/2} f(v) \delta_{u=|v|^2}\, dv\, du \\ &\le C_k \left(\int_{\mathbb{R}^d} |v|^k\, f(v)\, dv + \int_{\mathbb{R}^d} |v|^{2k}\, f(v)\, dz\right),\end{aligned}$$

from which we conclude. $\square$

Lemma 1.14. *Suppose $f \in L^p(\mathbb{R}^d)$ for some $p > 1$. Then $h^{*2} \in L^q(\mathbb{R}^{d+1})$ if*

(i) for $d=1$*:* $1<q<p$ *and* $q<\frac{2p}{p+1}$
(ii) for $d=2$*:* $q\le p$
(iii) for $d\ge 3$*: if* $f\in L_s(\mathbb{R}^d)$ $(s>0)$*, for* $q<p$ *and*

$$q=\frac{(d-2)(p-1)+sp}{(d-2)(p-1)+s}>1.$$

Proof. We compute first $h^{*2}(v,u)$ with $v,v'\in\mathbb{R}^d$ and $u,u'\in\mathbb{R}$.

$$\begin{aligned} h^{*2}(v,u) &= \int_{\mathbb{R}^d}\int_{\mathbb{R}} h(v-v',u-u')\,h(v',u')\,du'dv' \\ &= \int_{\mathbb{R}^d} f(v-v')\,f(v')\left\{\int_{\mathbb{R}} \delta_{u-u'=|v-v'|^2}\delta_{u'=|v'|^2}du'\right\}dv' \\ &= \int_{\mathbb{R}^d} f(v-v')\,f(v')\,\delta_{u=|v-v'|^2-|v'|^2}\,dv'. \end{aligned}$$

Moreover, we have

$$\delta_{u=|v-v'|^2-|v'|^2}=\delta_{u=2\left|\frac{v}{2}-v'\right|^2+\frac{|v|^2}{2}}.$$

Then we can compute the L^q norm of h^{*2},

$$\begin{aligned} &\int_{\mathbb{R}^d}\int_{\mathbb{R}}\left|h^{*2}(v,u)\right|^q dv\,du \\ &\quad= \int_{\mathbb{R}^d}\int_{\mathbb{R}}\left|\int_{\mathbb{R}^d} f(v-v')\,f(v')\,\delta_{u=2\left|\frac{v}{2}-v'\right|^2+\frac{|v|^2}{2}}\,dv'\right|^q dv\,du \\ &\quad\le \int_{\mathbb{R}^d}\int_{\mathbb{R}}\left|\left(\int_{\mathbb{R}^d}\delta_{\left|\frac{v}{2}-v'\right|^2=\frac{u}{2}-\frac{|v|^2}{4}}\,dv'\right)^{(q-1)/q}\left(\int_{\mathbb{R}^d} f(v-v')^q\,f(v')^q\,\delta_{\left|\frac{v}{2}-v'\right|^2=\frac{u}{2}-\frac{|v|^2}{4}}\,dv'\right)^{1/q}\right|^q dv\,du, \end{aligned} \tag{1.39}$$

where we used Holder's inequality.

We look to the integral over δ, using $w=\frac{v}{2}-v'$

$$\int_{\mathbb{R}^d}\delta_{|w|^2=\frac{u}{2}-\frac{|v|^2}{4}}\,dw=|\mathbb{S}^{d-1}|\int_{\mathbb{R}}\delta_{r^2=\frac{u}{2}-\frac{|v|^2}{4}}\,r^{d-1}\,dr$$

where we changed to polar coordinates and then, with $z=r^2$

$$\begin{aligned} \int_{\mathbb{R}^d}\delta_{|w|^2=\frac{u}{2}-\frac{|v|^2}{4}}\,dw &= \frac{|\mathbb{S}^{d-1}|}{2}\int_{\mathbb{R}}\delta_{z=\frac{u}{2}-\frac{|v|^2}{4}}\,z^{(d-2)/2}\,dz \\ &= \frac{|\mathbb{S}^{d-1}|}{2}\left(\frac{u}{2}-\frac{|v|^2}{4}\right)^{(d-2)/2}. \end{aligned} \tag{1.40}$$

Therefore we obtain, plugging (1.40) in (1.39) and using Fubbini,

$$\begin{aligned} &\int_{\mathbb{R}^d}\int_{\mathbb{R}}\left|h^{*2}(v,u)\right|^q dv\,du \\ &\quad\le \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(v-v')^q\,f(v')^q\left\{\int_{\mathbb{R}}\left[\frac{|\mathbb{S}^{d-1}|}{2}\left(\frac{u}{2}-\frac{|v|^2}{4}\right)^{(d-2)/2}\right]^{q-1}\delta_{u=2\left|\frac{v}{2}-v'\right|^2+\frac{|v|^2}{2}}\,du\right\}dv\,dv' \\ &\quad= \frac{|\mathbb{S}^{d-1}|^{q-1}}{2^{q-1}}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\left|\frac{v}{2}-v'\right|^{(d-2)(q-1)} f(v-v')^q\,f(v')^q\,dv\,dv' \qquad =: A \end{aligned}$$

Now we have the cases $d=1$, $d=2$ and $d\geq 3$:

(i) $d=1$. Splitting the expression, we have

$$\begin{aligned} A &\leq \int_{\left|\frac{v}{2}-v'\right|\leq 1} \frac{f(v-v')^q\, f(v')^q}{\left|\frac{v}{2}-v'\right|^{q-1}}\, dv\, dv' + \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(v-v')^q\, f(v')^q dv\, dv' \\ &=: T_1 + T_2. \end{aligned}$$

For the last estimate we have $T_2 \leq \|f\|_{L^q}^{2q} \leq \|f\|_{L^p}^{2q}$ (because $q<p$ and f is a probability measure), and for the first term we use Holder's inequality

$$T_1 \leq \left(\int_{\left|\frac{v}{2}-v'\right|\leq 1} \frac{1}{\left|\frac{v}{2}-v'\right|^{(q-1)p/(p-q)}}\, dv\, dv'\right)^{(p-q)/p} \left(\int_{\left|\frac{v}{2}-v'\right|\leq 1} f(v-v')^p\, f(v')^p\, dv\, dv'\right)^{q/p}.$$

Then, the first integral converges if $(q-1)p/(p-q)<1$, which give us $T_1 \leq C\|f\|_{L^p}^{2q}$ if

$$q < \frac{2p}{p+1}.$$

(ii) $d=2$. In this case we have

$$\begin{aligned} A &\leq \frac{|\mathbb{S}^1|^{q-1}}{2^{q-1}} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(v-v')^q\, f(v')^q dv\, dv' \\ &= \frac{|\mathbb{S}^1|^{q-1}}{2^{q-1}} \|f\|_{L^q}^{2q} \leq \frac{|\mathbb{S}^1|^{q-1}}{2^{q-1}} \|f\|_{L^p}^{2q}. \end{aligned}$$

(iii) $d=3$. We have, using $w=v-v'$ and $u=v'$

$$\begin{aligned} A &= \frac{|\mathbb{S}^{d-1}|^{q-1}}{2^{q-1}} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \left|\frac{v}{2}-v'\right|^{(d-2)(q-1)} f(v-v')^q\, f(v')^q dv\, dv' \\ &= \frac{|\mathbb{S}^{d-1}|^{q-1}}{2^{q-1}} \frac{1}{2^{(d-2)(q-1)}} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |w-u|^{(d-2)(q-1)}\, f(w)^q\, f(u)^q dw\, du \\ &\leq \frac{|\mathbb{S}^{d-1}|^{q-1}}{2^{(d-1)(q-1)}} \left\{ 2C \left(\int_{\mathbb{R}^d} |w|^{(d-2)(q-1)}\, f(w)^q\, dw\right) \left(\int_{\mathbb{R}^d} f(u)^q\, du\right) \right\} \\ &\leq C\|f\|_{L^q}^q \|f\|_{L^q_m}^q \end{aligned}$$

where we have used $|w-u|^{(d-2)(q-1)} \leq C\left(|w|^{(d-2)(q-1)} + |u|^{(d-2)(q-1)}\right)$ and $m=(d-2)(q-1)$.

Finally, we have $\|f\|_{L^q}^q \leq \|f\|_{L^p}^q$ and with the hypothesis $f\in L^p\cap L_s$, we have $\|f\|_{L^q_m}<\infty$ for $m=s(p-q)/(p-1)$ and $q<p$ (see Lemma 1.34 in Appendix 1.A.2), more precisely for

$$q = \frac{(d-2)(p-1)+sp}{(d-2)(p-1)+s} > 1.$$

$\square$

1.3.2 Asymptotic behaviour of Z'_N

In this section we shall study the behaviour of Z'_N when N goes to infinity. First of all, let us state a version of the Central Limit Theorem, also known as Berry-Esseen type theorem, which is the main ingredient of the proof of the asymptotic of Z'_N in Theorem 1.17. The proof of the CLT presented here is a slightly adaptation of [46, Theorem 4.6] (see also [12, Theorem 27]).

Theorem 1.15 (Central Limit Theorem)**.** *Let $g \in \mathbf{P}_3(\mathbb{R}^D)$ such that, for some integer $k \geq 1$, we have $g^{*k} \in L^p(\mathbb{R}^D)$ for some $p > 1$. Moreover, assume that*

$$\int_{\mathbb{R}^D} x\, g(x)\, dx = 0, \qquad \int_{\mathbb{R}^D} (x \otimes x)\, g(x)\, dx = I_D, \qquad \int_{\mathbb{R}^D} |x|^3 g(x)\, dx \leq C_3. \tag{1.41}$$

Then there exists a constant $C = C(D, p, \|g^{*k}\|_{L^p}) > 0$ and $N(k,p)$ such that for all $N > N(k,p)$ we have

$$\|g_N - \gamma\|_{L^\infty} = \sup_{x \in \mathbb{R}^D} |g_N(x) - \gamma(x)| \leq \frac{C}{\sqrt{N}},$$

where $g_N(x) = N^{D/2} g^{*N}(\sqrt{N}x)$ is the normalized N-convolution power of g.

In the sequel we will need the following lemma, and we refer again to [12, Proposition 26] and [46, Lemma 4.8] for its proof.

Lemma 1.16. *(i) Consider $g \in \mathbf{P}_3(\mathbb{R}^D)$ satisfying (1.41). Then, there exists $\delta \in (0,1)$ such that*

$$\forall\, \xi \in B(0,\delta) \qquad |\widehat{g}(\xi)| \leq e^{-|\xi|^2/4}.$$

(ii) Consider $g \in \mathbf{P}(\mathbb{R}^D) \cap L^p(\mathbb{R}^D)$ for $1 < p \leq \infty$. For any $\delta > 0$ there exists $\kappa(\delta) = \kappa(M_3(g), \|g\|_{L^p}, \delta) \in (0,1)$ such that

$$\sup_{|\xi| \geq \delta} |\widehat{g}(\xi)| \leq \kappa(\delta).$$

Proof of Theorem 1.15. We remark that

$$\widehat{g}_N(\xi) = \widehat{g}\left(\frac{\xi}{\sqrt{N}}\right)^N, \qquad \widehat{\gamma}_N(\xi) = \widehat{\gamma}\left(\frac{\xi}{\sqrt{N}}\right)^N.$$

We have $g^{*k} \in L^1 \cap L^p$, for $p \in (1,\infty]$, and then by the Hausdorff-Young inequality we deduce that $\widehat{(g^{*k})} = (\widehat{g})^k$ lies in $L^{p'} \cap L^\infty$ with $p' \in (1,\infty]$. Furthermore, $\widehat{g}_N(\xi) \in L^1$ for any $N \geq kp'$. Hence we shall use the inverse Fourier transform to write

$$\begin{aligned} |g_N(x) - \gamma(x)| &= (2\pi)^D \left| \int_{\mathbb{R}^D} e^{i\xi\cdot x} \left(\widehat{g}_N(\xi) - \widehat{\gamma}(\xi)\right)\, d\xi \right| \\ &\leq (2\pi)^D \int_{\mathbb{R}^D} |\widehat{g}_N(\xi) - \widehat{\gamma}(\xi)|\, d\xi. \end{aligned} \tag{1.42}$$

Spliting the last integral in low and high frequencies, we obtain

$$\begin{aligned}\int_{\mathbb{R}^D} |\widehat{g}_N(\xi) - \widehat{\gamma}(\xi)| \, d\xi &\leq \int_{|\xi|\geq\sqrt{N}\delta} |\widehat{g}_N(\xi)| \, d\xi + \int_{|\xi|\geq\sqrt{N}\delta} |\widehat{\gamma}(\xi)| \, d\xi \\ &\quad + \int_{|\xi|<\sqrt{N}\delta} |\widehat{g}_N(\xi) - \widehat{\gamma}(\xi)| \, d\xi \\ &=: T_1 + T_2 + T_3,\end{aligned}$$

for some $\delta \in (0,1)$.

For the first term, we write

$$\begin{aligned}T_1 &\leq \int_{|\xi|\geq\sqrt{N}\delta} \left|\widehat{g}\left(\frac{\xi}{\sqrt{N}}\right)\right|^N d\xi = N^{D/2} \int_{|\eta|\geq\delta} |\widehat{g}(\eta)| \, d\eta \\ &\leq N^{D/2} \left(\sup_{\eta\geq\delta} |\widehat{g}(\eta)^k|\right)^{N/k-p'} \int_{|\eta|\geq\delta} |\widehat{g}(\eta)^k|^{p'} \, d\eta \\ &\leq N^{D/2} \kappa(\delta)^{N/k-p'} C_{D,p} \|g^{*k}\|_{L^p}^{p'}\end{aligned}$$

where $\delta \in (0,1)$ is given by Lemma 1.16-(i) and $\kappa(\delta)$ is given by Lemma 1.16-(ii) applied to g^{*k} (because we have supposed only $g^{*k} \in L^p$). We get the same estimate for the second term, then we obtain that there exists a constant $C = C(D, p, \|g^{*k}\|_{L^p})$ such that

$$T_1 + T_2 \leq \frac{C}{\sqrt{N}}.$$

Finally, for the third term we have

$$T_3 = \int_{|\xi|<\sqrt{N}\delta} \frac{|\widehat{g}_N(\xi) - \widehat{\gamma}(\xi)|}{|\xi|^3} |\xi|^3 \, d\xi$$

and we can estimate

$$\begin{aligned}\frac{|\widehat{g}_N(\xi) - \widehat{\gamma}(\xi)|}{|\xi|^3} &= \frac{1}{N^{3/2}} \frac{|\widehat{g}(\xi/\sqrt{N})^N - \widehat{\gamma}(\xi/\sqrt{N})^N|}{|\xi/\sqrt{N}|^3} \\ &= \frac{1}{N^{3/2}} \frac{|\widehat{g}(\xi/\sqrt{N}) - \widehat{\gamma}(\xi/\sqrt{N})|}{|\xi/\sqrt{N}|^3} \times \left|\sum_{k=0}^{N-1} \widehat{g}(\xi/\sqrt{N})^k \widehat{\gamma}(\xi/\sqrt{N})^{(N-k-1)}\right|.\end{aligned}$$

Moreover, point (i) in Lemma 1.16 implies

$$\left|\sum_{k=0}^{N-1} \widehat{g}(\xi/\sqrt{N})^k \widehat{\gamma}(\xi/\sqrt{N})^{(N-k-1)}\right| \leq \sum_{k=0}^{N-1} e^{-\frac{k|\xi|^2}{4N}} e^{-\frac{(N-k-1)|\xi|^2}{4N}} \leq N e^{-\frac{|\xi|^2}{8}}.$$

Hence, we obtain

$$\begin{aligned}T_3 &\leq \frac{1}{N^{3/2}} \left(\sup_{\eta} \frac{|\widehat{g}(\eta) - \widehat{\gamma}(\eta)|}{|\eta|^3}\right) \int_{\mathbb{R}^D} N e^{-\frac{|\xi|^2}{8}} |\xi|^3 \, d\xi \\ &\leq \frac{1}{\sqrt{N}} \left(M_3(g) + M_3(\gamma)\right) C_D,\end{aligned}$$

and we finish the proof gathering the estimates of T_1, T_2 and T_3 togheter with (1.42). $\square$

With these results we are able to state the following theorem about the asymptotic behaviour of Z'_N.

Theorem 1.17. *Consider $f \in \mathbf{P}_6(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$, with $p > 1$, satisfying (1.35). Then we have*

$$Z'_N(f; r, z) = \frac{\sqrt{2d}}{\Sigma \mathcal{E}^{d/2}} \frac{(dN)^{\frac{d(N-1)-2}{2}}}{\left(r^2 - \frac{|z|^2}{N}\right)^{\frac{d(N-1)-2}{2}}} \frac{e^{-\frac{dN}{2}}}{e^{-\frac{r^2}{2}}}$$
$$\times \left[\exp\left(-\frac{|z|^2}{2\mathcal{E}N} - \frac{(r^2 - NE)^2}{2\Sigma^2 N} \right) + O\left(1/\sqrt{N}\right) \right]$$

and in the particular case $r^2 = dN$ and $z = 0$, we have

$$Z'_N(f; \sqrt{dN}, 0) = \frac{\sqrt{2d}}{\Sigma \mathcal{E}^{d/2}} \left[\exp\left(-\frac{N(d-E)^2}{2\Sigma^2} \right) + O\left(1/\sqrt{N}\right) \right].$$

Proof. Let us introduce

$$g(v, u) = \Sigma\, \mathcal{E}^{d/2}\, h(\mathcal{E}^{1/2} v, E + \Sigma u) \in \mathbf{P}(\mathbb{R}^{d+1}),$$

with $v \in \mathbb{R}^d$ and $u \in \mathbb{R}$. Since h lies in $\mathbf{P}_3(\mathbb{R}^{d+1})$ by Lemma 1.13 and $h^{*2} \in L^q(\mathbb{R}^{d+1})$ for some $q \in (1, p)$ thanks to Lemma 1.14, we have $g \in \mathbf{P}_3(\mathbb{R}^{d+1})$ and $g^{*2} \in L^q(\mathbb{R}^{d+1})$.

Moreover g verifies (by construction)

$$\int_{\mathbb{R}^{d+1}} y\, g(y)\, dy = 0\,, \quad \int_{\mathbb{R}^{d+1}} (y \otimes y)\, g(y)\, dy = I_{d+1},$$

where I_{d+1} is the identity matrix in dimension $d + 1$.

We can now apply Theorem 1.15 to g, which implies that there exists $C > 0$ and N_0 such that for all $N > N_0$,

$$\sup_{(v,u) \in \mathbb{R}^d \times \mathbb{R}} |g_N(v, u) - \gamma(v, u)| \leq \frac{C}{\sqrt{N}},$$

where $g_N(v, u) = N^{(d+1)/2} g^{*N}(\sqrt{N} v, \sqrt{N} u)$ is the normalized N-convolution power of g, with

$$g^{*N}(\sqrt{N} v, \sqrt{N} u) = \Sigma\, \mathcal{E}^{d/2}\, h^{*N}(\mathcal{E}^{1/2} \sqrt{N} v, NE + \Sigma \sqrt{N} u),$$

and

$$\gamma(v, u) = \frac{e^{-|v|^2/2}}{(2\pi)^{d/2}} \frac{e^{-u^2/2}}{(2\pi)^{1/2}}$$

is the Gaussian measure in dimension $d + 1$ (recall that we have $v \in \mathbb{R}^d$ and $u \in \mathbb{R}$). It follows that

$$\sup_{(v,u) \in \mathbb{R}^d \times \mathbb{R}} \left| h^{*N}(v, u) - \frac{\Sigma^{-1} \mathcal{E}^{-d/2}}{N^{(d+1)/2}} \gamma\left(\mathcal{E}^{-1/2} N^{-1/2} v, \frac{u - NE}{\Sigma \sqrt{N}} \right) \right| \leq \frac{C}{\sqrt{N}} \frac{\Sigma^{-1} \mathcal{E}^{-d/2}}{N^{(d+1)/2}}. \tag{1.43}$$

Gathering (1.43) and (1.38) we obtain

$$Z_N(f;r,z) = \frac{2\,N^{d/2}\left(r^2-\frac{|z|}{N^2}\right)^{1/2}}{|\mathcal{S}^N(r,z)|}\frac{\Sigma^{-1}\mathcal{E}^{-d/2}}{N^{(d+1)/2}}\frac{1}{(2\pi)^{(d+1)/2}}\left[\exp\left(-\frac{|z|^2}{2\mathcal{E}N}-\frac{(r^2-NE)^2}{2\Sigma^2 N}\right)+O\left(1/\sqrt{N}\right)\right].$$

Using (1.20) we have

$$Z_N(f;r,z) = \frac{2\,N^{d/2}\left(r^2-\frac{|z|}{N^2}\right)^{1/2}}{\left|\mathbb{S}^{d(N-1)-1}\right|}\left(r^2-\frac{|z|^2}{N}\right)_+^{-\frac{d(N-1)-1}{2}}\frac{\Sigma^{-1}\mathcal{E}^{-d/2}}{N^{(d+1)/2}}\frac{1}{(2\pi)^{(d+1)/2}} \times\left[\exp\left(-\frac{|z|^2}{2\mathcal{E}N}-\frac{(r^2-NE)^2}{2\Sigma^2 N}\right)+O\left(1/\sqrt{N}\right)\right].$$

Thanks to the formula

$$|\mathbb{S}^{n-1}| = \frac{2\pi^{n/2}}{\Gamma(n/2)}$$

and to Stirling's formula,

$$\Gamma\left(an+b\right) = \sqrt{2\pi}\,(an)^{\frac{an+b-1}{2}}\,e^{-an}\left(1+O(1/n)\right),$$

we have

$$\Gamma\left(\frac{d(N-1)}{2}\right) = \sqrt{2\pi}\,(dN)^{\frac{d(N-1)-1}{2}}\,2^{-\frac{d(N-1)-1}{2}}\,e^{-\frac{dN}{2}}\left(1+O(1/N)\right)$$

and then

$$Z_N(f;r,z) = \frac{\sqrt{2d}}{\Sigma\mathcal{E}^{d/2}}\left(\frac{e^{-\frac{r^2}{2}}}{(2\pi)^{\frac{dN}{2}}}\right)\frac{(dN)^{\frac{d(N-1)-2}{2}}}{\left(r^2-\frac{|z|^2}{N}\right)^{\frac{d(N-1)-2}{2}}}\frac{e^{-\frac{dN}{2}}}{e^{-\frac{r^2}{2}}} \times\left[\exp\left(-\frac{|z|^2}{2\mathcal{E}N}-\frac{(r^2-NE)^2}{2\Sigma^2 N}\right)+O\left(1/\sqrt{N}\right)\right],$$

which implies for the case $r^2 = dN$ and $z = 0$

$$Z_N'(f;\sqrt{dN},0) = \frac{\sqrt{2d}}{\Sigma\mathcal{E}^{d/2}}\left[\exp\left(-\frac{N(d-E)^2}{2\Sigma^2}\right)+O\left(1/\sqrt{N}\right)\right].$$

$\square$

1.3.3 Conditioned tensor product

Consider now

$$F^N = [f^{\otimes N}]_{\mathcal{S}_B^N} = \frac{f^{\otimes N}}{Z_N(f;\sqrt{dN},0)}\,\gamma^N$$

the restriction of the N-fold tensor of f to the Boltzmann's sphere $\mathcal{S}_B^N$, where f verifies (1.35) with $E = d$, more precisely with

$$E = \int |v|^2 f = d,$$

i.e. f has the same second order moment that γ.

We have then the following theorem, which is a precise version of point (i) in Theorem 1.3.

Theorem 1.18. *Consider $f \in \mathbf{P}_6(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$, with $p > 1$. Then, the sequence of probability measure $F^N \in \mathbf{P}(\mathcal{S}_\mathcal{B}^N)$ defined by $F^N = [f^{\otimes N}]_{\mathcal{S}_\mathcal{B}^N}$ is f-chaotic.*

More precisely, for any fixed ℓ there exists a constant $C := C(\ell) > 0$ such that for $N \geq \ell + 1$ we have

$$W_1(F_\ell^N, f^{\otimes \ell}) \leq \|F_\ell^N - f^{\otimes \ell}\|_{L^1_1} \leq \frac{C}{\sqrt{N}}.$$

Proof. With the notation $V = (v_1, \dots, v_N) \in \mathbb{R}^{dN}$, $V_\ell = (v_i)_{1 \leq i \leq \ell}$, $V_{\ell,N} = (v_i)_{\ell+1 \leq i \leq N}$ and $\bar{V}_\ell = \sum_{i=1}^{\ell} v_i$, we have from the definition of F^N

$$\begin{aligned} F^N(dV) &= \frac{f^{\otimes N}(V)\gamma^N(dV)}{Z_N(f;\sqrt{dN},0)} \\ &= \frac{f^{\otimes \ell}}{\gamma^{\otimes \ell}}(V_\ell)\,\frac{1}{Z_N'(f;\sqrt{dN},0)}\,\frac{f^{\otimes N-\ell}}{\gamma^{\otimes N-\ell}}(V_{\ell,N})\,\gamma^N(dV). \end{aligned}$$

We recall that $\gamma^N = \gamma^N_{\sqrt{dN},0}$ and we have

$$\gamma^N_{\sqrt{dN},0}(dV) = \gamma_\ell^N(dV_\ell)\,\gamma^{N-\ell}_{\sqrt{dN-|V_\ell|^2},z}(dV_{\ell,N})$$

where $z = -\sum_{i=1}^{\ell} v_i = -\bar{V}_\ell$. We fix $\ell \geq 1$ and $N \geq \ell + 1$, then we have

$$\begin{aligned} F_\ell^N(V_\ell) &= \int_{\mathbb{R}^{d(N-\ell)}} F^N(V)\,dV_{\ell,N} \\ &= \frac{f^{\otimes \ell}}{\gamma^{\otimes \ell}}(V_\ell)\,\frac{\gamma_\ell^N(V_\ell)}{Z_N'(f;\sqrt{dN},0)}\int_{\mathcal{S}^{N-\ell}\left(\sqrt{dN-|V_\ell|^2},z\right)} \frac{f^{\otimes N-\ell}}{\gamma^{\otimes N-\ell}}(V_{\ell,N})\,\gamma^{N-\ell}_{\sqrt{dN-|V_\ell|^2},z}(dV_{\ell,N}) \\ &= \frac{f^{\otimes \ell}}{\gamma^{\otimes \ell}}(V_\ell)\,\frac{Z_{N-\ell}'\left(f;\sqrt{dN-|V_\ell|^2},-\bar{V}_\ell\right)}{Z_N'(f;\sqrt{dN},0)}\,\gamma_\ell^N(V_\ell). \end{aligned}$$

Let us first compute the ratio betwenn $Z'_{N-\ell}$ and Z'_N, by Theorem 1.17 we have

$$\frac{Z'_{N-\ell}\left(f;\sqrt{dN-|V_\ell|^2},-\bar{V}_\ell\right)}{Z'_N(f;\sqrt{dN},0)} = \frac{(d(N-\ell))^{\frac{d(N-\ell-1)-2}{2}}}{\left(dN-|V_\ell|^2-\frac{|\bar{V}_\ell|^2}{N-\ell}\right)^{\frac{d(N-\ell-1)-2}{2}}}\frac{e^{-\frac{d(N-\ell)}{2}}}{e^{-\frac{(dN-|V_\ell|^2)}{2}}} \times\left[\exp\left(-\frac{|\bar{V}_\ell|^2}{2\mathcal{E}(N-\ell)}-\frac{(d\ell-|V_\ell|^2)^2}{2\Sigma^2(N-\ell)}\right)+O\left(N^{-1/2}\right)\right].$$

Using the later expression with Lemma 1.10 one obtains

$$\begin{aligned} F_\ell^N(V_\ell) &= \frac{f^{\otimes\ell}}{\gamma^{\otimes\ell}}(V_\ell)\,\frac{(d(N-\ell))^{\frac{d(N-\ell-1)-2}{2}}}{\left(dN-|V_\ell|^2-\frac{|\bar{V}_\ell|^2}{N-\ell}\right)^{\frac{d(N-\ell-1)-2}{2}}}\,\frac{e^{\frac{d\ell}{2}}}{e^{\frac{|V_\ell|^2}{2}}} \\ &\quad\times\left[\exp\left(-\frac{|\bar{V}_\ell|^2}{2\mathcal{E}(N-\ell)}-\frac{(d\ell-|V_\ell|^2)^2}{2\Sigma^2(N-\ell)}\right)+O\left(N^{-1/2}\right)\right] \\ &\quad\times\frac{\left|\mathbb{S}^{d(N-\ell-1)-1}\right|}{\left|\mathbb{S}^{d(N-1)-1}\right|}\,\frac{\left(dN-|V_\ell|^2-\frac{|\bar{V}_\ell|^2}{N-\ell}\right)_+^{\frac{d(N-\ell-1)-2}{2}}}{(dN)^{\frac{d(N-1)-2}{2}}\left(\frac{N-\ell}{N}\right)^{\frac{d}{2}}} \\ &= f^{\otimes\ell}\left[\exp\left(-\frac{|\bar{V}_\ell|^2}{2\mathcal{E}(N-\ell)}-\frac{(d\ell-|V_\ell|^2)^2}{2\Sigma^2(N-\ell)}\right)+O\left(N^{-1/2}\right)\right]\mathbf{1}_{dN-|V_\ell|^2-\frac{|\bar{V}_\ell|^2}{N-\ell}>0} \\ &\quad\times\frac{\left|\mathbb{S}^{d(N-\ell-1)-1}\right|}{\left|\mathbb{S}^{d(N-1)-1}\right|}\,\frac{(d(N-\ell))^{\frac{d(N-\ell-1)-2}{2}}}{(dN)^{\frac{d(N-1)-2}{2}}}\left(\frac{N}{N-\ell}\right)^{d/2}(2\pi e)^{\frac{d\ell}{2}}. \end{aligned}$$

Since

$$\left(\frac{N}{N-\ell}\right)^{d/2}=O(1),$$

we have

$$F_\ell^N(V_\ell)=f^{\otimes\ell}(V_\ell)\,\theta_1^N(V_\ell)\,\theta_2^N(V_\ell) \tag{1.44}$$

with

$$\begin{aligned} \theta_1^N &= \left[\exp\left(-\frac{|\bar{V}_\ell|^2}{2\mathcal{E}(N-\ell)}-\frac{(d\ell-|V_\ell|^2)^2}{2\Sigma^2(N-\ell)}\right)+O\left(N^{-1/2}\right)\right]\mathbf{1}_{dN-|V_\ell|^2-\frac{|\bar{V}_\ell|^2}{N-\ell}>0}, \\ \theta_2^N &= \frac{\left|\mathbb{S}^{d(N-\ell-1)-1}\right|}{\left|\mathbb{S}^{d(N-1)-1}\right|}\,\frac{(d(N-\ell))^{\frac{d(N-\ell-1)-2}{2}}}{(dN)^{\frac{d(N-1)-2}{2}}}\,(2\pi e)^{\frac{d\ell}{2}}. \end{aligned} \tag{1.45}$$

Thanks to Stirling's formula again, we obtain

$$\frac{\left|\mathbb{S}^{d(N-\ell-1)-1}\right|}{\left|\mathbb{S}^{d(N-1)-1}\right|}=\left(\frac{dN}{2\pi}\right)^{\frac{dl}{2}}\left(1+O(N^{-1})\right),\qquad \theta_2^N=1+O(N^{-1}).$$

Moreover we can easily see by (1.45) that $\|\theta_1^N\|_{L^\infty} \leq C$ uniformly in N, and

$$\begin{aligned} |\theta_1^N(V_\ell) - 1| &= |\theta_1^N(V_\ell) - 1|\mathbf{1}_{|V_\ell|\leq R} + |\theta_1^N(V_\ell) - 1|\mathbf{1}_{|V_\ell|\geq R} \\ &\leq \left| \left(\frac{|\bar{V}_\ell|^2}{2\mathcal{E}(N-\ell)} + \frac{(d\ell - |V_\ell|^2)^2}{2\Sigma^2(N-\ell)} \right) + O\left(1/\sqrt{N}\right) \right| \mathbf{1}_{|V_\ell|\leq R} + C\frac{|V_\ell|^b}{R^b}\mathbf{1}_{|V_\ell|\geq R} \\ &\leq C\left(\frac{R^2}{N} + \frac{R^4}{N} + O\left(1/\sqrt{N}\right) \right) \mathbf{1}_{|V_\ell|\leq R} + C\frac{|V_\ell|^b}{R^b}\mathbf{1}_{|V_\ell|\geq R}, \end{aligned} \tag{1.46}$$

for some $R > 0$ and $b \geq 0$.

Finally, choosing $R = N^{1/8}$ and $b = 4$ one has

$$\begin{aligned} \|F_\ell^N - f^{\otimes \ell}\|_{L_1^1} &= \|(\theta_1^N \theta_2^N - 1) f^{\otimes \ell}\|_{L_1^1} \\ &\leq (\theta_2^N - 1)\|\theta_1^N f^{\otimes \ell}\|_{L_1^1} + \|(\theta_1^N - 1) f^{\otimes \ell}\|_{L_1^1} \\ &\leq \frac{C}{N}\|f^{\otimes \ell}\|_{L_1^1} + \frac{C}{\sqrt{N}}\|f^{\otimes \ell}\|_{L_1^1} + \frac{C}{\sqrt{N}}\|f^{\otimes \ell}\|_{L_5^1} \\ &\leq \frac{C\ell}{N}\|f\|_{L_1^1} + \frac{C\ell}{\sqrt{N}}\|f\|_{L_1^1} + \frac{C\ell}{\sqrt{N}}\|f\|_{L_5^1}. \end{aligned}$$

$\square$

1.4 Entropic and Fisher's information chaos

We recall that in the Subsection 1.1.2 we defined the relative entropy and relative Fisher's information of a probability measure. Moreover, we defined stronger notions of chaos, namely the entropic chaos in Definition 1.2 and the Fisher's information chaos in Definition 1.4. We prove in this section precise versions of point (ii) in Theorem 1.3, Theorem 1.6 and Theorem 1.7.

1.4.1 Entropic chaos for the conditioned tensor product

We shall study now the entropic chaoticity of the probability measure $F^N = [f^{\otimes N}]_{\mathcal{S}_\mathcal{B}^N}$ with quantitative rate in the following theorem, which is a precise version of point (*ii*) of Theorem 1.3.

Theorem 1.19. *Let $f \in \mathbf{P}_6(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$ for some $p > 1$ verify $\int v f = 0$ and $\int |v|^2 f = d$. Then, the sequence of probabilities $F^N := [f^{\otimes N}]_{\mathcal{S}_\mathcal{B}^N} \in \mathbf{P}(\mathcal{S}_\mathcal{B}^N)$ is entropically f-chaotic. More precisely, there exists $C > 0$ such that we have*

$$\left| \frac{1}{N} H(F^N|\gamma^N) - H(f|\gamma) \right| \leq \frac{C}{\sqrt{N}}.$$

Proof. We write

$$
\begin{aligned}
\frac{1}{N}\, H(F^N|\gamma^N) &= \frac{1}{N} \int_{\mathcal{S}^N_{\mathcal{B}}} \left(\log \frac{dF^N}{d\gamma^N} \right) dF^N \\
&= \frac{1}{N} \int_{\mathcal{S}^N_{\mathcal{B}}} \left(\log \frac{f^{\otimes N}}{Z'_N(f; \sqrt{dN}, 0)\, \gamma^{\otimes N}} \right) dF^N \\
&= \int_{\mathbb{R}^d} \left(\log \frac{f}{\gamma} \right) dF^N_1 - \frac{1}{N} \log Z'_N(f; \sqrt{dN}, 0).
\end{aligned}
$$

Thanks to the assumptions on f, we can use Theorem 1.17 to obtain

$$
\frac{1}{N}\, H(F^N|\gamma^N) = \int_{\mathbb{R}^d} \left(\log \frac{f}{\gamma} \right) dF^N_1 + O(1/N).
$$

Using (1.44)-(1.45) we have $F^N_1(v) = \theta^N_1(v)\, \theta^N_2(v)\, f(v)$ or more precisely

$$
F^N_1(v) = f(v) \left(e^{-\frac{|v|^2}{2N} - \frac{|v|^4}{2N}} + O\left(1/\sqrt{N}\right) \right) (1 + O(1/N)) =: \theta^N(v)\, f(v),
$$

and then

$$
\frac{1}{N}\, H(F^N|\gamma^N) - H(f|\gamma) = \int_{\mathbb{R}^d} (\theta^N - 1)\, f \left(\log \frac{f}{\gamma} \right) + O(1/N). \tag{1.47}
$$

We estime now the first term of the right-hand side, denoted by T,

$$
\begin{aligned}
|T| &\le \int_{\mathbb{R}^d} |\theta^N - 1|\, f\, |\log \gamma|\, dv + \int_{\mathbb{R}^d} |\theta^N - 1|\, f\, |\log f|\, dv \\
&\le \int_{\mathbb{R}^d} |\theta^N - 1|\, f\, C(1 + |v|^2)\, dv + \int_{\mathbb{R}^d} |\theta^N - 1|\, f |\log f|\, dv \\
&=: T_1 + T_2.
\end{aligned}
$$

We recall that (already computed in equation (1.46))

$$
|\theta^N - 1| \le C \left(\frac{R^2}{N} + \frac{R^4}{N} + \frac{1}{\sqrt{N}} \right) \mathbf{1}_{|v| \le R} + C \frac{|v|^k}{R^k} \mathbf{1}_{|V_\ell| \ge R}
$$

for some $k \ge 0$ and $R > 0$. Then, for the first term we have

$$
\begin{aligned}
|T_1| &\le \int_{B_R} |\theta^N - 1|\, f\ (1 + |v|^2) + \int_{B_R^C} |\theta^N - 1|\, f\, (1 + |v|^2) \\
&\le \left(\frac{R^2}{N} + \frac{R^4}{N} + \frac{1}{\sqrt{N}} \right) \|f\|_{L^1_2} + \frac{1}{R^k} \Big(M_k(f) + M_{k+2}(f) \Big) \\
&\le \frac{C_f}{\sqrt{N}}
\end{aligned}
$$

where we have chosen $R = N^{1/8}$ and $k = 4$.

For the last term T_2, define $A > 1$ and $B_R = \{v \in \mathbb{R}^d;\ |v| \le R\}$, then we have

$$\begin{aligned}|T_2| \le &\int_{B_R} |\theta^N - 1|\, f\,|\log f| + \int_{B_R^C} |\theta^N - 1|\, f\,|\log f|\, \mathbf{1}_{f\ge A}\\ &+ \int_{B_R^C} |\theta^N - 1|\, f\,|\log f|\, \mathbf{1}_{1\le f\le A} + \int_{B_R^C} |\theta^N - 1|\, f\,|\log f|\, \mathbf{1}_{e^{-|v|^2}\le f\le 1}\\ &+ \int_{B_R^C} |\theta^N - 1|\, f\,|\log f|\, \mathbf{1}_{0\le f\le e^{-|v|^2}}.\end{aligned}$$

Now we compute each one of this five terms. First, we deduce that

$$|T_{2,1}| \le \left(\frac{R^2}{N} + \frac{R^4}{N} + \frac{1}{\sqrt{N}}\right) \int_{B_R} f\,|\log f| = \left(\frac{R^2}{N} + \frac{R^4}{N} + \frac{1}{\sqrt{N}}\right) C_f.$$

For the second term, we use that $f\,|\log f| \le f^{(1+p)/2} \le f^p/A^{(p-1)/2}$ over $\{f \ge A, |v| \ge R\}$, and then

$$|T_{2,2}| \le \frac{\|f\|^p_{L^p}}{A^{(p-1)/2}}.$$

Using $f\,|\log f| \le f\,|\log A|$ over $\{1 \le f \le A, |v| \ge R\}$ for the third one, we obtain

$$|T_{2,3}| \le \frac{\log A}{R^k}\, M_k(f).$$

Thanks to $f\,|\log f| \le f|v|^2 \le f|v|^{m+2}/R^m$ over $\{e^{-|v|^2} \le f \le 1, |v| \ge R\}$, we get

$$|T_{2,4}| \le \frac{1}{R^m}\, M_{m+2}(f).$$

Finally, by $f\,|\log f| \le 4\sqrt{f} \le 4e^{-|v|^2/2}$ over $\{0 \le f \le e^{-|v|^2}, |v| \ge R\}$

$$|T_{2,4}| \le C\, e^{-R}.$$

Putting togheter all this terms, we have

$$\begin{aligned}|T_2| &\le \left(\frac{R^2}{N} + \frac{R^4}{N} + \frac{1}{\sqrt{N}}\right) C_f + \frac{\|f\|^p_{L^p}}{A^{(p-1)/2}} + \frac{\log A}{R^k}\, M_k(f) + \frac{M_{m+2}(f)}{R^m} + C\, e^{-R}\\ &\le \frac{C_f}{\sqrt{N}}\end{aligned}$$

choosing $A^{(p-1)/2} = R^k$, $R = N^{1/8}$, $k = 6$ and $m = 4$.

We have then $|T| \le C\, N^{-1/2}$ and we conclude plugging it in (1.47).

$\square$

1.4.2 Relations between the different notions of chaos

First of all, we start with the following lemma and we refer to [12, 46, 59] and the references therein for a proof.

Lemma 1.20. *For all probabilities $\mu, \nu \in \mathbf{P}(Z)$ on a locally compact metric space, we have*

$$H(\mu|\nu) = \sup_{\varphi \in C_b(Z)} \left\{ \int_Z \varphi \, d\mu - \log \left(\int_Z e^{\varphi} \, d\nu \right) \right\}$$
$$= \sup_{\varphi \in C_b(Z),\, \int_Z e^{\varphi}\, d\nu = 1} \int_Z \varphi \, d\mu.$$

The following theorem is an adaptation of [12, Theorem 17], where the same result is proved for probability measures on the usual sphere $\mathbb{S}^{N-1}(\sqrt{N})$ in $\mathbb{R}^N$.

Theorem 1.21. *Consider $g \in \mathbf{P}_6(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$, for some $p \in (1, \infty]$, where g satisfies $\int vg = 0$ and $\int |v|^2 g = d$. Consider G^N a probability measure on $\mathcal{S}^N_{\mathcal{B}}$ such that for some positive integer ℓ, we have $G^N_\ell \rightharpoonup \pi_\ell$ in $\mathbf{P}(\mathbb{R}^{d\ell})$ when N goes to infinity.*

Then, we have

$$\frac{1}{\ell} H(\pi_\ell | g^{\otimes \ell}) \leq \liminf_{N \to \infty} \frac{1}{N} H\left(G^N | \left[g^{\otimes N} \right]_{\mathcal{S}^N_{\mathcal{B}}} \right).$$

Proof. Let fix a function $\varphi := \varphi(v_1, \ldots, v_\ell) \in C_b(\mathbb{R}^{d\ell})$ such that

$$\int_{\mathbb{R}^{d\ell}} e^{\varphi} \, g^{\otimes \ell} = 1, \qquad H(\pi_\ell | g^{\otimes \ell}) \leq \int_{\mathbb{R}^{d\ell}} \varphi \, d\pi_\ell + \varepsilon \tag{1.48}$$

for some $\varepsilon > 0$, which is possible thanks to Lemma 1.20. We introduce the function

$$\Phi(v_1, ..., v_N) := \varphi(v_1, \ldots, v_\ell) + \cdots + \varphi(v_{(m-1)\ell+1}, \ldots, v_{m\ell}),$$

where m is the integer part of N/ℓ, i.e. $N = m\ell + r$ with $0 \leq r \leq \ell - 1$. Thanks again to Lemma 1.20 we have

$$\frac{1}{N} H\left(G^N | \left[g^{\otimes N} \right]_{\mathcal{S}^N_{\mathcal{B}}} \right) \geq \frac{1}{N} \int_{\mathcal{S}^N_{\mathcal{B}}} \Phi \, G^N(dV) - \frac{1}{N} \log \left(\int_{\mathcal{S}^N_{\mathcal{B}}} e^{\Phi} \, d\left[g^{\otimes N} \right]_{\mathcal{S}^N_{\mathcal{B}}} \right).$$

For the first term of the right-hand side, using the symmetry of G^N and the convergence of its ℓ-marginal, we have

$$\frac{1}{N} \int_{\mathcal{S}^N_{\mathcal{B}}} \Phi \, G^N(dV) = \frac{m}{N} \int_{\mathbb{R}^{d\ell}} \varphi \, dG^N_\ell \xrightarrow[N \to \infty]{} \frac{1}{\ell} \int_{\mathbb{R}^{d\ell}} \varphi \, d\pi_\ell.$$

We note that the second term of the right-hand side can be written in the following way

$$\int_{\mathcal{S}^N_{\mathcal{B}}} e^{\Phi} \, d\left[g^{\otimes N} \right]_{\mathcal{S}^N_{\mathcal{B}}} = \frac{1}{Z'_N(g; \sqrt{dN}, 0)} \int_{\mathcal{S}^N_{\mathcal{B}}} e^{\Phi} \left(\frac{g}{\gamma} \right)^{\otimes N} \, d\gamma^N$$

since
$$\left[g^{\otimes N}\right]_{\mathcal{S}^N_{\mathcal{B}}} = \frac{g^{\otimes N}}{Z_N(g;\sqrt{dN},0)}\,\gamma^N.$$

Applying Theorem 1.17 and thanks to $\int |v|^2\, g = d$ we get
$$Z'_N(g;\sqrt{dN},0) = \frac{\sqrt{2d}}{\Sigma(g)}\left(1+O(1/\sqrt{N})\right),$$
where $\Sigma(g)$ is given by (1.35) applied to g, and then
$$\lim_{N\to\infty}\left(\frac{1}{N}\,\log Z'_N(g;\sqrt{dN},0)\right) = 0. \tag{1.49}$$

For the other term, denoting $u = (v_1,\dots,v_{m\ell})$, $w = (v_{m\ell+1},\dots,v_N)$ and $\bar w = v_{m\ell+1}+\cdots+v_N$, we write
$$\begin{aligned}
&\int_{\mathcal{S}^N(\sqrt{dN},0)} e^{\Phi}\left(\frac{g}{\gamma}\right)^{\otimes N} d\gamma^N \\
&= \int_{\mathbb{R}^{dr}} \frac{|\mathbb{S}^{d(N-r-1)-1}|}{|\mathbb{S}^{d(N-1)}|}\, \frac{\left(dN-|w|^2-\frac{|\bar w|^2}{N-r}\right)^{\frac{d(N-r-1)-2}{2}}}{(dN)^{\frac{d(N-1)-2}{2}}}\left(\frac{N}{N-r}\right)^{\frac{d}{2}}\left(\frac{g}{\gamma}\right)^{\otimes r} \\
&\quad\times\left\{\int_{\mathcal{S}^{\ell m}(\sqrt{dN-|w|^2},-\bar w)}\left(\frac{e^{\varphi}\,g^{\otimes\ell}}{\gamma^{\otimes\ell}}\right)^{\otimes m} d\gamma^N_{\sqrt{dN-|w|^2},-\bar w}\right\} dw
\end{aligned}$$
where the integral in dw have to be taken over the region
$$\{w\in\mathbb{R}^{dr}\,|\,dN-|w|^2-|\bar w|^2/(\ell m)>0\}.$$

We recognize that the last integral is equal to $Z'_m\left(e^{\varphi}\,g^{\otimes\ell};\sqrt{dN-|w|^2},-\bar w\right)$ (where Z'_m is a multi-dimensional version of Z'_N, obtained replacing N by $m\ell$) and by Theorem 1.17 we have
$$\begin{aligned}
&Z'_m\left(e^{\varphi}\,g^{\otimes\ell};\sqrt{dN-|w|^2},-\bar w\right) \\
&\quad= O(1)\times\frac{(d\ell m)^{\frac{d(\ell m-1)-2}{2}}}{\left(dN-|w|^2-\frac{|\bar w|^2}{\ell m}\right)^{\frac{d(\ell m-1)-2}{2}}}\,\frac{e^{-\frac{d\ell m}{2}}}{e^{-\frac{(dN-|w|^2)}{2}}}
\end{aligned}$$
and using (1.27), we get
$$\begin{aligned}
\int_{\mathcal{S}^N(\sqrt{dN},0)} e^{\Phi}\left(\frac{g}{\gamma}\right)^{\otimes N} d\gamma^N &= C\int_{\mathbb{R}^{dr}} e^{-\frac{|w|^2}{2}}\left(\frac{g}{\gamma}\right)^{\otimes r} dw \\
&= O(1)\times(2\pi)^{dr/2}\int_{\mathbb{R}^{dr}} g^{\otimes r}dw = O(1).
\end{aligned}$$

With these estimates at hand, we can deduce

$$\liminf_{N\to\infty}\left(-\frac{1}{N}\,\log\int_{\mathcal{S}^N(\sqrt{dN},0)} e^{\Phi}\left(\frac{g}{\gamma}\right)^{\otimes N}\,d\gamma^N\right)\geq 0$$

and together with (1.49) we obtain

$$\liminf_{N\to\infty}\frac{1}{N}\,H(G^N|\left[g^{\otimes N}\right]_{\mathcal{S}^N_{\mathcal{B}}})\geq\frac{1}{\ell}\int_{\mathbb{R}^{d\ell}}\varphi\,d\pi_\ell\geq\frac{1}{\ell}\,H(\pi_\ell|g^{\otimes\ell})-\varepsilon.$$

Since ε is arbitrary, we can conclude letting $\varepsilon\to 0$.

□

Our aim now is to give an analogous result of Theorem 1.21 for the Fisher's information. However the strategy here is different, it is not based on the asymptotic behaviour of Z'_N like before, but on a geometric approach following [46], where this analogous result is proved in the Kac's sphere setting. To this purpose, firstly we shall present some results to conclude with the Theorem 1.23.

Consider $W=(w_1,\dots,w_N)\in\mathbb{R}^{dN}$ and $V=(v_1,\dots,v_N)\in\mathcal{S}^N_{\mathcal{B}}$, where we recall that $v_i=(v_{i,\alpha})_{1\leq\alpha\leq d}$, $w_i=(w_{i,\alpha})_{1\leq\alpha\leq d}\in\mathbb{R}^d$ for all $1\leq i\leq N$.

Let P_h be the projection on the hyperplane $\{X\in\mathbb{R}^{dN}\ ;\ \sum_{i=1}^N x_i=0\}$, then it can be computed in the following way

$$P_hW=W-\sum_{\alpha=1}^{d}\left(W\cdot\frac{e^N_\alpha}{|e^N_\alpha|}\right)\frac{e^N_\alpha}{|e^N_\alpha|},$$

where $e^N_\alpha=(e_\alpha,\dots,e_\alpha)\in\mathbb{R}^{dN}$ with $e_\alpha=(\delta_{\alpha\beta})_{1\leq\beta\leq d}\in\mathbb{R}^d$. Since $|e^N_\alpha|=\sqrt{N}$ we obtain

$$P_hW=W-\frac{1}{N}\sum_{\alpha=1}^{d}(W\cdot e^N_\alpha)\,e^N_\alpha. \tag{1.50}$$

Moreover, the projection P_s on the sphere $\{X\in\mathbb{R}^{dN}\ ;\ \sum_{i=1}^N|x_i|^2=dN\}$ is given by

$$P_sW=\sqrt{dN}\,\frac{W}{|W|}. \tag{1.51}$$

Hence the projection $P_{\mathcal{S}}$ on the Boltzmann's sphere $\mathcal{S}^N_{\mathcal{B}}$ can be computed as the composition of the others, i.e. $P_{\mathcal{S}}=P_s\circ P_h$, more precisely

$$\begin{aligned}P_{\mathcal{S}}W&=(P_s\circ P_h)W\\&=\sqrt{dN}\,\frac{P_hW}{|P_hW|}\\&=\sqrt{dN}\,\frac{W-\frac{1}{N}\sum_{\alpha=1}^d(W\cdot e^N_\alpha)\,e^N_\alpha}{\left|W-\frac{1}{N}\sum_{\alpha=1}^d(W\cdot e^N_\alpha)\,e^N_\alpha\right|},\end{aligned} \tag{1.52}$$

or in coordinates, for $1 \le j \le N$ and $1 \le \beta \le d$,

$$(P_{\mathcal{S}} W)_{j,\beta} = \frac{\sqrt{dN}}{\left| W - \frac{1}{N} \sum_{\alpha=1}^{d} (W \cdot e_\alpha^N)\, e_\alpha^N \right|} \left(w_{j,\beta} - \frac{1}{N} \sum_{k=1}^{N} w_{k,\beta} \right). \tag{1.53}$$

Consider $V \in \mathcal{S}_{\mathcal{B}}^N$ and a smooth function F defined on $\mathcal{S}_{\mathcal{B}}^N$. Then the gradient ∇_h on $\{X \in \mathbb{R}^{dN} \; ; \; \sum_{i=1}^N x_i = 0\}$ is (recall that ∇ stands for the usual gradient on $\mathbb{R}^{dN}$)

$$\nabla_h F(V) = \nabla F(V) - \frac{1}{N} \sum_{i=1}^{N} \sum_{\alpha=1}^{d} \partial_{v_{i,\alpha}} F(V)\, e_\alpha^N.$$

Moreover, the gradient ∇_s on the sphere $\{X \in \mathbb{R}^{dN} \; ; \; \sum_{i=1}^N |x_i|^2 = dN\}$ is given by

$$\nabla_s F(V) = \nabla F(V) - \left(\frac{V}{|V|} \cdot \nabla F(V) \right) \frac{V}{|V|}.$$

Combining them we can compute the gradient on $\mathcal{S}_{\mathcal{B}}^N$, which is given by

$$\begin{aligned}
\nabla_{\mathcal{S}} F(V) &= \nabla_h F(V) - \left(\frac{V}{|V|} \cdot \nabla_h F(V) \right) \frac{V}{|V|} \\
&= \nabla F(V) - \frac{1}{N} \sum_{i=1}^{N} \sum_{\alpha=1}^{d} \partial_{v_{i,\alpha}} F(V)\, e_\alpha^N \\
&\quad - \left[V \cdot \nabla F(V) - \frac{1}{N} \sum_{i=1}^{N} \sum_{\alpha=1}^{d} \partial_{v_{i,\alpha}} F(V)\, (e_\alpha^N \cdot V) \right] \frac{V}{|V|^2} \\
&= \nabla F(V) - \frac{1}{N} \sum_{i=1}^{N} \sum_{\alpha=1}^{d} \partial_{v_{i,\alpha}} F(V)\, e_\alpha^N - [V \cdot \nabla F(V)] \frac{V}{|V|^2},
\end{aligned} \tag{1.54}$$

since $e_\alpha^N \cdot V = \sum_{i=1}^N v_{i,\alpha} = 0$ because $V \in \mathcal{S}_{\mathcal{B}}^N$.

Let Φ be a smooth vector field on $\mathbb{R}^{dN}$, which written in composants is $\Phi(V) = (\Phi_1(V), \dots, \Phi_N(V))$ with $\Phi_i(V) = (\Phi_{i,1}(V), \dots, \Phi_{i,d}(V))$ for $1 \le i \le N$. We denote by $\mathrm{div}_{\mathcal{S}}$ the divergence on $\mathcal{S}_{\mathcal{B}}^N$, then it can be computed in the following way

$$\mathrm{div}_{\mathcal{S}}\, \Phi(V) = \sum_{j=1}^{N} \sum_{\beta=1}^{d} \nabla_{\mathcal{S}} \Phi_{j,\beta}(V) \cdot e_{j,\beta},$$

where $e_{j,\beta} = (\delta_{jk} \delta_{\beta\gamma})_{(1 \le k \le N)(1 \le \gamma \le d)} \in \mathbb{R}^{dN}$. Using (1.54) and after some simplifications we obtain

$$\mathrm{div}_{\mathcal{S}}\, \Phi(V) = \mathrm{div}\, \Phi(V) - \frac{1}{N} \sum_{j=1}^{N} \sum_{\beta=1}^{d} \sum_{i=1}^{N} \partial_{v_{i,\beta}} \Phi_{j,\beta}(V) - \sum_{j=1}^{N} \sum_{\beta=1}^{d} V \cdot \nabla \Phi_{j,\beta}(V)\, \frac{v_{j,\beta}}{|V|^2}. \tag{1.55}$$

Lemma 1.22. *Consider a function F and a vector field Φ, smooth enough, defined on $\mathcal{S}^N_{\mathcal{B}}$. Then the following integration by parts formula on $\mathcal{S}^N_{\mathcal{B}}$ holds*

$$\int_{\mathcal{S}^N_{\mathcal{B}}} \left\{ \nabla_{\mathcal{S}} F(V) \cdot \Phi(V) + F(V) \operatorname{div}_{\mathcal{S}} \Phi(V) - \frac{d(N-1)-1}{dN} F(V)\Phi(V) \cdot V \right\} d\gamma^N(V) = 0.$$

Proof. The proof presented here is an adaptation of [46, Lemma 4.16]. Let χ be a smooth function with compact support on $\mathbb{R}_+$ and define for $V \in \mathbb{R}^{dN}$

$$\phi(V) := \chi(|P_h V|)\, (F \circ P_{\mathcal{S}})(V)\, (\Phi \circ P_{\mathcal{S}})(V).$$

We can compute $\operatorname{div} \phi(V)$ and after some simplifications using the formulæ for the projections (1.50) and (1.52), the gradient (1.54) and the divergence (1.55) on $\mathcal{S}^N_{\mathcal{B}}$ we get

$$\begin{aligned}\operatorname{div} \phi(V) = {} & \frac{\chi'(|P_h V|)}{\sqrt{dN}}\, F(P_{\mathcal{S}} V)\, P_{\mathcal{S}} V \cdot \Phi(P_{\mathcal{S}} V) \\ & + \chi(|P_h V|)\, \nabla_{\mathcal{S}} F(P_{\mathcal{S}} V) \cdot \Phi(P_{\mathcal{S}} V)\, \frac{\sqrt{dN}}{|P_h V|} \\ & + \chi(|P_h V|)\, F(P_{\mathcal{S}} V)\, \operatorname{div}_{\mathcal{S}} \Phi(P_{\mathcal{S}} V)\, \frac{\sqrt{dN}}{|P_h V|}.\end{aligned} \tag{1.56}$$

Integrating (1.56) we get

$$\begin{aligned}& \int_{\mathbb{R}^{dN}} F(P_{\mathcal{S}} V)\, P_{\mathcal{S}} V \cdot \Phi(P_{\mathcal{S}} V)\, \frac{\chi'(|P_h V|)}{\sqrt{dN}}\, dV \\ & \quad + \int_{\mathbb{R}^{dN}} \Big[\nabla_{\mathcal{S}} F(P_{\mathcal{S}} V) \cdot \Phi(P_{\mathcal{S}} V) + F(P_{\mathcal{S}} V)\, \operatorname{div}_{\mathcal{S}} \Phi(P_{\mathcal{S}} V) \Big]\, \chi(|P_h V|)\, \frac{\sqrt{dN}}{|P_h V|}\, dV = 0.\end{aligned}$$

Using the change of coordinates $V = (v_1, \ldots, v_N) \to U = (u_1, \ldots, u_N)$ given by Lemma 1.9 and then the variables $w = \sum_{i=1}^N |u_i|^2$ and $z = \sqrt{N} u_N$, we obtain that the last expression is equal to

$$\begin{aligned}& \int_0^\infty \!\! \int_{\mathbb{R}^d} \left\{ \frac{|\mathbb{S}^{d(N-1)-1}|}{2\, N^{d/2}} \left(w - \frac{|z|^2}{N} \right)^{\frac{d(N-1)-2}{2}} \int_{\mathcal{S}^N(w,z)} F(V)\, V \cdot \Phi(V)\, d\gamma^N_{w,z} \right\} \frac{\chi'\left(\sqrt{w - \frac{|z|^2}{N}} \right)}{\sqrt{dN}}\, dz\, dw \\ & + \int_0^\infty \!\! \int_{\mathbb{R}^d} \left\{ \frac{|\mathbb{S}^{d(N-1)-1}|}{2\, N^{d/2}} \left(w - \frac{|z|^2}{N} \right)^{\frac{d(N-1)-2}{2}} \right. \\ & \qquad \left. \int_{\mathcal{S}^N(w,z)} \Big[\nabla_{\mathcal{S}} F(P_{\mathcal{S}} V) \cdot \Phi(P_{\mathcal{S}} V) + F(P_{\mathcal{S}} V)\, \operatorname{div}_{\mathcal{S}} \Phi(P_{\mathcal{S}} V) \Big]\, d\gamma^N_{w,z} \right\} \chi\left(\sqrt{w - \frac{|z|^2}{N}} \right) \frac{\sqrt{dN}}{\sqrt{w - \frac{|z|^2}{N}}}\, dz\, dw,\end{aligned}$$

and then we get

$$\int_0^\infty\!\!\int_{\mathbb{R}^d} \left(w - \frac{|z|^2}{N}\right)^{\frac{d(N-1)-2}{2}} \frac{\chi'\left(\sqrt{w-\frac{|z|^2}{N}}\right)}{dN}\, dz\, dw \left(\int_{\mathcal{S}^N_{\mathcal{B}}} F(V)\, V\cdot\Phi(V)\, d\gamma^N\right)$$
$$+ \int_0^\infty\!\!\int_{\mathbb{R}^d} \left(w - \frac{|z|^2}{N}\right)^{\frac{d(N-1)-3}{2}} \chi\left(\sqrt{w-\frac{|z|^2}{N}}\right) dz\, dw \left(\int_{\mathcal{S}^N_{\mathcal{B}}} \left[\nabla_{\mathcal{S}} F(V)\cdot\Phi(V) + F(V)\,\mathrm{div}_{\mathcal{S}}\,\Phi(V)\right] d\gamma^N\right)$$
$$= 0.$$

Since we have

$$\int_0^\infty\!\!\int_{\mathbb{R}^d} \left(w - \frac{|z|^2}{N}\right)^{\frac{d(N-1)-2}{2}} \chi'\left(\sqrt{w-\frac{|z|^2}{N}}\right) dz\, dw =$$
$$-\,[d(N-1)-1]\int_0^\infty\!\!\int_{\mathbb{R}^d} \left(w - \frac{|z|^2}{N}\right)^{\frac{d(N-1)-3}{2}} \chi\left(\sqrt{w-\frac{|z|^2}{N}}\right) dz\, dw,$$

we obtain the result

$$\int_{\mathcal{S}^N_{\mathcal{B}}} \left\{\nabla_{\mathcal{S}} F(V)\cdot\Phi(V) + F(V)\,\mathrm{div}_{\mathcal{S}}\,\Phi(V) - \frac{d(N-1)-1}{dN}\, F(V)\Phi(V)\cdot V\right\} d\gamma^N(V) = 0.$$

□

With these results at hand we are able to state the following theorem, which is the Fisher's information version of Theorem 1.21 and the proof is an adaptation of [46, Theorem 4.15].

Theorem 1.23. *Consider G^N a probability measure on $\mathcal{S}^N_{\mathcal{B}}$ such that for some positive integer ℓ, we have $G^N_\ell \rightharpoonup \pi_\ell$ in $\mathbf{P}(\mathbb{R}^{d\ell})$ when N goes to infinity.*

Then, we have

$$\frac{1}{\ell}\, I(\pi_\ell|\gamma^{\otimes\ell}) \le \liminf_{N\to\infty} \frac{1}{N}\, I(G^N|\gamma^N).$$

Proof. Let us denote $G^N =: g^N\gamma^N$. Using [46] we have the following representation formula

$$I(G^N|\gamma^N) = \int_{\mathcal{S}^N_{\mathcal{B}}} |\nabla_{\mathcal{S}} \log g^N|^2 g^N\, d\gamma^N$$
$$= \sup_{\Phi\in C^1_b(\mathbb{R}^{dN};\mathbb{R}^{dN})} \int_{\mathcal{S}^N_{\mathcal{B}}} \left(\nabla_{\mathcal{S}} \log g^N\cdot\Phi - \frac{|\Phi|^2}{4}\right) g^N\, d\gamma^N$$

and we obtain by Lemma 1.22

$$I(G^N|\gamma^N) = \sup_{\Phi\in C^1_b(\mathbb{R}^{dN};\mathbb{R}^{dN})} \int_{\mathcal{S}^N_{\mathcal{B}}} \left(\frac{d(N-1)-1}{dN}\,\Phi(V)\cdot V - \mathrm{div}_{\mathcal{S}}\,\Phi(V) - \frac{|\Phi(V)|^2}{4}\right) g^N\, d\gamma^N. \tag{1.57}$$

Furthermore for π_ℓ we have, also from [46],

$$I(\pi_\ell|\gamma^{\otimes\ell}) = \sup_{\varphi\in C_b^1(\mathbb{R}^{d\ell};\mathbb{R}^{d\ell})} \int_{\mathbb{R}^{d\ell}} \left(\varphi\cdot V_\ell - \operatorname{div}\varphi - \frac{|\varphi|^2}{4}\right)\pi_\ell.$$

Let us fix $\varepsilon > 0$ and choose φ such that

$$\frac{1}{\ell}\, I(\pi_\ell|\gamma^{\otimes\ell}) - \varepsilon \leq \frac{1}{\ell}\int_{\mathbb{R}^{d\ell}} \left(\varphi\cdot V_\ell - \operatorname{div}\varphi - \frac{|\varphi|^2}{4}\right)\pi_\ell$$

Denote $N = q\ell + r$, $0 \leq r < \ell$, and define $V_N = (V_{\ell,1},\dots,V_{\ell,q},V_r)$. Choosing $\Phi(V_N) := (\varphi(V_{\ell,1}),\dots,\varphi(V_{\ell,q}),0) \in C_b^1(\mathbb{R}^{dN};\mathbb{R}^{dN})$ we obtain from (1.57) and the symmetry of G^N

$$\begin{aligned}\frac{1}{N}\, I(G^N|\gamma^N) &\geq \frac{1}{N}\int_{\mathcal{S}^N_{\mathcal{B}}} \left(\frac{d(N-1)-1}{dN}\,\Phi(V_N)\cdot V_N - \operatorname{div}_{\mathcal{S}}\Phi(V_N) - \frac{|\Phi(V_N)|^2}{4}\right) G^N(dV_N)\\ &\geq \frac{q}{N}\int_{\mathbb{R}^{d\ell}} \left(\frac{d(N-1)-1}{dN}\,\varphi(V_\ell)\cdot V_\ell - \operatorname{div}\varphi(V_\ell) - \frac{|\varphi(V_\ell)|^2}{4}\right) G^N_\ell(dV_\ell) + \frac{R(N)}{N},\end{aligned}$$

with

$$R(N) = \int_{\mathbb{R}^{d\ell}} \sum_{k=1}^{\ell}\sum_{i=1}^{\ell}\sum_{\beta=1}^{d}\left(\frac{1}{N}\,\partial_{v_{i,\beta}}\varphi_{k,\beta} + \frac{1}{dN}(\partial_{v_{i,\beta}}\varphi_{k,\beta})v_{i,\beta}v_{k,\beta}\right) G^N_\ell(dV_\ell).$$

The last expression is bounded if $\nabla\varphi$ decreases rapidly enough at infinity. Hence, passing to the limit we obtain

$$\begin{aligned}\liminf_{N\to\infty}\frac{1}{N}\, I(G^N|\gamma^N) &\geq \frac{1}{\ell}\int_{\mathbb{R}^{d\ell}} \left(\varphi\cdot V_\ell - \operatorname{div}\varphi - \frac{|\varphi|^2}{4}\right)\pi_\ell\\ &\geq \frac{1}{\ell} I(\pi_\ell|\gamma^{\otimes\ell}) - \varepsilon,\end{aligned}$$

and we conclude letting $\varepsilon \to 0$. □

We can prove now precise versions of implications $(i) \Rightarrow (ii)$ and $(iii) \Rightarrow (iv)$ of Theorem 1.6 as follows.

Theorem 1.24. *Consider $G^N \in \mathbf{P}(\mathcal{S}^N_{\mathcal{B}})$ such that $G^N_1 \rightharpoonup f$ in $\mathbf{P}(\mathbb{R}^d)$. We have the following properties:*

(i) If $H(f|\gamma) < \infty$ and $\lim_{N\to\infty}\frac{1}{N}\, H(G^N|\gamma^N) = H(f|\gamma)$, then G^N is f-Kac's chaotic.

(ii) If $I(f|\gamma) < \infty$ and $\lim_{N\to\infty}\frac{1}{N}\, I(G^N|\gamma^N) = I(f|\gamma)$, then G^N is f-Kac's chaotic.

Proof. Let us fix $\ell \in \mathbb{N}^*$. Since $G^N_1 \rightharpoonup f$ in $\mathbf{P}(\mathbb{R}^d)$ we know by [70, Proposition 2.2] that G^N is tight. Then there exists a subsequence $G^{N'}$ and $\pi_\ell \in \mathbf{P}(\mathbb{R}^{d\ell})$ such that $G^{N'}_\ell \rightharpoonup \pi_\ell$ in $\mathbf{P}(\mathbb{R}^{d\ell})$, when N' goes to infinity (and in particular $\pi_1 = f$).

(i) By Theorem 1.21 we have

$$\frac{1}{\ell}\, H(\pi_\ell|\gamma^{\otimes \ell}) \leq \liminf_{N'\to\infty} \frac{1}{N'}\, H(G^{N'}|\gamma^{N'}) = H(f|\gamma).$$

Since we also have the reverse inequality by superadditivity of the entropy functional, we obtain

$$\begin{aligned} H(\pi_\ell|\gamma^{\otimes \ell}) - \ell H(f|\gamma) &= \int \pi_\ell \log \frac{\pi_\ell}{\gamma^{\otimes \ell}} - \ell \int f \log \frac{f}{\gamma} \\ &= \int \pi_\ell \log \frac{\pi_\ell}{\gamma^{\otimes \ell}} - \int \pi_\ell \log \frac{f^{\otimes \ell}}{\gamma^{\otimes \ell}} \\ &= \int f^{\otimes \ell} \left(\frac{\pi_\ell}{f^{\otimes \ell}} \log \frac{\pi_\ell}{f^{\otimes \ell}} - \frac{\pi_\ell}{f^{\otimes \ell}} + 1 \right) \\ &= 0, \end{aligned}$$

which implies $\pi_\ell = f^{\otimes \ell}$ a.e. on $\{f^{\otimes \ell} > 0\}$, since the function $z \mapsto z \log z - z + 1$ is equal to 0 in $z = 1$. Thanks to $\pi_\ell, f^{\otimes \ell} \in \mathbf{P}(\mathbb{R}^{d\ell})$, we obtain

$$\int_{\{f^{\otimes \ell}>0\}} \pi_\ell = \int_{\{f^{\otimes \ell}>0\}} f^{\otimes \ell} = 1.$$

It follows that $\pi_\ell = f^{\otimes \ell}$ a.e on $\mathbb{R}^{d\ell}$, so the whole sequence G^N_ℓ converges to $f^{\otimes \ell}$ and thus G^N is f-chaotic.

(ii) The proof of point (ii) being similar, thanks to Theorem 1.23 and the superadditivity of the Fisher's information [11], we skip it.

□

Recall another notion of entropic chaos stated in (1.18), as proposed in [12, Theorem 9 and Open Problem 11] and [62, Remark 7.11], for $G^N \in \mathbf{P}(\mathcal{S}^N_{\mathcal{B}})$ and $f \in \mathbf{P}_6 \cap L^p(\mathbb{R}^d)$ with $p > 1$, we consider the following property

$$\lim_{N\to\infty} \frac{1}{N}\, H\left(G^N|[f^{\otimes N}]_{\mathcal{S}^N_{\mathcal{B}}}\right) = 0. \tag{1.58}$$

Let us now investigate the relation between condition (1.58) and the entropic chaos (Definition 1.2) in the following result, which shows that, under some assumptions on f, they are equivalent.

Theorem 1.25. *Let $f \in \mathbf{P}_6(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ and $G^N \in \mathbf{P}(\mathcal{S}^N_{\mathcal{B}})$ such that $G^N_1 \rightharpoonup f$. Suppose further that $f(v_1) \geq \exp(-\alpha|v_1|^2 + \beta)$ for some $\alpha > 0$ and $\beta \in \mathbb{R}$. Then the following asserstions are equivalent:*

(i) $\displaystyle\lim_{N\to\infty} \frac{1}{N}\, H\left(G^N|[f^{\otimes N}]_{\mathcal{S}^N_{\mathcal{B}}}\right) = 0$;

(ii) $\displaystyle\lim_{N\to\infty} \frac{1}{N}\, H(G^N|\gamma^N) = H(f|\gamma)$.

Remark 1.26. We remark that both conditions (i) and (ii) imply that G^N is f-chaotic. Indeed, in [12, Theorem 19] is proved that (i) implies the f-chaoticity of G^N in the Kac's sphere framework, the generalization to the Boltzmann's sphere case is straightforward. Finally, the fact that condition (ii) implies that G^N is f-chaotic follows from Theorem 1.24.

Proof. Denote $G^N =: g^N\gamma^N$ and $F^N = [f^{\otimes N}]_{\mathcal{S}_{\mathcal{B}}^N} =: f^N\gamma^N$. Then we write

$$\begin{aligned} H(G^N|\gamma^N) &= \int_{\mathcal{S}_{\mathcal{B}}^N}\left(\log\frac{g^N}{f^N}\right) g^N\, d\gamma^N + \int_{\mathcal{S}_{\mathcal{B}}^N}\left(\log f^N\right) g^N\, d\gamma^N \\ &= H(G^N|[f^{\otimes N}]_{\mathcal{S}_{\mathcal{B}}^N}) + \int \log f^{\otimes N}\, dG^N - \int \log \gamma^{\otimes N}\, dG^N - \log Z_N'(f;\sqrt{dN},0) \\ &= H(G^N|[f^{\otimes N}]_{\mathcal{S}_{\mathcal{B}}^N}) + N\int_{\mathbb{R}^d}\log f\, dG_1^N + \frac{dN}{2}(\log 2\pi+1) - \log Z_N'(f;\sqrt{dN},0) \end{aligned} \tag{1.59}$$

using the symmetry of G^N, the explicit formula for $\gamma^{\otimes N}$ and the fact that $M_2(G^N) = dN$. Since $M_2(f) = d$, we obtain

$$\frac{1}{N}H(G^N|\gamma^N) - H(f|\gamma) = \frac{1}{N}H(G^N|[f^{\otimes N}]_{\mathcal{S}_{\mathcal{B}}^N}) + \int_{\mathbb{R}^d}(G_1^N - f)\log f - \frac{1}{N}\log Z_N'(f;\sqrt{dN},0).$$

The third term of the right-hand side goes to 0 as $N\to\infty$ thanks to Theorem 1.17. Hence we only need to prove that the second term of the right-hand side vanishes as $N\to\infty$, which implies that (i) is equivalent to (ii).

With the assumptions on f we obtain $|\log f| \le \log\|f\|_{L^\infty} + \alpha|v|^2 + \beta \le C_1(1+|v|^2)$. Consider $R>1$ and we have

$$\int_{|v|>R}(1+|v|^2)f < \frac{1}{R^4}\int_{|v|>R}|v|^4 f + \frac{1}{R^4}\int_{|v|>R}|v|^6 f \le C_2R^{-4}.$$

Let χ_R be a smooth function such that $0\le\chi_R\le 1$, $\chi_R(v) = 1$ for $|v|\le R$ and $\chi_R(v) = 0$ for $|v|\ge R+1$. We can split the integral to be estimated in the following way

$$\int_{\mathbb{R}^d}(G_1^N - f)\log f = \int_{\mathbb{R}^d}\chi_R(G_1^N - f)\log f + \int_{\mathbb{R}^d}(1-\chi_R)(G_1^N - f)\log f. \tag{1.60}$$

Let us show first that $H(G_1^N) = \int G_1^N\log G_1^N$ is bounded. If we assume condition (ii) then $N^{-1}H(G^N|\gamma^N)$ is bounded. On the other hand, if we assume (i), from (1.59) we have

$$\frac{1}{N}H(G^N|\gamma^N) \le \frac{1}{N}H(G^N|[f^{\otimes N}]_{\mathcal{S}_{\mathcal{B}}^N}) + \log\|f\|_{L^\infty} + \frac{dN}{2}(\log 2\pi + 1) - \frac{1}{N}\log Z_N'(f;\sqrt{dN},0),$$

and again $N^{-1}H(G^N|\gamma^N)$ is bounded. Moreover, we obtain thanks to [3] that

$$H(G_1^N|\gamma_1^N) \le C\,\frac{H(G^N|\gamma^N)}{N}$$

for some $C > 0$ and can write

$$H(G_1^N|\gamma) = H(G_1^N|\gamma_1^N) + \int \log \frac{\gamma_1^N}{\gamma}\, G_1^N,$$

which is bounded thanks to the explicit computation of γ_1^N in Lemma 1.10 and to the Lemma 1.11. We deduce, since $H(G_1^N|\gamma) = H(G^N) + d(\log 2\pi + 1)/2$, that $H(G_1^N)$ is bounded either if we assume (i) or (ii).

Then, for the first term of (1.60), since $\chi_R \log f$ is a bounded function, G_1^N converges weakly to f in $\mathbf{P}(\mathbb{R}^d)$ and $H(G_1^N)$ is bounded, we obtain that $\int \chi_R(G_1^N - f)\log f \to 0$ as $N \to \infty$. For the second term of (1.60) we write (recall that $\int(1+|v|^2)G_1^N = 1 + d = \int(1+|v|^2)f$)

$$\begin{aligned}\left|\int_{\mathbb{R}^d}(1-\chi_R)(G_1^N - f)\log f\right| &\leq C_1 \int_{\mathbb{R}^d}(1-\chi_R)(1+|v|^2)(G_1^N + f)\\ &\leq C_1 C_2 R^{-4} + C_1(1+d) - C_1\int_{\mathbb{R}^d}\chi_R(1+|v|^2)G_1^N.\end{aligned}$$

The function $\chi_R(1+|v|^2)$ being bounded and continuous, we know that $\int \chi_R(1+|v|^2)(G_1^N - f) \to 0$ as $N \to \infty$. Thus passing to the limit in the last expression we obtain

$$\begin{aligned}\limsup_{N\to\infty}\left|\int_{\mathbb{R}^d}(1-\chi_R)(G_1^N - f)\log f\right| &\leq C_1C_2R^{-4} + C_1(1+d) - C_1\int_{\mathbb{R}^d}(\chi_R)(1+|v|^2)f\\ &\leq 2C_1C_2R^{-4}\end{aligned}$$

which concludes the proof letting $R \to \infty$.

□

Remark 1.27. In the setting of the Kac's sphere (usual sphere $\mathbb{S}^{N-1}(\sqrt{N})$), we find in [12, Theorem 21] a proof of (i) implies (ii) without the assumption $f(v_1) \geq \exp(-\alpha|v_1|^2 + \beta)$. We can adapt it to our case in the following way.

Proof of $(i) \Rightarrow (ii)$. We write from (1.59) and for $\delta > 0$

$$\frac{1}{N}H(G^N|\gamma^N) \leq \frac{1}{N}H(G^N|[f^{\otimes N}]_{\mathcal{S}_{\mathcal{B}}^N}) + \int \log(f+\delta)\,G_1^N + \frac{d}{2}(\log 2\pi + 1) - \frac{1}{N}\log Z_N'(f;\sqrt{dN},0).$$

Since $\log(f+\delta)$ is a bounded function thanks to $f \in L^\infty$, $H(G_1^N)$ is bounded and $G_1^N \rightharpoonup f$ in $\mathbf{P}(\mathbb{R}^d)$ we have $\int \log(f+\delta)\,G_1^N \to \int\log(f+\delta)\,f$ as $N \to \infty$. We can pass to the limit $N \to \infty$ to obtain

$$\limsup_{N\to\infty}\frac{1}{N}H(G^N|\gamma^N) \leq \int \log(f+\delta)\,f + \frac{d}{2}(\log 2\pi + 1).$$

Now letting $\delta \to 0$, by dominated convergence we obtain

$$\limsup_{N\to\infty}\frac{1}{N}H(G^N|\gamma^N) \leq \int f\log f + \frac{d}{2}(\log 2\pi + 1) = H(f|\gamma),$$

and we conclude with this estimate togheter with

$$H(f|\gamma) \leq \liminf_{N\to\infty}\frac{1}{N}H(G^N|\gamma^N)$$

from Theorem 1.21. □

1.4.3 On a more general class of chaotic probabilities

In the subsection 1.4.1 we have constructed a particular probability measure on $\mathcal{S}^N_{\mathcal{B}}$ that is entropically chaotic. Hence, a natural question is whether it is true for a more general class of probabilities on the Boltzmann's sphere. Theorem 1.31, which is a precise version of $(ii) \Rightarrow (iii)$ in Theorem 1.6, gives an answer with a quantitative rate.

First of all, let us present some results concerning different forms of measuring chaos that will be useful in te sequel.

Lemma 1.28. *Consider $f, g \in \mathbf{P}(\mathbb{R}^d)$ and $F^N, G^N \in \mathbf{P}(\mathbb{R}^{dN})$. Let us define $M_k(F,G) := M_k(F) + M_k(G)$.*

For any $k \geq 2$ we have

$$W_2(f,g) \leq 2^{\frac{3}{2}}\, M_k(f,g)^{\frac{1}{2(k-1)}}\, W_1(f,g)^{\frac{k-2}{2(k-1)}} \tag{1.61}$$

and

$$\frac{W_2(F^N,G^N)}{\sqrt{N}} \leq 2^{\frac{3}{2}} \left(\frac{M_k(F^N,G^N)}{N}\right)^{\frac{1}{2(k-1)}} \left(\frac{W_1(F^N,G^N)}{N}\right)^{\frac{k-2}{2(k-1)}}. \tag{1.62}$$

The proof of Lemma 1.28 come from [62, Lemma 4.1] for (1.61) and (1.62) is a simple generalization of (1.61) to the case of N variables.

We denote by $\overline{W}_1$ the MKW distance (1.12) defined with a bounded distance in $\mathbb{R}^d$, more precisely, for all $f, g \in \mathbf{P}_1(\mathbb{R}^d)$,

$$\overline{W}_1(f,g) = \inf_{\pi \in \Pi(f,g)} \int_{\mathbb{R}^d \times \mathbb{R}^d} \min\{|x-y|, 1\}\, \pi(dx,dy).$$

Consider $G^N \in \mathbf{P}(\mathbb{R}^{dN})$ and $f \in \mathbf{P}(\mathbb{R}^d)$. We define then $\widehat{G}^N$, $\delta_f \in \mathbf{P}(\mathbf{P}(\mathbb{R}^d))$ by, for all $\Phi \in C_b(\mathbf{P}(\mathbb{R}^d))$,

$$\begin{aligned} &\int_{\mathbf{P}(\mathbb{R}^d)} \Phi(\rho)\, \widehat{G}^N(d\rho) = \int_{\mathbb{R}^{dN}} \Phi(\mu^N_V)\, G^N(dV), \qquad \mu^N_V = \frac{1}{N}\sum_{i=1}^N \delta_{v_i} \in \mathbf{P}(\mathbb{R}^d), \\ &\int_{\mathbf{P}(\mathbb{R}^d)} \Phi(\rho)\, \delta_f(d\rho) = \Phi(f). \end{aligned} \tag{1.63}$$

Furthermore, $\mathcal{W}$ stands for the Wasserstein distance on $\mathbf{P}(\mathbf{P}(\mathbb{R}^d))$. More precisely, for some distance D on $\mathbf{P}(\mathbb{R}^d)$ we define

$$\forall \mu,\nu \in \mathbf{P}(\mathbf{P}(\mathbb{R}^d)), \quad \mathcal{W}_D(\mu,\nu) := \inf_{\pi \in \Pi(\mu,\nu)} \int_{\mathbf{P}(\mathbb{R}^d)\times\mathbf{P}(\mathbb{R}^d)} D(f,g)\, d\pi(f,g).$$

In the particular case of $\widehat{G}^N$ and δ_f we have $\Pi(\widehat{G}^N, \delta_f) = \{\widehat{G}^N \otimes \delta_f\}$ and then

$$\mathcal{W}_D\left(\widehat{G}^N, \delta_f\right) = \int_{\mathbb{R}^{dN}} D(\mu^N_V, f)\, G^N(dV). \tag{1.64}$$

We have the following result from [46].

Lemma 1.29. *Consider $f, g \in \mathbf{P}(\mathbb{R}^d)$ and $F^N, G^N \in \mathbf{P}(\mathcal{S}^N_{\mathcal{B}})$. Let us define $M_k(F,G) := M_k(F) + M_k(G)$.*

(i) For any $k > 2$ we have

$$W_2(f,g) \le 2^{\frac{3}{2}}\, M_k(f,g)^{\frac{1}{k}}\, \overline{W}_1(f,g)^{\frac{1}{2}-\frac{1}{k}} \tag{1.65}$$

and

$$\frac{W_2(F^N,G^N)}{\sqrt{N}} \le 2^{\frac{3}{2}} \left(\frac{M_k(F^N,G^N)}{N}\right)^{\frac{1}{k}} \left(\frac{\overline{W}_1(F^N,G^N)}{N}\right)^{\frac{1}{2}-\frac{1}{k}}. \tag{1.66}$$

(ii) For any $0 < \alpha_1 < 1/(d+1)$ and $k > d(\alpha_1^{-1} - d - 1)^{-1}$ there exists a constant $C := C(d,\alpha_1,k) > 0$ such that

$$\mathcal{W}_{\overline{W}_1}(\widehat{G}^N, \delta_f) \le C M_k(G_1^N, f)^{1/k} \left(\overline{W}_1(G_2^N, f^{\otimes 2}) + \frac{1}{N}\right)^{\alpha_1}. \tag{1.67}$$

(iii) For any $0 < \alpha_2 < 1/d'$ and $k > d'(\alpha_2^{-1} - d')^{-1}$, with $d' := \max(d,2)$, there exists a constant $C := C(d,\alpha_2,k) > 0$ such that

$$\left|\overline{W}_1(G^N, f^{\otimes N}) - \mathcal{W}_{\overline{W}_1}(\widehat{G}^N, \delta_f)\right| \le C\, \frac{M_k(f)^{1/k}}{N^{\alpha_2}}. \tag{1.68}$$

The equations (1.65) and (1.66) come from [46, Lemmas 2.1 and 2.2], and (1.67)-(1.68) are proved in [46, Theorem 1.2].

As a consequence of Lemma 1.29 we have the following result.

Lemma 1.30. *Consider $G^N \in \mathbf{P}(\mathcal{S}^N_{\mathcal{B}})$ and $f \in \mathbf{P}(\mathbb{R}^d)$ such that $M_k(G_1^N)$ and $M_k(f)$ are finite, for $k > 2$. Let us denote $\mathcal{M}_k := M_k(G_1^N) + M_k(f)$.*

Then for any $0 < \alpha_1 < 1/(d+1)$ and $\alpha_1 < k(dk + d + k)^{-1}$, $0 < \alpha_2 < 1/d'$ and $\alpha_2 < k(d'k + d')^{-1}$, with $d' := \max(d,2)$, there exists a constant $C := C(d,k,\alpha_1,\alpha_2)$ such that

$$\frac{W_2(G^N, f^{\otimes N})}{\sqrt{N}} \le C \mathcal{M}_k^{\frac{1}{k}} \left(\overline{W}_1(G_2^N, f^{\otimes 2})^{\alpha_1} + N^{-\alpha_1} + N^{-\alpha_2}\right)^{\frac{1}{2}-\frac{1}{k}}$$

Proof. First of all, we remark that $N^{-1} M_k(G^N)$ is equivalent to $M_k(G_1^N)$ since G^N is symmetric. Then, using Lemma 1.29 we have

$$\begin{aligned}
\frac{W_2(G^N, f^{\otimes N})}{\sqrt{N}} &\le 2^{\frac{2}{3}} \mathcal{M}_k^{\frac{1}{k}} \left(\frac{\overline{W}_1(G^N, f^{\otimes N})}{N}\right)^{\frac{1}{2}-\frac{1}{k}} \\
&\le 2^{\frac{2}{3}} \mathcal{M}_k^{\frac{1}{k}} \left(C \frac{M_k(f)^{\frac{1}{k}}}{N^{\alpha_2}} + \mathcal{W}_{\overline{W}_1}\left(\widehat{G}^N, \delta_f\right)\right)^{\frac{1}{2}-\frac{1}{k}} \\
&\le 2^{\frac{2}{3}} C \mathcal{M}_k^{\frac{1}{k}} \left(N^{-\alpha_2} + \left(\overline{W}_1(G_2^N, f^{\otimes 2}) + N^{-1}\right)^{\alpha_1}\right)^{\frac{1}{2}-\frac{1}{k}}
\end{aligned}$$

where we have used successively (1.66), (1.68) and (1.67), with α_1 and α_2 defined as above.

□

We can now state a precise version of $(ii) \Rightarrow (iii)$ in Theorem 1.6.

Theorem 1.31. *Consider $G^N \in \mathbf{P}(\mathcal{S}^N_{\mathcal{B}})$. Moreover we suppose that G^N is f-chaotic, for some $f \in \mathbf{P}(\mathbb{R}^d)$, and also that*

$$M_k(G^N_1) \leq C_1,\ k \geq 6, \qquad \frac{1}{N}\, H(G^N|\gamma^N) \leq C_2, \qquad \frac{1}{N}\, I(G^N|\gamma^N) \leq C_3.$$

Then G^N is entropically f-chaotic. More precisely, there exists $C = C(C_1, C_2, C_3) > 0$ and for any $\beta < (k-2)[4(dk+d+k)]^{-1}$ a constant $C' := C'(\beta)$ such that

$$\left| \frac{1}{N}\, H(G^N|\gamma^N) - H(f|\gamma) \right| \leq C \left(\frac{W_2(G^N, f^{\otimes N})}{\sqrt{N}} + C' N^{-\beta} \right).$$

Proof. First of all, thanks to Theorem 1.21 (with $g = \gamma$ and $\ell = 1$) we have

$$H(f|\gamma) \leq \liminf_{N\to\infty} \frac{1}{N}\, H(G^N|\gamma^N) \leq C_2$$

and thanks to Theorem 1.23

$$I(f|\gamma) \leq \liminf_{N\to\infty} \frac{1}{N}\, I(G^N|\gamma^N) \leq C_3,$$

which implies that $I(f) < \infty$. Indeed, $I(f|\gamma) = I(f) + M_2(f) - 2d$, from which we conclude.

Furthermore, since $I(f) \leq C$, f lies in $L^p(\mathbb{R}^d)$ for some $p > 1$ by Sobolev embeddings. Moreover $M_k(f) < \infty$ for some $k \geq 6$ since $M_k(G^N_1)$ is bounded and $G^N_1 \rightharpoonup f$ weakly in $\mathbf{P}(\mathbb{R}^d)$. We have then all the conditions on f to construct $F^N = [f^{\otimes N}]_{\mathcal{S}^N_{\mathcal{B}}}$ satisfying Theorems 1.18 and 1.19.

Let us denote

$$F^N = \frac{f^{\otimes N}}{Z_N(f; \sqrt{dN}, 0)}\, \gamma^N =: f^N \gamma^N$$

and we compute the relative Fisher's information with respect to γ^N

$$\frac{1}{N}\, I(F^N|\gamma^N) = \frac{1}{N} \int_{\mathcal{S}^N_{\mathcal{B}}} \frac{\left|\nabla_{\mathcal{S}} f^N\right|^2}{f^N}\, d\gamma^N$$

where we recall that $\nabla_{\mathcal{S}}$ is the tangent component to the sphere $\mathcal{S}^N_{\mathcal{B}}$ of the usual gradient ∇ in $\mathbb{R}^{dN}$. Since $|\nabla_{\mathcal{S}} f^N|^2 \leq |\nabla f^N|^2$, let us compute the usual gradient of f^N

$$\begin{aligned} \frac{\left|\nabla f^N\right|^2}{f^N} &= \sum_{i=1}^N \frac{\left|\nabla_{\mathbb{R}^d} f^N\right|^2}{f^N} \\ &= \frac{1}{Z_N(f; \sqrt{dN}, 0)} \sum_{i=1}^N \frac{|\nabla_i f_i|^2}{f_i} f_1 \cdots f_{i-1}\, f_{i+1} \cdots f_N \end{aligned}$$

where $f_i = f(v_i)$.

We can return to the Fisher's information to obtain

$$\begin{aligned}
\frac{1}{N}\, I(F^N|\gamma^N) &\le \frac{1}{N} \int_{\mathcal{S}^N_{\mathcal{B}}} \frac{\left|\nabla f^N\right|^2}{f^N}\, d\gamma^N \\
&= \frac{1}{N} \int_{\mathcal{S}^N_{\mathcal{B}}} \frac{1}{Z_N(f;\sqrt{dN},0)} \sum_{i=1}^N \frac{|\nabla_i f_i|^2}{f_i} f_1 \cdots f_{i-1}\, f_{i+1} \cdots f_N\, d\gamma^N \\
&= \int_{\mathbb{R}^d} \frac{|\nabla_{v_1} f_1|^2}{f_1}\, \frac{Z_{N-1}(f;\sqrt{dN-|v_1|^2},-v_1)}{Z_N(f;\sqrt{dN},0)}\, d\gamma^N_1.
\end{aligned}$$

In the proof of Theorem 1.18 we computed the quantity

$$\frac{Z'_{N-1}(f;\sqrt{dN-|v_1|^2},-v_1)}{Z'_N(f;\sqrt{dN},0)}\, \gamma^N_1(v_1) = \theta^N_1(v_1)\gamma(v_1)$$

with $|\theta^N_1(v_1)| \le C'$. Now, we use the fact that

$$\frac{Z_{N-1}(f;\sqrt{dN-|v_1|^2},-v_1)}{Z_N(f;\sqrt{dN},0)} = \frac{1}{\gamma(v_1)}\, \frac{Z'_{N-1}(f;\sqrt{dN-|v_1|^2},-v_1)}{Z'_N(f;\sqrt{dN},0)}$$

to obtain

$$\frac{1}{N}\, I(F^N|\gamma^N) \le \int_{\mathbb{R}^d} \frac{|\nabla_{v_1} f_1|^2}{f_1}\, \theta^N_1(v_1)\, dv_1 \le C. \tag{1.69}$$

Since $\mathcal{S}^N_{\mathcal{B}}$ has positive Ricci curvature (because it has positive curvature), by [78, Theorem 30.22] and [53] the following HWI inequalities hold

$$\begin{aligned}
H(F^N|\gamma^N) - H(G^N|\gamma^N) &\le \frac{\pi}{2}\sqrt{I(F^N|\gamma^N)}\, W_2(F^N,G^N), \\
H(G^N|\gamma^N) - H(F^N|\gamma^N) &\le \frac{\pi}{2}\sqrt{I(G^N|\gamma^N)}\, W_2(F^N,G^N).
\end{aligned} \tag{1.70}$$

Remark 1.32. In the original HWI inequality, the 2-MKW distance is defined with the geodesic distance on $\mathcal{S}^N_{\mathcal{B}}$, however here we use on $\mathcal{S}^N_{\mathcal{B}}$ the Euclidean distance inherited from $\mathbb{R}^{dN}$. Fortunately, these distance are equivalent, hence the HWI inequality holds in our case adding a factor $\pi/2$ on the right-hand side.

Multiplying both sides by $1/N$ we obtain

$$\begin{aligned}
\frac{1}{N}\, H(F^N|\gamma^N) - \frac{1}{N}\, H(G^N|\gamma^N) &\le \frac{\pi}{2}\sqrt{\frac{I(F^N|\gamma^N)}{N}}\, \frac{W_2(F^N,G^N)}{\sqrt{N}}, \\
\frac{1}{N}\, H(G^N|\gamma^N) - \frac{1}{N}\, H(F^N|\gamma^N) &\le \frac{\pi}{2}\sqrt{\frac{I(G^N|\gamma^N)}{N}}\, \frac{W_2(F^N,G^N)}{\sqrt{N}}.
\end{aligned}$$

Since $N^{-1}I(F^N|\gamma^N)$ and $N^{-1}I(G^N|\gamma^N)$ are bounded, we deduce

$$\left|\frac{1}{N}\, H(F^N|\gamma^N) - H(G^N|\gamma^N)\right| \le C\, \frac{W_2(F^N,G^N)}{\sqrt{N}}. \tag{1.71}$$

Finally, we write

$$\left|\frac{1}{N}\,H(G^N|\gamma^N) - H(f|\gamma)\right| \le \left|\frac{1}{N}\,H(G^N|\gamma^N) - \frac{1}{N}\,H(F^N|\gamma^N)\right| \\ + \left|\frac{1}{N}\,H(F^N|\gamma^N) - H(f|\gamma)\right|$$

and thanks to the later estimate (1.71) with the triangle inequality for the first term of the right-hand side and Theorem 1.19 for the second one, we obtain

$$\left|\frac{1}{N}\,H(G^N|\gamma^N) - H(f|\gamma)\right| \le C\left(\frac{W_2(G^N,f^{\otimes N})}{\sqrt{N}} + \frac{W_2(F^N,f^{\otimes N})}{\sqrt{N}} + \frac{1}{\sqrt{N}}\right). \qquad (1.72)$$

Now we have to estimate the second term of the right-hand side. Hence, thanks to Lemma 1.30 we have

$$\frac{W_2(F^N,f^{\otimes N})}{\sqrt{N}} \le C'\mathcal{M}_k^{\frac{1}{k}}\left(\overline{W}_1(F_2^N,f^{\otimes 2})^{\alpha_1} + N^{-\alpha_1} + N^{-\alpha_2}\right)^{\frac{1}{2}-\frac{1}{k}},$$

and from Theorem 1.18 we have $\overline{W}_1(F_2^N,f^{\otimes 2}) \le W_1(F_2^N,f^{\otimes 2}) \le CN^{-1/2}$, which yields

$$\frac{W_2(F^N,f^{\otimes N})}{\sqrt{N}} \le C'\mathcal{M}_k^{\frac{1}{k}}\left(N^{-\alpha_1/2} + N^{-\alpha_2}\right)^{\frac{1}{2}-\frac{1}{k}} \\ \le C'N^{-\frac{\alpha_1}{2}(\frac{1}{2}-\frac{1}{k})},$$

with $\alpha_1 < k(dk+d+k)^{-1}$. We conclude putting this last estimate in (1.72). □

We give a possible answer to [12, Open problem 11] in the Boltzmann's sphere framework, which is a precise version of Theorem 1.7.

Theorem 1.33. *Consider $G^N \in \mathbf{P}(\mathcal{S}_\mathcal{B}^N)$ such that G^N is f-chaotic, for some $f \in \mathbf{P}(\mathbb{R}^d)$, and suppose that*

$$M_k(G_1^N) \le C,\ k > 2, \qquad \frac{1}{N}\,I(G^N|\gamma^N) \le C. \qquad (1.73)$$

Suppose further that

$$f \in L^\infty(\mathbb{R}^d) \qquad \textit{and} \qquad f(v_1) \ge \exp(-a|v_1|^2) \qquad (1.74)$$

for some constant $a > 0$.

Then for any fixed ℓ, there exists a constant $C = C(d,\ell,\|f\|_{L^\infty}, M_k(G_1^N), N^{-1}I(G^N|\gamma^N)) > 0$ such that for all $N \ge \ell+1$ we have

$$H(G_\ell^N|f^{\otimes\ell}) \le C\,W_1(G_\ell^N,f^{\otimes\ell})^{\theta(\ell,d,k)},$$

where $\theta(\ell,d,k) = k[d\ell(k+3)+2k+4]^{-1}$. As a consequence, $H(G_\ell^N|f^{\otimes\ell}) \to 0$ when $N \to \infty$ and condition (1.19) holds.

As discussed in the introduction just after Theorem 1.7, assumptions (1.73)-(1.74) of Theorem 1.33 are natural in the case of Maxwellian molecules since they are propagated in time. However, the conditioned tensor product assumption can be made at initial time for the Boltzmann model but it is not propagated. As a consequence of this theorem, we shall obtain that condition (1.19) is propagated under the master equation for Maxwellian molecules (see point (iv) of Theorem 1.8 below).

Proof of Theorem 1.33. We write

$$\begin{aligned} H(G_\ell^N|f^{\otimes\ell}) &= \Big[H(G_\ell^N|\gamma^{\otimes\ell}) - H(f^{\otimes\ell}|\gamma^{\otimes\ell})\Big] + \int (G_\ell^N - f^{\otimes\ell}) \log \gamma^{\otimes\ell} \\ &\quad + \int (f^{\otimes\ell} - G_\ell^N) \log f^{\otimes\ell} \\ &=: T_1 + T_2 + T_3. \end{aligned}$$

Let us split the proof in several steps.

Step 1. For the first term we use the HWI inequality on $\mathbb{R}^{d\ell}$ [67],

$$T_1 = H(G_\ell^N|\gamma^{\otimes\ell}) - H(f^{\otimes\ell}|\gamma^{\otimes\ell}) \le \sqrt{I(G_\ell^N|\gamma^{\otimes\ell})}\, W_2(G_\ell^N, f^{\otimes\ell}).$$

Let us first show that the Fisher's information $I(G_\ell^N|\gamma^{\otimes\ell})$ is bounded thanks to $N^{-1} I(G^N|\gamma^N) \le C$. Thanks to [3, Example 2] (see also [16] for related inequalities) there exists some constant $C' > 0$ such that

$$\frac{I(G_\ell^N|\gamma_\ell^N)}{\ell} \le C' \, \frac{I(G^N|\gamma^N)}{N}.$$

We write then

$$\begin{aligned} I(G_\ell^N|\gamma_\ell^N) &= \int \Big|\nabla \log G_\ell^N - \nabla \log \gamma_\ell^N\Big|^2 \, G_\ell^N \\ &= I(G_\ell^N) + \int \Big[2\Delta \log \gamma_\ell^N + |\nabla \log \gamma_\ell^N|^2\Big] \, G_\ell^N, \end{aligned} \tag{1.75}$$

and then we deduce that

$$I(G_\ell^N) \le I(G_\ell^N|\gamma_\ell^N) + \int \Big[2\Delta \log \gamma_\ell^N + |\nabla \log \gamma_\ell^N|^2\Big]_- \, G_\ell^N \tag{1.76}$$

is bounded thanks to explicit computation of γ_ℓ^N in Lemma 1.10. We conclude that $I(G_\ell^N|\gamma^{\otimes\ell})$ is bounded since $M_2(G_\ell^N) = d\ell$ and writing

$$\begin{aligned} I(G_\ell^N|\gamma^{\otimes\ell}) &= I(G_\ell^N) + \int \Big[2\Delta \log \gamma^{\otimes\ell} + |\nabla \log \gamma^{\otimes\ell}|^2\Big] \, G_\ell^N \\ &= I(G_\ell^N) + M_2(G_\ell^N) - 2d\ell = I(G_\ell^N) - d\ell. \end{aligned} \tag{1.77}$$

Moreover, we have thanks to Lemma 1.28 applied for $G_\ell^N, f^{\otimes\ell} \in \mathbf{P}(\mathbb{R}^{d\ell})$

$$W_2(G_\ell^N, f^{\otimes\ell}) \le C\, M_k(G_\ell^N, f^{\otimes\ell})^{\frac{1}{2(k-1)}} \, W_1(G_\ell^N, f^{\otimes\ell})^{\frac{k-2}{2(k-1)}},$$

where $M_k(G_\ell^N, f^{\otimes \ell}) := M_k(G_\ell^N) + M_k(f^{\otimes \ell})$. We conclude then

$$T_1 \le C\, M_k(G_\ell^N, f^{\otimes \ell})^{\frac{1}{2(k-1)}}\, W_1(G_\ell^N, f^{\otimes \ell})^{\frac{k-2}{2(k-1)}}. \tag{1.78}$$

Step 2. Let us denote by B_R the ball centered at origin with radius $R > 0$ on $\mathbb{R}^{d\ell}$, by B_R^c its complementary and let $v = (v_1, \dots, v_\ell) \in \mathbb{R}^{d\ell}$. Since $\log \gamma^{\otimes \ell} = -(d/2)\log 2\pi - |v|^2/2$, we can write

$$T_2 = \frac{1}{2}\int_{B_R} (f^{\otimes \ell} - G_\ell^N)|v|^2 + \frac{1}{2}\int_{B_R^c} (f^{\otimes \ell} - G_\ell^N)|v|^2.$$

The function $\phi(v) = |v|^2$ lies in $\mathrm{Lip}(B_R)$ with $\|\nabla\phi\|_{L^\infty(B_R)} = 2R$. We obtain then

$$\begin{aligned}
\int_{B_R} (f^{\otimes \ell} - G_\ell^N)|v|^2 &\le 2R \sup_{\|\phi\|_{\mathrm{Lip}(B_R)}\le 1} \left\{ \int \phi(f^{\otimes \ell} - G_\ell^N) \right\} \\
&\le 2R \sup_{\|\phi\|_{\mathrm{Lip}(\mathbb{R}^{d\ell})}\le 1} \left\{ \int \phi(f^{\otimes \ell} - G_\ell^N) \right\} \\
&= 2R\, W_1(G_\ell^N, f^{\otimes \ell}),
\end{aligned} \tag{1.79}$$

where the last equality comes from the duality form for the W_1 distance (see for instance [78]). Next we write

$$\int_{B_R^c} (f^{\otimes \ell} - G_\ell^N)|v|^2 \le \frac{1}{R^{k-2}} \int_{B_R^c} (f^{\otimes \ell} + G_\ell^N)|v|^k = \frac{M_k(G_\ell^N, f^{\otimes \ell})}{R^{k-2}}. \tag{1.80}$$

Choosing R such that (1.79) is equal to (1.80) we get

$$T_2 \le 2^{\frac{k-2}{k-1}}\, M_k(G_\ell^N, f^{\otimes \ell})^{\frac{1}{k-1}}\, W_1(G_\ell^N, f^{\otimes \ell})^{\frac{k-2}{k-1}}. \tag{1.81}$$

Step 3. Finally, let us investigate the third term T_3. We write

$$T_3 = \int_{B_R} (f^{\otimes \ell} - G_\ell^N)\log f^{\otimes \ell} + \int_{B_R^c} (f^{\otimes \ell} - G_\ell^N)\log f^{\otimes \ell}. \tag{1.82}$$

For the first integral in (1.82) we have, since $f \in L^\infty$ and $f^{\otimes \ell}(v) \ge e^{-a|v|^2}$,

$$\int_{B_R} (f^{\otimes \ell} - G_\ell^N)\log f^{\otimes \ell} \le \left(\ell \log\|f\|_{L^\infty(B_R)} + aR^2 \right) \|f^{\otimes \ell} - G_\ell^N\|_{L^1(B_R)}.$$

Let $g = f^{\otimes \ell} - G_\ell^N$ and consider a mollifier ρ_ε, i.e. $\rho_\varepsilon(v) = \varepsilon^{-d\ell}\rho(\varepsilon^{-1}v)$, $\rho \in C_c^\infty(\mathbb{R}^{d\ell})$ with $\rho \ge 0$, $\int \rho = 1$ and $\operatorname{supp}\rho \subset B_1$. Then we have

$$\|g\|_{L^1(B_R)} \le \|g * \rho_\varepsilon\|_{L^1(B_R)} + \|g * \rho_\varepsilon - g\|_{L^1(B_R)}.$$

For the first term we obtain

$$\begin{aligned}\|g * \rho_\varepsilon\|_{L^1(B_R)} &= \int_{B_R} \left\{ \int |\rho_\varepsilon(w-v)|\,|f^{\otimes\ell}(v) - G_\ell^N(v)|\,dv \right\} dw \\ &\le \|\nabla\rho_\varepsilon\|_{L^\infty(B_R)}\, W_1(G_\ell^N, f^{\otimes\ell}) \int_{B_R} dw \\ &\le \frac{C}{\varepsilon^{d\ell+1}}\, R^{d\ell}\, W_1(G_\ell^N, f^{\otimes\ell}).\end{aligned}$$

Moreover, for the second one we have

$$\|g * \rho_\varepsilon - g\|_{L^1(B_R)} \le \varepsilon \|\nabla g\|_{L^1} \le \varepsilon \left(\|\nabla f^{\otimes\ell}\|_{L^1} + \|\nabla G_\ell^N\|_{L^1} \right).$$

By Theorem 1.23, we have $I(f^{\otimes\ell}|\gamma^{\otimes\ell}) \le C$ and then we deduce that $\|\nabla f^{\otimes\ell}\|_{L^1}$ is finite. Moreover, the boundness of $I(G_\ell^N)$ (see (1.76)) implies that $\|\nabla G_\ell^N\|_{L^1}$ is also finite. We have then

$$\begin{aligned}\|f^{\otimes\ell} - G_\ell^N\|_{L^1(B_R)} &\le \frac{C}{\varepsilon^{d\ell+1}}\, R^{d\ell}\, W_1(G_\ell^N, f^{\otimes\ell}) + C\varepsilon \\ &\le C\, R^{\frac{d\ell}{d\ell+2}}\, W_1(G_\ell^N, f^{\otimes\ell})^{\frac{1}{d\ell+2}},\end{aligned}$$

where we have optimized ε.

For the second integral in (1.82) we have

$$\int_{B_R^c} (f^{\otimes\ell} - G_\ell^N) \log f^{\otimes\ell} \le \ell \log \|f\|_{L^\infty} \frac{M_k(G_\ell^N, f^{\otimes\ell})}{R^k}.$$

We conclude then, optimizing in R,

$$\begin{aligned}T_3 &\le C \left(\ell \log \|f\|_{L^\infty(B_R)} + aR^2 \right) R^{\frac{d\ell}{d\ell+2}}\, W_1(G_\ell^N, f^{\otimes\ell})^{\frac{1}{d\ell+2}} + \ell \log \|f\|_{L^\infty} \frac{M_k(G_\ell^N, f^{\otimes\ell})}{R^k} \\ &\le C\, W_1(G_\ell^N, f^{\otimes\ell})^{\frac{k}{d\ell(k+3)+2k+4}}.\end{aligned} \tag{1.83}$$

Finally, gathering (1.78), (1.81) and (1.83), we obtain

$$\begin{aligned}H(G_\ell^N|f^{\otimes\ell}) &\le C \left(W_1(G_\ell^N, f^{\otimes\ell})^{\frac{k-2}{2(k-1)}} + W_1(G_\ell^N, f^{\otimes\ell})^{\frac{k-2}{k-1}} + W_1(G_\ell^N, f^{\otimes\ell})^{\frac{k}{d\ell(k+3)+2k+4}} \right) \\ &\le C\, W_1(G_\ell^N, f^{\otimes\ell})^{\frac{k}{d\ell(k+3)+2k+4}},\end{aligned}$$

where $C = C(d, \ell, \|f\|_{L^\infty}, M_k(G_1^N), N^{-1} I(G^N|\gamma^N))$.

□

1.5 Application to the Boltzmann equation

We can apply our results to the spatially homogeneous Boltzmann equation (equations (1.5) and (1.4) in Section 1.1) with true Maxwellian molecules (1.8).

We prove now Theorem 1.8.

Proof of Theorem 1.8 (i). We found the proof in [62, Theorem 7.10]. □

Proof of Theorem 1.8 (ii). First of all, from [62, Theorem 5.1], for all $t \geq 0$, G_t^N is f_t-chaotic. Now, we split the proof in several steps.

Step 1. Let G_0^N be built as in Theorem 1.18, i.e. $G_0^N = [f_0^{\otimes N}]_{\mathcal{S}_{\mathcal{B}}^N}$, which is possible since $f_0 \in \mathbf{P}_6(\mathbb{R}^d)$ and $I(f_0|\gamma)$ is finite. We know from [62, Lemma 7.4] that for all $t \geq 0$ the normalized Fisher's information $N^{-1} I(G_t^N|\gamma^N)$ is bounded since $N^{-1} I(G_t^N|\gamma^N) \leq N^{-1} I(G_0^N|\gamma^N)$ and the later one is bounded by construction (see equation (1.69)). Moreover, $M_6(\Pi_1(G_0^N))$ is bounded by construction, thus for all $t \geq 0$, $M_6(\Pi_1(G_t^N))$ is also bounded thanks to [62, Lemma 5.3] .

We can then apply Theorem 1.31 to G_t^N (taking $G^N = G_t^N$ and $f = f_t$ in the notation of that theorem) and we obtain that for any $\beta < (k-2)[4(dk+d+k)]^{-1}$ there exists $C' = C'(\beta)$ such that

$$\left| \frac{1}{N} H(G_t^N|\gamma^N) - H(f_t|\gamma) \right| \leq C C' \left(\frac{W_2(G_t^N, f_t^{\otimes N})}{\sqrt{N}} + N^{-\beta} \right). \tag{1.84}$$

We have then to estimate the first term of the right-hand side and we shall use the result of propagation of chaos proved in [62].

Step 2. Thanks to the result of propagation of chaos in [62, Theorems 5.1 and 5.2] we have, for $s > 2 + d/4$,

$$\sup_{t \geq 0} \left\| \Pi_2(G_t^N) - f_t^{\otimes 2} \right\|_{H^{-s}} \leq C \, \mathcal{W}_{W_2} \left(\widehat{G}_0^N, \delta_{f_0} \right) \tag{1.85}$$

where we recall that $\widehat{G}_0^N$, $\delta_{f_0} \in \mathbf{P}(\mathbf{P}(\mathbb{R}^d))$ are defined in (1.63) and $\mathcal{W}_{W_2}(\widehat{G}_0^N, \delta_{f_0})$ in (1.64), more precisely

$$\mathcal{W}_{W_2}\left(\widehat{G}_0^N, \delta_{f_0}\right) = \int_{\mathbb{R}^{dN}} W_2(\mu_V^N, f_0) \, G_0^N(dV).$$

We recall that we want to estimate the first term of the right-hand side of (1.84) and we shall explain how we can obtain it from (1.85). On the one hand, for the right-hand side of (1.85) we shall obtain a estimate of the type

$$\mathcal{W}_{W_2}\left(\widehat{G}_0^N, \delta_{f_0}\right) \leq C \, \left[W_1(\Pi_2(G_0^N), f_0^{\otimes 2}) + N^{-\theta_2} \right]^{\theta_1}$$

since we can estimate $W_1(\Pi_2(G_0^N), f_0^{\otimes 2})$ from Theorem 1.18. On the other hand, for the left-hand side of (1.85), we shall deduce an estimate like

$$\frac{1}{\sqrt{N}} \, W_2(G_t^N, f_t^{\otimes N}) \leq C \left\| \Pi_2(G_t^N) - f_t^{\otimes 2} \right\|_{H^{-s}}^{\theta_3}$$

to be able to conclude.

Step 3. First of all, we deduce from (1.65) in Lemma 1.29,

$$\mathcal{W}_{W_2}\left(\widehat{G}_0^N, \delta_{f_0}\right) \leq 2^{\frac{2}{3}} \mathcal{M}_k^{\frac{1}{k}} \mathcal{W}_{\overline{W}_1}\left(\widehat{G}_0^N, \delta_{f_0}\right)^{\frac{1}{2}-\frac{1}{k}}.$$

Then, thanks to (1.67) in Lemma 1.29 we obtain

$$\mathcal{W}_{W_2}\left(\widehat{G}_0^N, \delta_{f_0}\right) \leq 2^{\frac{2}{3}} \mathcal{M}_k^{\frac{1}{k}} \left(C_{\alpha_1} \mathcal{M}_k^{\frac{1}{k}} \left(\overline{W}_1(\Pi_2(G_0^N), f_0^{\otimes 2}) + N^{-1}\right)^{\alpha_1}\right)^{\frac{1}{2}-\frac{1}{k}},$$

and using Theorem 1.18, which tell us $\overline{W}_1(\Pi_2(G_0^N), f_0^{\otimes 2}) \leq CN^{-1/2}$, we deduce

$$\mathcal{W}_{W_2}\left(\widehat{G}_0^N, \delta_{f_0}\right) \leq C_{\alpha_1} N^{-\frac{\alpha_1}{2}\left(\frac{1}{2}-\frac{1}{k}\right)}, \tag{1.86}$$

where we recall that $\alpha_1 < k(dk+d+k)^{-1}$.

Step 4. Thanks to [46, Lemma 2.1] applied to $\Pi_2(G_t^N)$ and $f_t^{\otimes 2} \in \mathbf{P}(\mathbb{R}^{2d})$, for any $s > d/2$ (with $d \geq 2$) there exists $C := C(d,s)$ such that

$$\overline{W}_1(\Pi_2(G_t^N), f_t^{\otimes 2}) \leq C M_k(\Pi_2(G_t^N), f_t^{\otimes 2})^{\frac{2d}{2d+2ks}} \left\| \Pi_2(G_t^N) - f_t^{\otimes 2} \right\|_{H^{-s}}^{\frac{2k}{2d+2ks}}.$$

Furthermore, from Lemma 1.30 we obtain that there exists a constant $C := C(d,k,\alpha_1,\alpha_2)$ such that

$$\frac{W_2(G_t^N, f_t^{\otimes N})}{\sqrt{N}} \leq C\mathcal{M}_k^{\frac{1}{k}} \left(\overline{W}_1(\Pi_2(G_t^N), f_t^{\otimes 2})^{\alpha_1} + N^{-\alpha_1} + N^{-\alpha_2}\right)^{\frac{1}{2}-\frac{1}{k}}.$$

Finally, gathering these two estimates with (1.85) and (1.86) we obtain that there exists $C := C(d,s,\alpha_1,\alpha_2,M_k(f_0),M_k(\Pi_1(G_0^N)))$ such that

$$\begin{aligned}\frac{W_2(G_t^N, f_t^{\otimes N})}{\sqrt{N}} &\leq C\left(N^{-\alpha_1^2\left(\frac{k}{d+ks}\right)\left(\frac{1}{2}-\frac{1}{k}\right)} + N^{-\alpha_1} + N^{-\alpha_2}\right)^{\frac{1}{2}-\frac{1}{k}} \\ &\leq CN^{-\epsilon},\end{aligned} \tag{1.87}$$

where

$$\begin{aligned}\epsilon &= \alpha_1^2\left(\frac{k}{d+ks}\right)\left(\frac{1}{2}-\frac{1}{k}\right)^2 \\ &< \left(\frac{k-2}{2(dk+d+k)}\right)^2 \frac{k}{d+ks} \\ &< \left(\frac{k-2}{2(dk+d+k)}\right)^2 \frac{4k}{dk+4d+8k}\end{aligned}$$

using $\alpha_1 < k(dk+d+k)^{-1}$ and $s > 2 + d/4$ from (1.85). We conclude taking $k=6$ and gathering (1.87) with (1.84).

$\square$

Proof of Theorem 1.8 (*iii*). The proof is a consequence of points (*i*) and Theorem 1.25. Since we have $f_0 \in \mathbf{P}_6 \cap L^\infty(\mathbb{R}^d)$, $f_0(v_1) \geq \exp(-\alpha|v_1|^2 + \beta)$ and

$$\lim_{N\to\infty} \frac{1}{N} H(G_0^N | [f_0^{\otimes N}]_{\mathcal{S}_{\mathcal{B}}^N}) = 0,$$

Theorem 1.25 implies that G_0^N is entropically f_0-chaotic. Moreover, for all $t > 0$ the solution f_t is bounded by below by a Maxwellian, i.e. $f_t(v_1) \geq \exp(-\bar{\alpha}|v_1|^2 + \bar{\beta})$ for $\bar{\alpha} > 0$ and $\bar{\beta} \in \mathbb{R}$, and also lies in $\mathbf{P}_6 \cap L^\infty(\mathbb{R}^d)$ (see for example [76] and the references therein). By point (*i*), for all $t > 0$ the solution G_t^N is entropically f_t-chaotic, then applying once more Theorem 1.25 we deduce that

$$\lim_{N\to\infty} \frac{1}{N} H(G_t^N | [f_t^{\otimes N}]_{\mathcal{S}_{\mathcal{B}}^N}) = 0.$$

□

Proof of Theorem 1.8 (*iv*). The proof is a consequence of Theorem 1.33. From the assumptions on f_0 and G_0^N, we conclude by Theorem 1.33 that G_0^N satisfies condition (1.19)

$$\forall\, \ell \in \mathbb{N}, \qquad \lim_{N\to\infty} H(\Pi_\ell(G_0^N) | f_0^{\otimes \ell}) = 0.$$

As already said in Step 1 of the proof of point (*iv*) of Theorem 1.8, for all $t \geq 0$, the normalized Fisher' information $N^{-1} I(G_t^N|\gamma^N)$ is bounded, as well as $M_k(\Pi_1(G_t^N))$. Furthermore, for all $t \geq 0$, we have $f_t \in L^\infty(\mathbb{R}^d)$ and $f_t(v_1) \geq \exp(-\bar{\alpha}|v_1|^2 + \bar{\beta})$ for some $\bar{\alpha} > 0$ and $\bar{\beta} \in \mathbb{R}$ (see point (*iii*) above). Hence, using once more Theorem 1.33, we conclude that for all $t \geq 0$, G_t^N satisfies condition (1.19)

$$\forall\, \ell \in \mathbb{N}, \qquad \lim_{N\to\infty} H(\Pi_\ell(G_t^N) | f_t^{\otimes \ell}) = 0.$$

□

1.A Auxiliary results

We prove here some auxiliary results used in Section 1.2 and Section 1.3.

1.A.1 Change of variables

We present the proof of Lemma 1.9 in Section 1.2.

Proof of Lemma 1.9. Thanks to (1.21) we have

$$|u_N|^2 = \frac{1}{N}\left(\sum_{i=1}^N |v_i|^2 + 2\sum_{i=1}^{N-1}\sum_{j>i}^N v_i \cdot v_j\right)$$

and, for $1 \le k \le N-1$,

$$|u_k|^2 = \frac{1}{k(k+1)} \left(\sum_{i=1}^{k} |v_i|^2 + 2 \sum_{i=1}^{k-1} \sum_{j>i}^{k} v_i \cdot v_j + k^2 |v_{k+1}|^2 - 2k \sum_{i=1}^{k} v_i \cdot v_{k+1} \right).$$

We deduce from these estimates that $|u_1|^2 + \cdots + |u_N|^2 =: I_1 + I_2$ with

$$\begin{aligned} I_1 &= \sum_{k=1}^{N-1} \left(\frac{1}{k(k+1)} \sum_{i=1}^{k} |v_k|^2 + \frac{k}{k+1} |v_{k+1}|^2 \right) + \frac{1}{N} \sum_{i=1}^{N} |v_i|^2 \\ &=: \sum_{k=1}^{N-1} A_k + A_N \end{aligned}$$

and

$$\begin{aligned} I_2 &= 2 \left[\sum_{k=1}^{N-1} \left(\frac{1}{k(k+1)} \sum_{i=1}^{k-1} \sum_{j=i+1}^{k} v_i \cdot v_j - \frac{1}{k+1} \sum_{i=1}^{k} v_i \cdot v_{k+1} \right) - \frac{1}{N} \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} v_i \cdot v_j \right] \\ &=: 2 \left[\sum_{k=1}^{N-1} B_k + B_N \right]. \end{aligned}$$

First of all, looking to I_1 we easily see that $|v_N|^2$ appears only in A_{N-1} and A_N, so its coefficient is $(N-1)/N + 1/N = 1$. For m such that $2 \le m \le N-1$, $|v_m|^2$ appears in $A_{m-1}, A_m, \ldots, A_{N-1}$ and A_N, hence its coefficient is given by

$$\frac{m-1}{m} + \sum_{j=m}^{N-1} \frac{1}{j(j+1)} + \frac{1}{N} = 1.$$

The coefficient of $|v_1|^2$ is the same of $|v_2|^2$ since there is no A_0. We conclude then

$$I_1 = |v_1|^2 + \cdots + |v_N|^2.$$

We can compute I_2 in the same way. For $1 \le m \le N-1$, $v_m \cdot v_N$ appears only in B_{N-1} and B_N, so its coefficient is $-1/N + 1/N = 0$. Moreover, for $1 \le m < p \le N-1$, $v_m \cdot v_p$ appears in $B_{p-1}, B_p, \ldots, B_{N-1}$ and B_N, hence its coefficient is given by

$$-\frac{1}{p} + \sum_{j=p}^{N-1} \frac{1}{p(p+1)} + \frac{1}{N} = 0.$$

Finally, we conclude that $|u_1|^2 + \cdots + |u_N|^2 = |v_1|^2 + \cdots + |v_N|^2 = r^2$ and $u_N = z/\sqrt{N}$ follows easily from (1.21).

The last point to prove is that the Jacobien is equal to one. To simplify we consider $d = 1$, the general case being the same. Consider the matrix M_N that represents the linear application in (1.21), i.e. $M_N u = v$, where $u = (u_1, \ldots, u_N) \in \mathbb{R}^N$ and $v = (v_1, \ldots, v_N) \in \mathbb{R}^N$.

We claim that $\det(M_N) = 1$. Indeed we have

$$M_N = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & \cdots & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ \frac{1}{\sqrt{(N-1)N}} & \cdots & \cdots & \frac{1}{\sqrt{(N-1)N}} & -\frac{(N-1)}{\sqrt{(N-1)N}} \\ \frac{1}{\sqrt{N}} & \cdots & \cdots & \cdots & \frac{1}{\sqrt{N}} \end{pmatrix}$$

and it can be written in the form $M_N = D_N A_N$ with a diagonal matrix D_N,

$$M_N = \begin{pmatrix} \frac{1}{\sqrt{2}} & & & & \\ & \frac{1}{\sqrt{6}} & & & \\ & & \ddots & & \\ & & & \frac{1}{\sqrt{(N-1)N}} & \\ & & & & \frac{1}{\sqrt{N}} \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 & \cdots & 0 \\ 1 & 1 & -2 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 1 & \cdots & \cdots & 1 & -(N-1) \\ 1 & \cdots & \cdots & \cdots & 1 \end{pmatrix}.$$

Let us prove the claim by recurrence. For $N = 2$ is clear that $\det(D_2) = 1/2$ and $\det(A_2) = 2$, which implies $\det(M_2) = 1$. Then, supposing that $\det(M_{N-1}) = 1$ we have

$$\det(M_{N-1}) = \left(\prod_{k=1}^{N-2} \frac{1}{\sqrt{k(k+1)}} \times \frac{1}{\sqrt{(N-1)}} \right) \det(A_{N-1}) = 1 \tag{1.88}$$

since $\det(D_{N-1})$ is easily computed. Moreover, we have the following relation $\det(A_N) = N \det(A_{N-1})$. Hence we deduce that

$$\begin{aligned} \det(M_N) &= \left(\prod_{k=1}^{N-1} \frac{1}{\sqrt{k(k+1)}} \times \frac{1}{\sqrt{N}} \right) \det(A_N) \\ &= \left(\prod_{k=1}^{N-2} \frac{1}{\sqrt{k(k+1)}} \times \frac{1}{\sqrt{(N-1)N}} \times \frac{1}{\sqrt{N}} \right) N \det(A_{N-1}) \\ &= 1 \end{aligned}$$

thanks to (1.88), which concludes the proof of the claim.

□

1.A.2 Regularity lemma

Lemma 1.34. *Let $f \in \mathbf{P}(\mathbb{R}^d)$. Suppose $f \in L^p \cap L_s(\mathbb{R}^d)$ for $p > 1$ and $s > 0$. Then $f \in L^q_m(\mathbb{R}^d)$ with $q < p$ and $m = s(p-q)(p-1)$.*

Proof. Let us compute the L^q_m norm of f,

$$\begin{aligned} \|f\|^q_{L^q_m} &= \int \left(1 + |v|^2\right)^{m/2} f(v)^q \, dv \\ &\leq C \left(\int f(v)^q \, dv + \int |v|^m f(v)^q \, dv \right). \end{aligned}$$

For the first term we have $\|f\|_{L^q}^q \le \|f\|_{L^p}^q$ and for the second one we obtain

$$\int |v|^m f(v)^q \, dv \le \left(\int |v|^{mr/(r-1)} f(v)^{(q-\alpha)r/(r-1)} \right)^{(r-1)/r} \left(\int f(v)^{\alpha r} \right)^{1/r}$$

by Holder's inequality for some $r > 1$ and $0 < \alpha < q$. Now choosing $r = p/\alpha$ and choosing α such that $(q-\alpha)r/(r-1) = 1$, i.e. $\alpha = p(q-1)/(p-1)$ we obtain

$$\int |v|^m f(v)^q \, dv \le \left(\int |v|^{m(p-1)/(p-q)} f(v) \right)^{(p-q)/(p-1)} \left(\int f(v)^p \right)^{(q-1)/(p-1)}.$$

Finally, choosing $m = s(p-q)/(p-1)$ we conclude with

$$\|f\|_{L^q_m}^q \le C \left(\|f\|_{L^p}^q + \|f\|_{L_s}^{(p-q)/(p-1)} \|f\|_{L^p}^{p(q-1)/(p-1)} \right).$$

□

Chapitre 2

Chaos and entropic chaos in Kac's model without high moments

ABSTRACT. In this paper we present a new local Lévy Central Limit Theorem, showing convergence to stable states that are not necessarily the Gaussian, and use it to find new and intuitive entropically chaotic families with underlying one-particle function that has moments of order 2α, with $1 < \alpha < 2$. We also discuss a lower semi continuity result for the relative entropy with respect to our specific family of functions, and use it to show a form of stability property for entropic chaos in our settings.

2.1 Introduction

One of the most influential equations in the kinetic theory of gases, describing the evolution in time of the distribution function of a dilute gas, is the so-called Boltzmann equation. While widely used, the Boltzmann equation poses two fundamental questions in Kinetic Theory, pertaining to the spatially homogeneous case: The validity of the equation and the rate of convergence to equilibrium in it.

In his 1956 paper, [49], Kac attempted to give a partial solution to these two problems. Kac introduced a many-particle model, consisting of N indistinguishable particle with one dimensional velocities, undergoing binary collision and constrained to the energy sphere $\mathbb{S}^{N-1}(\sqrt{N})$, which we will call 'the Kac's sphere'. Kac's evolution equation is given by

$$\partial_t F_N\left(v_1, \dots, v_N\right) = -N(I-Q)F_N\left(v_1, \dots, v_N\right), \tag{2.1}$$

where F_N represents the probability density function of the N particles, and the gain term Q is given by

$$QF\left(t, v_1, \dots, v_N\right) = \frac{1}{2\pi}\frac{2}{N(N-1)}\sum_{i<j}\int_0^{2\pi} F\left(t, v_1, \dots, v_i(\theta), \dots, v_j(\theta), \dots, v_N\right) d\theta, \tag{2.2}$$

with

$$v_i(\theta) = v_i \cos(\theta) + v_j \sin(\theta), \quad v_j(\theta) = -v_i \sin(\theta) + v_j \cos(\theta). \tag{2.3}$$

Motivated by Boltzmann's 'Stosszahlansatz' assumption, Kac defined the concept of *Chaoticity* (what he called 'the Boltzmann property' in his paper) which measures the asymptotic independence of a finite, fixed, number of particles, as the number of total particles goes to infinity. In its modern variant the definition of chaoticity is:

Definition 2.1. Let X be a Polish space. A family of symmetric probability measures on X^N, $\{\mu_N\}_{N\in\mathbb{N}}$, is called $\mu-$chaotic, where μ is a probability measure on X, if for any $k \in \mathbb{N}$

$$\lim_{N\to\infty} \Pi_k\left(\mu_N\right) = \mu^{\otimes k}, \tag{2.4}$$

where $\Pi_k(\mu_N)$ is the $k-$th marginal of μ_N and the limit is in the weak topology.

In the context of Kac's work, the symmetric family of measures under investigation is supported on Kac's sphere and is given by $\mu_N = F_N d\sigma^N$, where $d\sigma^N$ is the uniform probability measure on Kac's sphere. The limit measure μ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$ with a probability density function f. We say that the family $\{F_N\}_{N\in\mathbb{N}}$ is $f-$chaotic in that particular setting. As an interesting remark, we note that it is known, see for instance [70], that it is enough to check the marginals for $k = 1, 2$ in order to conclude chaoticity.

Using a beautiful combinatorial argument, Kac showed that the property of chaoticity propagates with his evolution equation, i.e. if $\{F_N\left(0, v_1, \dots, v_N\right)\}_{N\in\mathbb{N}}$ is f_0-chaotic then the solution to equation (2.1), $\{F_N\left(t, v_1, \dots, v_N\right)\}_{N\in\mathbb{N}}$ is f_t-chaotic, where f_t solves a caricature of the Boltzmann equation.

While Kac's model is not entirely realistic (as it doesn't conserve momentum) and his limit equation wasn't the Boltzmann equation, the ideas presented in his paper were powerful enough that McKean managed to extend them to the $d-$dimensional case (see [57]). Under similar condition to those presented by Kac, McKean construct a similar $N-$particle model from which the *real* spatially homogeneous Boltzmann equation arose as mean field limit for many cases. We will not discuss this model in this work, and refer the interested reader to [15, 35, 57] for more information.

Giving a partial answer to the validation of the Boltzmann equation, Kac set out to try and find a partial solution to the rate of convergence as well. Using his linear model he conjectured that the rate of convergence to equilibrium in the natural L^2 norm would be exponential, with a rate that is independent of the number of particles (the so-called spectral gap problem). While this proved to be true eventually, it is easy to see that for very natural chaotic families the norm of the initial datum depends *very* strongly on N - exponentially so, making the possibility of taking a limit impossible. This effect is due to the multiplicative nature of the L^2 norm and the definition of chaoticity.

A different kind of 'distance' was needed, one that respects the property of chaoticity. To that end the concept of the relative entropy was invoked.

Definition 2.2. Given two probability measures, μ, ν, on a Polish space X, we define the relative entropy of μ with respect to ν, $H(\mu|\nu)$, as

$$H(\mu|\nu) = \int_X h \log h d\nu, \tag{2.5}$$

where $h = \frac{d\mu}{d\nu}$, and $H(\mu|\nu) = \infty$ if μ is not absolutely continuous with respect to ν.

Definition 2.3. Given a probability density function F_N on Kac's sphere we define the *entropy of* F_N to be

$$H_N(F_N) = H\left(F_N d\sigma^N | d\sigma^N\right) = \int_{\mathbb{S}^{N-1}\left(\sqrt{N}\right)} F_N \log F_N d\sigma^N. \tag{2.6}$$

The reason for this choice of a distance functional lies with the so-called *extensivity* property of the entropy: In a very intuitive way, we'd like to think that 'nice' $f-$chaotic families behave like $F_N \approx f^{\otimes N}$, as such

$$H_N(F_N) \approx N \int_{\mathbb{R}} f(v) \log\left(\frac{f(v)}{\gamma(v)}\right) dv, \tag{2.7}$$

where γ is the standard Gaussian on $\mathbb{R}$, giving a *linear dependence* in N, instead of an exponential one. This intuition was defined formally in [12], where the authors investigated the entropy functional on the Kac's sphere:

Definition 2.4. A symmetric $\mu-$chaotic family of probability measures on Kac's sphere, $\{\mu_N\}_{N\in\mathbb{N}}$, is called entropically chaotic if

$$\lim_{N\to\infty} \frac{H_N\left(\mu_N | d\sigma^N\right)}{N} = H\left(\mu|\gamma\right), \tag{2.8}$$

where $d\sigma^N$ is the uniform probability measure on Kac's sphere and $H(\mu|\gamma)$ is the relative entropy of μ and $\gamma(v)dv$.

As before, in the setting of Kac's model we use the measures $\mu_N = F_N d\sigma^N$ and $\mu = f(x)dx$, and say that $\{F_N\}_{N\in\mathbb{N}}$ is $f-$entropically chaotic if (2.8) is satisfied. The concept of entropic chaoticity is much stronger than that of chaoticity as it involves the correlation between arbitrary number of particles. This was shown to be true in [12] and we will verify it in our particular setting as well later on in this paper.

At this point we'd like to mention that the notion of chaoticity, and the use of a rescaled relative entropy between two measures on X^N and its connection to the relative entropy of the limit measures, doesn't solely lie in the realm of Kac's Model. Indeed, in [6] the authors have investigated propagation of chaos for the mean field Gibbs measure with potential F using exactly such ideas and a type of lower semi continuity result for the relative entropy.

Of particular interest to the study of chaoticity, and entropic chaoticity, are special measures that are obtained by a tensorising a measure, μ, on $\mathbb{R}$, and restricting the tensor product to Kac's sphere. In our particular study, much like that of [12], we'll be interested in measures on Kac's sphere with probability density function

$$F_N\left(v_1,\dots,v_N\right)=\frac{f^{\otimes N}\left(v_1,\dots .v_N\right)}{\mathcal{Z}_N\left(f,\sqrt{N}\right)}, \tag{2.9}$$

where f is a probability density function on $\mathbb{R}$ and the so-called *normalisation function*, $\mathcal{Z}_N\left(f,r\right)$, is defined by

$$\mathcal{Z}_N\left(f,r\right)=\int_{\mathbb{S}^{N-1}(r)} f^{\otimes N}d\sigma_r^N, \tag{2.10}$$

with $d\sigma_r^N$ the uniform probability measure on $\mathbb{S}^{N-1}(r)$. In what follows we will call probability density functions F_N of the form (2.9) *conditioned tensorisation of* f. It is worth to mention that Kac himself considered such functions, and have shown that they are chaotic when f has very strong integrability conditions.

The question of whether or not conditioned tensorisation of the function f is well defined rests heavily on the concentration of the tensorised measure $f^{\otimes N}$ on Kac's sphere. The main technical tool that is required is a local central limit theorem that shows exactly how $\mathcal{Z}_N(f,r)$ behaves asymptotically, for any $r>0$. In [12], the authors have managed to prove that:

Theorem 2.5. *Let f be a probability density on $\mathbb{R}$ such that $f\in L^p(\mathbb{R})$ for some $p>1$, $\int_{\mathbb{R}} x^2f(x)=1$ and $\int_{\mathbb{R}} x^4f(x)dx<\infty$. Then*

$$\mathcal{Z}_N(f,\sqrt{u})=\frac{2}{\sqrt{N}\Sigma\left|\mathbb{S}^{N-1}\right|u^{\frac{N-2}{2}}}\left(\frac{e^{-\frac{(u-N)^2}{2N\Sigma^2}}}{\sqrt{2\pi}}+\lambda_N(u)\right), \tag{2.11}$$

where $\Sigma^2=\int_{\mathbb{R}} v^4f(v)dv-1$ and $\sup_u|\lambda_N(u)|\underset{N\to\infty}{\longrightarrow}0$.

The above approximation yielded more than just an explanation to why our definition is appropriate. Once proven, the above easily proves the following, which can also be found in [12]:

Theorem 2.6. *Let f be a probability density on $\mathbb{R}$ such that $f\in L^p(\mathbb{R})$ for some $p>1$, $\int_{\mathbb{R}} x^2f(x)=1$ and $\int_{\mathbb{R}} x^4f(x)dx<\infty$. Then the family of conditioned tensorisation of f, given by (2.9), is $f-$chaotic. Moreover, it is $f-$entropically chaotic.*

We'd like to mention at this point that the above theorems were extended to to McKean's model by the first author in [19].

The appearance of the fourth moment of the function f in Theorem 2.5 shouldn't be too surprising: As we're trying to measure fluctuation of the random variable $K_N=\sum_{i=1}^N V_i^2$ from its mean, N, a useful quantity to consider is the variance of K_N, pertaining to the fourth moment of the underlying function f. This, however, is not a necessary condition to be able to obtain the desired concentration result. In this work we will consider families of conditioned tensorisation of a function f, where the underlying generating function f has moment of order 2α, with $1 < \alpha < 2$.

The main approximation theorem of this paper, one that extends Theorem 2.5 and lies at the heart of many subsequent proofs, relies on concepts related to $\alpha-$stable processes. We will introduce them at this point so we'll be able to state our main results.

Definition 2.7. A random variable U is said to be $\alpha-$stable for $0 < \alpha < 2$, $\alpha \neq 1$ if

$$\frac{\sum_{i=1}^{n} X_i}{n^{\frac{1}{\alpha}}}$$

has the same probability distribution function as U, where X_i are independent copies of U. Equivalently, the characteristic function of U is of the form

$$\widehat{\gamma}_{C_S,\alpha,p,q}(\xi) = e^{-C_S|\xi|^\alpha \cdot \frac{\Gamma(3-\alpha)}{\alpha(\alpha-1)} \cos\left(\frac{\pi\alpha}{2}\right)\left(1+i\mathrm{sgn}(\xi)(p-q)\tan\frac{\alpha\pi}{2}\right)}, \tag{2.12}$$

with $C_S > 0$, $p, q \geq 0$ and $p + q = 1$.

In the above, and what is to follow, we have used the convention

$$\widehat{\varphi}(\xi) = \int_{\mathbb{R}} e^{-ix\xi}\varphi(x)dx. \tag{2.13}$$

for the characteristic function, $\widehat{\varphi}$, of a probability density φ. It is also worth to mention that some books, including Feller's, refer to above definition as *strict stability*.

Remark 2.8. Equation (2.12) can be rewritten in the form

$$\widehat{\gamma}_{\sigma,\alpha,\beta}(\xi) = e^{-\sigma|\xi|^\alpha\left(1+i\beta\mathrm{sgn}(\xi)\tan\frac{\alpha\pi}{2}\right)}, \tag{2.14}$$

where

$$\sigma = C_S \cdot \frac{\Gamma(3-\alpha)}{\alpha(\alpha-1)}\cos\left(\frac{\pi\alpha}{2}\right) > 0, \;\beta = p - q.$$

We will use both forms in accordance to the situation.

Definition 2.9. The *Domain of Attraction* (in short, DA) of $\gamma_{\sigma,\alpha,\beta}$ is the set of all real random variables X such that there exist sequences $\{a_n\}_{n\in\mathbb{N}} > 0$ and $\{b_n\}_{n\in\mathbb{N}} \in \mathbb{R}$ such that

$$\frac{\sum_{i=1}^{n} X_i}{a_n} - nb_n \underset{n\to\infty}{\longrightarrow} U, \tag{2.15}$$

where X_i are independent copies of X, U is the real random variable with characteristic function $\widehat{\gamma}_{\sigma,\alpha,\beta}$ and the limit is to be understood in the weak sense. Equivalently, one can prove that the DA of $\gamma_{\sigma,\alpha,\beta}$ is the set of all real random variables X, whose characteristic function $\widehat{\psi}$ satisfies

$$n\left(\widehat{\psi}\left(\frac{\xi}{a_n}\right)e^{-ib_n\xi} - 1\right) \underset{n\to\infty}{\longrightarrow} -\sigma|\xi|^\alpha\left(1 + i\beta\mathrm{sgn}(\xi)\tan\left(\frac{\pi\alpha}{2}\right)\right), \tag{2.16}$$

where $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\mathbb{N}}$ are sequences as in (2.15) (See [36]).

Definition 2.10. The *Natural Domain of Attraction* (in short, NDA) of $\gamma_{\sigma,\alpha,\beta}$ is the subset of the DA of $\widehat{\gamma}_{\sigma,\alpha,\beta}$ for which $a_n = n^{\frac{1}{\alpha}}$ and $b_n = 0$ are applicable as a sequences in (2.15).

Definition 2.11. The *Fourier Domain of Attraction* (in short, FDA) of $\gamma_{\sigma,\alpha,\beta}$ is the set of all real random variables X whose characteristic function $\widehat{\psi}$ satisfies

$$\widehat{\psi}(\xi) = 1 - \sigma|\xi|^{\alpha}\left(1 + i\beta \operatorname{sgn}(\xi)\tan\left(\frac{\pi\alpha}{2}\right)\right) + \eta_{\psi}(\xi), \tag{2.17}$$

where $\frac{\eta_\psi(\xi)}{|\xi|^\alpha} \in L^\infty$ and $\frac{\eta_\psi(\xi)}{|\xi|^\alpha} \underset{\xi\to 0}{\longrightarrow} 0$. The function η_ψ is called *the reminder function* of $\widehat{\psi}$.

The general local Lévy central limit theorem we prove in this paper the following:

Theorem 2.12. *Let g be the probability density function of a random real variable X. Assume that $g \in L^p(\mathbb{R})$ for some $p > 1$ and g is in the NDA of $\gamma_{\sigma,\alpha,\beta}$ for some $\sigma > 0$, β and $1 < \alpha < 2$. Assume in addition that g has finite moment of some order. Define*

$$g_N(x) = N^{\frac{1}{\alpha}} g^{*N}\left(N^{\frac{1}{\alpha}}x\right),$$

and

$$\gamma_{\sigma,\alpha,\beta}(x) = \frac{1}{2\pi}\int_{\mathbb{R}} \widehat{\gamma}_{\sigma,\alpha,\beta}(\xi) e^{i\xi x} d\xi. \tag{2.18}$$

Then, for any positive sequence $\{\beta_N\}_{N\to\infty}$ that converges to zero as N goes to infinity, any $\tau > 0$ and N large enough we have that

$$\begin{aligned} \|g_N - \gamma_{\sigma,\alpha,\beta}\|_\infty \leq C_{g,\alpha}\Bigg(& N^{\frac{1}{\alpha}}(1 - \beta_N^{2+\tau} + \phi_\tau(\beta_N))^{N-q} + e^{-\frac{\sigma N \beta_N^\alpha}{2}} \\ & + \omega_g(\beta_N) + 2\sigma\beta_N^\alpha\left(1 + \beta^2\tan^2\left(\frac{\pi\alpha}{2}\right)\right)\Bigg) = \epsilon_\tau(N), \end{aligned} \tag{2.19}$$

where

(i) $C_{g,\alpha} > 0$ *is a constant depending only on g, its moments and α.*

(ii) q *can be chosen to be the Hölder conjugate of* $\min(2,p)$.

(iii) ϕ_τ *satisfies*

$$\lim_{x\to 0}\frac{\phi_\tau(x)}{|x|^{2+\tau}} = 0,$$

(iv) η_g *is the reminder function of $\widehat{g}$, defined in Definition 2.11, and* $\omega_g(\beta) = \sup_{|x|\leq\beta}\frac{|\eta_g(x)|}{|x|^\alpha}$.

Section 2.3, where we prove the above theorem, also provides simple condition to check when a probability density function is in the NDA of some $\gamma_{\sigma,\alpha,\beta}$ as well as a simplified case of the general theorem, one we will use in most of our applications. Theorem 2.12 will allow us to show that:

Theorem 2.13. *Let f be a probability density such that $f \in L^p$ for some $p > 1$ and $\int x^2 f(x)dx = 1$. Let*

$$\nu_f(x) = \int_{-\sqrt{x}}^{\sqrt{x}} y^4 f(y)dy \tag{2.20}$$

and assume that $\nu_f(x) \underset{x\to\infty}{\sim} C_S x^{2-\alpha}$ for some $C_S > 0$ and $1 < \alpha < 2$. Then the family of conditioned tensorisation of f, given by (2.9), is $f-$chaotic. Moreover, it is $f-$entropically chaotic.

As a special case, one has that

Theorem 2.14. *Let f be a probability density such that $f \in L^p$ for some $p > 1$ and $\int x^2 f(x)dx = 1$. Assume in addition that*

$$f(x) \underset{x\to\infty}{\sim} \frac{D}{|x|^{1+2\alpha}}, \tag{2.21}$$

for some $1 < \alpha < 2$ and $D > 0$. Then the family of conditioned tensorisation of f, given by (2.9), is $f-$chaotic. Moreover, it is $f-$entropically chaotic.

The family of conditioned tensorisation of a function f,

$$\nu_N = F_N d\sigma^N, \tag{2.22}$$

where F_N is given by (2.9), plays an important role on Kac's sphere as an 'attractor of chaoticity'. The first clue to this is the following distorted lower semi continuity property:

Theorem 2.15. *Let f be a probability density such that $f \in L^p$ for some $p > 1$ and $\int x^2 f(x)dx = 1$. Let*

$$\nu_f(x) = \int_{-\sqrt{x}}^{\sqrt{x}} y^4 f(y)dy$$

and assume that $\nu_f(x) \underset{x\to\infty}{\sim} C_S x^{2-\alpha}$ for some $C_S > 0$ and $1 < \alpha < 2$. Let μ_N be a symmetric probability measure on Kac's sphere such that for some $k \in \mathbb{N}$

$$\Pi_k(\mu_N) \underset{N\to\infty}{\rightharpoonup} \mu_k, \tag{2.23}$$

where μ_k is a probability measure on $\mathbb{R}^k$. Then, we find that

(i) $\Pi_1(\mu_N) \underset{N\to\infty}{\rightharpoonup} \Pi_1(\mu_k) = \mu$ *and*

$$H(\mu|f) \le \liminf_{N\to\infty} \frac{H_N(\mu_N|\nu_N)}{N}, \tag{2.24}$$

where $H(\mu|f)$ is the relative entropy between μ and the measure $f(v)dv$.

(ii) For any $\delta > 0$ we have that

$$\begin{aligned} \liminf_{N\to\infty} \frac{H(\mu_N|\nu_N)}{N} \ge \frac{H(\mu_k|f^{\otimes k})}{k} - \limsup_{N\to\infty} \int_{\mathbb{R}} \log\left(f(v)+\delta\right) d\Pi_1\left(\mu_N\right)(v) \\ + \int \log\left(f(v)\right) d\mu(v) - \frac{1-\int |v|^2 d\mu(v)}{2}, \end{aligned} \tag{2.25}$$

where ν_N is given by (2.22).

Theorem 2.15 is the key to proving the following stability property of entropic chaoticity:

Theorem 2.16. *Let f be a probability density such that $f \in L^p$ for some $p > 1$ and $\int x^2 f(x)dx = 1$. Let*

$$\nu_f(x) = \int_{-\sqrt{x}}^{\sqrt{x}} y^4 f(y)dy$$

and assume that $\nu_f(x) \underset{x\to\infty}{\sim} C_S x^{2-\alpha}$ for some $C_S > 0$ and $1 < \alpha < 2$. Assume in addition that $f \in L^\infty(\mathbb{R})$. Then, if $\{\mu_N\}_{N\in\mathbb{N}}$ is a family of symmetric probability measures on Kac's sphere and

$$\lim_{N\to\infty} \frac{H(\mu_N|\nu_N)}{N} = 0, \tag{2.26}$$

where ν_N is given by (2.22), we have that μ_N is $f-$chaotic. Moreover, μ_N is $f-$entropically chaotic.

A different approach to the stability problem involves the concept of the relative Fisher information functional on $\mathbb{R}$, I, and on Kac's sphere, I_N:

Definition 2.17. Given two probability measures, μ, ν on $\mathbb{R}$, we define the relative Fisher information of μ with respect to ν, $I(\mu|\nu)$, as

$$I(\mu|\nu) = \int_{\mathbb{R}} \frac{|h'(x)|^2}{h(x)} d\nu(x) = 4\int_{\mathbb{R}} |\frac{d}{dx}\sqrt{h(x)}|^2 d\nu(x), \tag{2.27}$$

where $h = \frac{d\mu}{d\nu}$, and $I(\mu|\nu) = \infty$ if μ is not absolutely continuous with respect to ν.

Given two probability measures, μ_N, ν_N on Kac's sphere, we define the relative Fisher information of μ_N with respect to ν_N, $I_N(\mu_N|\nu_N)$, as

$$I_N(\mu_N|\nu_N) = \int_{\mathbb{S}^{N-1}(\sqrt{N})} \frac{|\nabla_S h|^2}{h} d\nu, \tag{2.28}$$

where $h = \frac{d\mu_N}{d\nu_N}$, and $I_N(\mu_N|\nu_N) = \infty$ if μ_N is not absolutely continuous with respect to ν_N. Here ∇_S denotes the components of the usual gradient on $\mathbb{R}^N$ that is tangential to Kac's sphere.

Theorem 2.18. *Let $\{\mu_N\}_{N\in\mathbb{N}}$ be a family of symmetric probability measures on Kac's sphere that is $f-$chaotic. Assume that there exists $C_S > 0$ and $1 < \alpha < 2$ such that*

$$\int_{-\sqrt{x}}^{\sqrt{x}} v_1^4 d\Pi_1(\mu_N)(v_1) \underset{x\to\infty}{\sim} C_S x^{2-\alpha} \tag{2.29}$$

uniformly in N, and that

$$\frac{H_N(\mu_N|\sigma^N)}{N} \leq C, \quad \frac{I_N(\mu_N|\sigma^N)}{N} \leq C \tag{2.30}$$

for all N. Then μ_N is $f-$entropically chaotic.

The presented work is structured as follows: In Section 2.2 we will present some preliminaries to the work, including known results on the normalisation function and marginals of probability measures on Kac's sphere. Section 2.3 will be focused on proving the newly found local Lévy Central Limit Theorem, described in Theorem 2.12, as well as giving a particular version of it (with additional conditions). While we will use it in Section 2.4 where we will prove Theorems 2.13 and 2.14, Section 2.3 is interesting in its own right. As such, we present it in a self contained way (referring to definitions presented in the introduction) in hope for it to be accessible to people who are not familiar with Kac's model. In Section 2.5 we will discuss the lower semi continuity property of processes of our type (Theorem 2.15) and prove the stability theorems, Theorems 2.16 and 2.18. Once all the proofs are done, Section 2.6 will give more details about the spectral gap problem, the entropy method, Cercignani's many body conjecture and explain the connection between it and the presented work. Section 2.7 will see closing remarks for our work, including some connection between the current work and Cercignani's many body conjecture, while the Appendix will discuss a quantitative Lévy type approximation theorem, and include some additional computation that would otherwise encumber the presentation of our paper.

Lastly, we'd like to mention some references for topics that we've touched here. For more information about the Boltzmann equation we refer the interested reader to [21, 62, 77, 76]. For more information about Kac's and McKean Model, the spectral gap problem and the related Entropy method and Cercignani's many body conjecture we refer the interested reader to [3, 13, 14, 12, 19, 34, 35, 33, 46, 48, 49, 51, 54, 57, 62, 77]. For more information about propagation of chaos, from view points of Analysis, Probability and PDE, we refer the interested reader to [4, 5, 6, 12, 19, 32, 33, 46, 62, 70], and for more information about the roles of central limit theorem in the study of Kac-like equation we refer the interested reader to [5, 40, 55].

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2.2 Preliminaries

2.2.1 The normalisation function

As discussed in the introduction, the normalisation function, $\mathcal{Z}_N\left(f, \sqrt{r}\right)$, plays an important role in the proofs of chaoticity and entropic chaoticity of distribution families of the form

$$F_N = \frac{f^{\otimes N}}{\mathcal{Z}_N\left(f, \sqrt{N}\right)}.$$

In this short subsection we will give a probabilistic interpretation to it, as well as explain why it is well defined under simple conditions on f.

Lemma 2.19. *Let f be a probability density function for the real random variable V. Then*

$$\mathcal{Z}_N\left(f, \sqrt{r}\right) = \frac{2h^{*N}(r)}{|\mathbb{S}^{N-1}| r^{\frac{N-2}{2}}} \tag{2.31}$$

where h be the associated probability density function for the real random variable V^2 and h^{*N} is the $N-$th iterated convolution of h.

Proof for the above lemma can be found in [12, 34], yet we present it here for completion.

Proof. Denote by $S_N = \sum_{i=1}^N V_i^2$ the sum of independent copies of the real random variable V^2. For any function $\varphi \in C_b\left(\mathbb{R}^N\right)$, depending only on $r = \sqrt{\sum_{i=1}^N v_i^2}$ we find that

$$\mathbb{E}\varphi = \int_{\mathbb{R}^N} \varphi\left(\sqrt{\sum_{i=1}^N v_i^2}\right) \Pi_{i=1}^N f(v_i) dv_1 \dots dv_N =$$

$$|\mathbb{S}^{N-1}| \int_0^\infty \varphi(r) r^{N-1} \left(\int_{\mathbb{S}^{N-1}(r)} \Pi_{i=1}^N f(v_i) d\sigma_r^N\right) dr = |\mathbb{S}^{N-1}| \int_0^\infty \varphi(r) r^{N-1} \mathcal{Z}_N\left(f, r\right) dr$$

On the other hand

$$\mathbb{E}\varphi = \int_0^\infty \varphi\left(\sqrt{r}\right) s_N(r) dr = 2\int_0^\infty r\varphi(r) s_N\left(r^2\right) dr.$$

Since the above is valid for any φ we conclude that

$$\mathcal{Z}_N\left(f, \sqrt{r}\right) = \frac{2s_N(r)}{|\mathbb{S}^{N-1}| r^{\frac{N-2}{2}}}.$$

A known fact from probability theory states that the density function for S_N, s_N, is given by

$$s_N(u) = h^{*N}(u)$$

where h^{*N} is the $N-$th iterated convolution of h. This completes the proof. □

Remark 2.20. It is easy to see that probability density function h, associated to the probability density function f as described in the above lemma, is given by

$$h(u) = \begin{cases} \frac{f\left(\sqrt{u}\right) + f\left(-\sqrt{u}\right)}{2\sqrt{u}} & u > 0 \\ 0 & u \leq 0 \end{cases} \tag{2.32}$$

As such, using the convexity of $t \to t^q$ for any $q > 1$, we find that if in addition $f \in L^p(\mathbb{R})$ then

$$\begin{aligned}\int h(u)^{p'} du &\leq \frac{1}{2}\int_0^\infty \frac{f(\sqrt{u})^{p'} + f(-\sqrt{u})^{p'}}{u^{\frac{p'}{2}}} du = \int_{\mathbb{R}} \frac{f(x)^{p'}}{x^{p'-1}} \\ &\leq \int_{[-1,1]} \frac{f(x)^{p'}}{x^{p'-1}} + \int_{\mathbb{R}} f(x)^{p'} dx \\ \leq \left(\int_{[-1,1]} f(x)^p dx\right)^{\frac{p'}{p}} &\left(\int_{[-1,1]} \frac{dx}{x^{\frac{p(p'-1)}{p-p'}}}\right)^{\frac{p-p'}{p'}} + \int_{f>1} f(x)^p dx + \int_{f<1} f(x) dx,\end{aligned} \tag{2.33}$$

where $p' < p$. Choosing $1 < p' < \frac{2p}{1+p}$ we find $h \in L^{p'}(\mathbb{R})$, showing that h itself gains extra integrability properties in this case. This will serve us later on in Section 2.4.

2.2.2 Marginals on Kac's sphere

By its definition, chaoticity depends strongly on understanding how finite marginal on Kac's sphere behave. In particular, in our presented cases, we'll be interested to find a simple formula for the $k-$th marginal of probability measures of the form $F_N d\sigma^N$. To do that we state the following simple lemma, whose proof we'll omit, but can be found in [34]:

Lemma 2.21. *Let F_N be an integrable function on $\mathbb{S}^{N-1}(r)$, then*

$$\int_{\mathbb{S}^{N-1}(r)} F_N d\sigma_r^N = \frac{|\mathbb{S}^{N-j-1}|}{|\mathbb{S}^{N-1}|}\frac{1}{r^{N-2}} \int \left(r^2 - \sum_{i=1}^j v_i^2\right)_+^{\frac{N-j-2}{2}}$$
$$\left(\int_{\mathbb{S}^{N-j-1}\left(\sqrt{r^2-\sum_{i=1}^j v_i^2}\right)} F_N d\sigma^{N-j}_{\sqrt{r^2-\sum_{i=1}^j v_i^2}}\right) dv_1 \dots dv_j,$$

where $g_+ = \max(g, 0)$ for a function g.

Using the above lemma, one can easily show the following:

Lemma 2.22. *Given a distribution function F_N on Kac's sphere, then the probability density function of the $k-$th marginal of the probability measure $F_N d\sigma^N$ is given by*

$$\begin{aligned}\Pi_k(F_N)(v_1, \dots, v_k) = \frac{|\mathbb{S}^{N-k-1}|}{|\mathbb{S}^{N-1}|}\frac{1}{N^{\frac{N-2}{2}}}\left(N - \sum_{i=1}^k v_i^2\right)_+^{\frac{N-k-2}{2}} \\ \left(\int_{\mathbb{S}^{N-k-1}\left(\sqrt{r^2-\sum_{i=1}^k v_i^2}\right)} F_N d\sigma^{N-k}_{\sqrt{r^2-\sum_{i=1}^j v_k^2}}\right).\end{aligned} \tag{2.34}$$

Next we show a simple condition for chaoticity, one we will use later on in Section 2.4:

Lemma 2.23. *Let $\{F_N\}_{N\in\mathbb{N}}$ be a family of distribution functions on Kac's sphere. Assume that there exists a distribution function f, on $\mathbb{R}$, such that*

$$\lim_{N\to\infty}\Pi_k(F_N)\,(v_1,\dots,v_k) = f^{\otimes k}\,(v_1,\dots,v_k) \tag{2.35}$$

pointwise for all $k\in\mathbb{N}$. Then

$$\lim_{N\to\infty}\left\|\Pi_k(F_N)\,(v_1,\dots,v_k) - f^{\otimes k}\,(v_1,\dots,v_k)\right\|_{L^1\left(\mathbb{R}^k\right)} = 0, \tag{2.36}$$

for all $k\in\mathbb{N}$, and n particular $\{F_N\}_{N\in\mathbb{N}}$ is $f-$chaotic.

The proof for this (and a more general statement) can be found in [33]. Since the proof is very simple we will add it here, for completion.

Proof. Let $k\in\mathbb{N}$ be fixed. Define $g_N = \Pi_k(F_N) + f^{\otimes k}$. By assumption (2.35) we know that

$$\lim_{N\to\infty} g_N = 2f^{\otimes k} = g,$$

pointwise and since $|\Pi_k(F_N) - f^{\otimes k}| \leq g_N$, and

$$\int_{\mathbb{R}^k} g_N\,(v_1,\dots,v_k)\,dv_1\dots dv_k = \int_{\mathbb{R}^k} g\,(v_1,\dots,v_k)\,dv_1\dots dv_k$$

for all N, we can use the generalised dominated convergence theorem to conclude (2.36). ☐

2.3 Lévy type local Central Limit Theorem

This section's purpose is the introduce a new local Lévy type central limit theorem, one that will help us in our investigation of families of conditioned tensorisation of a function f, extending results presented in [12]. The bulk of the work is inspired from [12] and [40] though there are some significant changes on which we will remark.

We start this section with an important technical theorem, taken from [40], which plays a crucial role in the proof of our local central limit theorem. The fact that it only works in $\mathbb{R}$ will affect the lower semi-continuity property, discussed in Section 2.5. At this point we advice the reader to review the definition of $\alpha-$stability, DA, NDA, FDA of $\gamma_{\sigma,\alpha,\beta}$ given in Definition 2.7, 2.9, 2.10, 2.11, as well as Remark 2.8 and equation (2.18).

Theorem 2.24. *For any $\widehat{\gamma}_{\sigma,\alpha,\beta}$ we have that the NDA equals the FDA.*

Due to its importance, we will present a full proof for this theorem. The proof relies on the following technical lemma (again, taken from [40]):

Lemma 2.25. *Let $g : \mathbb{R}\backslash\{0\} \to \mathbb{R}$ be a continuous function that satisfies $\lim_{n\to\infty} g\left(\frac{x}{n}\right) = 0$ for any $x \in \mathbb{R}\setminus\{0\}$. Then $\lim_{x\to 0} g(x) = 0$.*

We leave the proof to the Appendix, and show how one can prove Theorem 2.24 using it.

Proof of Theorem 2.24. We start with the easy direction. Assume that $\widehat{\psi}$ is in the FDA of $\gamma_{\sigma,\alpha,\beta}$. We have that

$$n\left(\widehat{\psi}\left(\frac{\xi}{n^{\frac{1}{\alpha}}}\right)-1\right) = -n\cdot\frac{\sigma|\xi|^\alpha}{n}\left(1+i\beta\mathrm{sgn}\left(\frac{\xi}{n^{\frac{1}{\alpha}}}\right)\tan\left(\frac{\pi\alpha}{2}\right)\right)+n\eta\left(\frac{\xi}{n^{\frac{1}{\alpha}}}\right).$$

$$= -\sigma|\xi|^\alpha\left(1+i\beta\mathrm{sgn}(\xi)\tan\left(\frac{\pi\alpha}{2}\right)\right)+|\xi|^\alpha\cdot\frac{\eta\left(\frac{\xi}{n^{\frac{1}{\alpha}}}\right)}{\left(\frac{\xi}{n^{\frac{1}{\alpha}}}\right)^\alpha},$$

concluding the desired result.

Conversely, assume that $\widehat{\psi}$ is in the NDA of $\gamma_{\sigma,\alpha,\beta}$ and define

$$\eta(\xi) = \widehat{\psi}(\xi) - 1 + \sigma|\xi|^\alpha\left(1+i\beta\mathrm{sgn}(\xi)\tan\left(\frac{\pi\alpha}{2}\right)\right).$$

We have that for any $\xi \neq 0$

$$\frac{\eta\left(\frac{\xi}{n^{\frac{1}{\alpha}}}\right)}{\left|\frac{\xi}{n^{\frac{1}{\alpha}}}\right|^\alpha} = \frac{1}{|\xi|^\alpha}\left(n\left(\widehat{\psi}\left(\frac{\xi}{n^{\frac{1}{\alpha}}}\right)-1\right)+\sigma|\xi|^\alpha\left(1+i\beta\mathrm{sgn}(\xi)\tan\left(\frac{\pi\alpha}{2}\right)\right)\right)\underset{n\to\infty}{\longrightarrow} 0.$$

Defining $g(\xi) = \frac{\eta(\xi)}{|\xi|^\alpha}$ we find that g is continuous on $\mathbb{R}\setminus\{0\}$ and

$$g\left(\frac{\xi}{n^{\frac{1}{\alpha}}}\right)\underset{n\to\infty}{\longrightarrow} 0$$

for any $\xi \neq 0$. A simple modification of Lemma 2.25 proves that $\lim_{\xi\to 0} g(\xi) = 0$. This also shows, since η is continuous, that $\frac{\eta(\xi)}{|\xi|^\alpha}$ is bounded around $\xi = 0$. For $|\xi| > \delta$ we have that

$$\frac{|\eta(\xi)|}{|\xi|^\alpha} \leq \frac{2}{\delta^\alpha} + \sigma\left(1+|\beta|\left|\tan\left(\frac{\pi\alpha}{2}\right)\right|\right),$$

proving that $\frac{\eta(\xi)}{|\xi|^\alpha} \in L^\infty$, and the result follows. □

Theorem 2.24 gives us a very convenient approximation for the characteristic function of any real random variable in the NDA of $\gamma_{\sigma,\beta,\alpha}$, one we will use quite strongly.

Next, we're like to find some simple conditions for when a real random variable belongs to the NDA of $\gamma_{\sigma,\beta,\alpha}$. This is given by a theorem from Feller's book, [36]:

Theorem 2.26. *Let F be a probability distribution function of a real random variable, X, that has zero mean, and let $1 < \alpha < 2$. Denote by*

$$\mu(x) = \int_{-x}^{x} y^2 F(dy). \tag{2.37}$$

If

(i)

$$\mu(x) \underset{x \longrightarrow \infty}{\sim} x^{2-\alpha} L(x), \tag{2.38}$$

where L is slowly varying (i.e. $\frac{L(tx)}{L(x)} \underset{x\to\infty}{\longrightarrow} 1$ for any $t > 0$).

(ii)

$$\begin{aligned} \frac{1-F(x)}{1-F(x)+F(-x)} &\underset{x\to\infty}{\longrightarrow} p, \\ \frac{F(-x)}{1-F(x)+F(-x)} &\underset{x\to\infty}{\longrightarrow} q. \end{aligned} \tag{2.39}$$

(iii) There exists a sequence $\{a_n\}_{n\in\mathbb{N}} > 0$ such that

$$\frac{n\mu(a_n)}{a_n^2} \underset{n\to\infty}{\longrightarrow} C_S. \tag{2.40}$$

Then X is in the DA of $\gamma_{C_S,\alpha,p,q}$ with $\{a_n\}_{n\in\mathbb{N}}$ found in (iii) and $b_n = 0$.

Remark 2.27. It is worth mentioning that a similar, less restrictive theorem, holds in the case $0 < \alpha < 1$. Since we will not use it in this work, we decided to exclude it from this section. For more information we refer the interested reader to [36].

Remark 2.28. Of particular interest to us are the following cases:

— If in condition (i) of Theorem 2.26 one has that $L(x) \underset{x\to\infty}{\sim} C_S$ then the sequence

$$a_n = n^{\frac{1}{\alpha}}$$

will be suitable for condition (iii) of the same theorem.

— If the probability distribution function, $F(x)$, is supported in $[\kappa, \infty)$ for some $\kappa \in \mathbb{R}$ then condition (ii) of Theorem 2.26 is immediately satisfied with $p = 1$ and $q = 0$.

We now turn our attention to the proof of Theorem 2.12. The main idea of the proof is to evaluate the supremum of the difference between the probability density functions using inversion formula and their characteristic functions. An integral will emerge, one we will have to divide into two domains: frequencies that are close to zero, and frequencies that are 'far' from zero. The domain of frequencies that are almost zero will be taken care of by requiring that the characteristic function would be in the NDA of some stable distribution. The other frequencies will be dealt with presently:

Theorem 2.29. *Let g be a probability density function on $\mathbb{R}$ such that*

$$E_\lambda = \int_{\mathbb{R}} |x|^\lambda g(x)dx < \infty, \tag{2.41}$$

for some $\lambda > 0$, and

$$H(g) = \int_{\mathbb{R}} g(x)\log g(x)dx < \infty. \tag{2.42}$$

Then for any $\beta > 0$, there exists $\eta = \eta(\beta, H(g), E_\lambda) > 0$ such that if $|\xi| > \beta$ then $|\widehat{g}(\xi)| \leq 1-\eta$. Moreover, given $\tau > 0$ one can get the estimation

$$|\widehat{g}(\xi)| \leq 1 - \beta^{2+\tau} + \phi_\tau(\beta), \tag{2.43}$$

for $\beta < \beta_0$ small enough, where $\frac{\phi_\delta(\tau)}{\beta^{2+\tau}} \underset{\beta\to 0}{\longrightarrow} 0$.

Remark 2.30. The proof of the first part of the above theorem, to be presented shortly, is very similar to the proof found in [12]. The novelty of our approach manifests itself in (2.43), where an explicit distance from 1 is given. The surprising part is that to show this estimation no new machinery is required, only an intermediate approximation.

Proof. For a given $\xi \in \mathbb{R}$ we can find a $z \in \mathbb{R}$ such that

$$|\widehat{g}(\xi)| = \widehat{g}(\xi)e^{-i\xi z}.$$

By the definition of the Fourier transform, and the fact that $\widehat{g}(0) = 1$, we have that

$$|\widehat{g}(\xi)| = \int_{\mathbb{R}} g(x)e^{-i(x+z)\xi}dx = 1 - \int_{\mathbb{R}} g(x)\left(1 - e^{-i(x+z)\xi}\right)dx.$$

Since $|\widehat{g}|$ is real we find that

$$\begin{aligned} |\widehat{g}(\xi)| &= 1 - \int_{\mathbb{R}} g(x)\left(1-\cos\left((x+z)\xi\right)\right)dx \\ &\leq 1 - \int_B g(x)\left(1-\cos\left((x+z)\xi\right)\right)dx \end{aligned} \tag{2.44}$$

for any measurable set B.

Define:

$$B_{\delta,R} = \{x \in [-R,R] \mid 1-\cos\left((z+x)\xi\right) \leq \delta\},$$

where δ and R are to be specified later. From its definition, and (2.44), we conclude that

$$\begin{aligned} |\widehat{g}(\xi)| &\leq 1 - \int_{[-R,R]\setminus B_{\delta,R}} g(x)\left(1-\cos\left((x+z)\xi\right)\right)dx \\ &\leq 1 - \delta\int_{[-R,R]\setminus B_{\delta,R}} g(x)dx. \end{aligned} \tag{2.45}$$

Next we notice that $x \in B_{\delta,R}$ if and only if $x \in [-R,R]$ and

$$|(z+x)\xi + 2\pi k| \leq \arccos(1-\delta)$$

for some $k \in \mathbb{Z}$. Since $\arccos(1-\delta) \leq \sqrt{2\delta}$ we conclude that if $x \in B_{\delta,R}$ then, for some $k \in \mathbb{Z}$,

$$\left| x - \left(\frac{2\pi k}{\xi} - z \right) \right| \leq \frac{\sqrt{2\delta}}{|\xi|}. \tag{2.46}$$

We denote by I_k the closed intervals centred in $\frac{2\pi k}{\xi} - z$, with radius $\frac{\sqrt{2\delta}}{|\xi|}$. Since the distance between the centres of any two I_k-s is at least $\frac{2\pi}{|\xi|}$, while the length of each interval is at most $\frac{2}{|\xi|}$, if we pick $\delta < \frac{1}{2}$, we conclude that the intervals I_k-s are mutually disjoint.

From (2.46) we see that the set $B_{\delta,R}$ is contained in a union of I_k-s.
Let n be the number of $k \in \mathbb{Z}$ such that $\frac{2\pi k}{\xi} - z \in [-R,R]$. All such $k-$s, but possibly the biggest and smallest k, satisfy that $\left[\frac{2\pi(k-1)}{|\xi|}, \frac{2\pi k}{|\xi|}\right] \subset [-R,R]$. Thus,

$$(n-2) \cdot \frac{2\pi}{|\xi|} = \sum_k \left| \left[\frac{2\pi(k-1)}{|\xi|}, \frac{2\pi k}{|\xi|} \right] \right| \leq 2R.$$

With $|\cdot|$ denoting the Lebesgue measure, we conclude that

$$|B_{\delta,R}| \leq n \cdot \frac{\sqrt{2\delta}}{|\xi|} \leq \left(\frac{R}{\pi} + \frac{2}{|\xi|} \right) \sqrt{2\delta} \leq \frac{R}{\pi} \left(1 + \frac{2\pi}{R\beta} \right) \sqrt{2\delta}. \tag{2.47}$$

At this point we will use the entropy and moment conditions on g to connect between the known value $|B_{\delta,R}|$ and the desired value $\int_{B_{\delta,R}} g(x)dx$. To do that we will use the relative entropy (see Definition 2.5) and the following known inequality:

$$\mu(B) \leq \frac{2H(\mu|\nu)}{\log\left(1 + \frac{H(\mu|\nu)}{\nu(B)}\right)}, \tag{2.48}$$

where μ and ν are regular probability measure on $\mathbb{R}$ and B is a measurable set.

Define

$$d\mu(x) = \frac{\chi_{[-R,R]}(x) g(x)}{\int_{[-R,R]} g(x)dx} dx, \quad d\nu(x) = \frac{\chi_{[-R,R]}(x)}{2R} dx. \tag{2.49}$$

We have that $\frac{d\mu}{d\nu}(x) = \frac{2R\chi_{[-R,R]}(x)g(x)}{\int_{[-R,R]} g(x)dx}$ and

$$\begin{aligned} H(\mu|\nu) &= \int_{[-R,R]} \log\left(\frac{2Rg(x)}{\int_{[-R,R]} g(x)dx} \right) \frac{g(x)}{\int_{[-R,R]} g(x)dx} dx \\ &= \log(2R) - \log\left(\int_{[-R,R]} g(x)dx \right) + \frac{1}{\int_{[-R,R]} g(x)dx} \int_{[-R,R]} g(x) \log g(x) dx \\ &\leq \log(2R) - \log\left(1 - \frac{E_\lambda}{R^\lambda}\right) + \frac{1}{1 - \frac{E_\lambda}{R^\lambda}} \int g(x) |\log g(x)| dx. \end{aligned} \tag{2.50}$$

We have used the fact that

$$\int_{[-R,R]} g(x)dx = 1 - \int_{|x|>R} g(x)dx \geq 1 - \frac{1}{R^\lambda} \int_{|x|>R} |x|^\lambda g(x)dx \geq 1 - \frac{E_\lambda}{R^\lambda}. \tag{2.51}$$

We will now turn our attention to the term $\int g(x)|\log(g(x))|dx$. For *any* positive function $\psi(x)$, we have that

$$\psi(x)\left(\frac{g(x)}{\psi(x)}\log\left(\frac{g(x)}{\psi(x)}\right)-\frac{g(x)}{\psi(x)}+1\right)\geq 0.$$

Thus, for any measurable set A we have that

$$\int_A g(x)\log g(x)dx \geq \int_A g(x)\log\psi(x)dx + \int_A g(x) - \int_A \psi(x)dx,$$

when the right hand side is finite. Choosing $\psi(x)=e^{-|x|^\lambda}$ and $A=\{g<1\}$ we find that

$$\begin{aligned}\left|\int_{g<1} g(x)\log g(x)dx\right| &= -\int_{g<1} g(x)\log g(x)\\ \leq \int_{g<1}|x|^\lambda g(x)dx - \int_{g<1} g(x)dx &+ \int_{g<1}\psi(x)dx < E_\lambda + C_\lambda.\end{aligned} \tag{2.52}$$

where $C_\lambda=\int\psi(x)dx$. Since

$$\int g(x)|\log(g(x)| = H(g) - 2\int_{g<1} g(x)\log g(x)dx.$$

we conclude that

$$H(\mu|\nu)\leq \log(2R)-\log\left(1-\frac{E_\lambda}{R^\lambda}\right)+\frac{H(g)+2E_\lambda+2C_\lambda}{1-\frac{E_\lambda}{R^\lambda}}. \tag{2.53}$$

Together with (2.47) and (2.48) we find that

$$\mu(B_{\delta,R})\leq \frac{2\log(2R)-2\log\left(1-\frac{E_\lambda}{R^\lambda}\right)+\frac{2H(g)+4E_\lambda+4C_\lambda}{1-\frac{E_\lambda}{R^\lambda}}}{\log\left(1+\frac{\log(2R)-\log\left(1-\frac{E_\lambda}{R^\lambda}\right)+\frac{H(g)+2E_\lambda+2C_\lambda}{1-\frac{E_\lambda}{R^\lambda}}}{\frac{1}{2\pi}\left(1+\frac{2\pi}{R\beta}\right)\sqrt{2\delta}}\right)}. \tag{2.54}$$

Next, we notice that

$$\int_{[-R,R]\setminus B_{\delta,R}} g(x)dx = \left(\int_{[-R,R]} g(x)dx\right)\mu\left([-R,R]\setminus B_{\delta,R}\right)\geq\left(1-\frac{E_\lambda}{R^\lambda}\right)\left(1-\mu(B_{\delta,R})\right)$$

which, along with (2.45) and (2.54) gives us the following control:

$$|\widehat{g}(\xi)|\leq 1-\delta\cdot\left(1-\frac{E_\lambda}{R^\lambda}\right)\left(1-\frac{2\log(2R)-2\log\left(1-\frac{E_\lambda}{R^\lambda}\right)+\frac{2H(g)+4E_\lambda+4C_\lambda}{1-\frac{E_\lambda}{R^\lambda}}}{\log\left(1+\frac{\log(2R)-\log\left(1-\frac{E_\lambda}{R^\lambda}\right)+\frac{H(g)+2E_\lambda+2C_\lambda}{1-\frac{E_\lambda}{R^\lambda}}}{\frac{1}{2\pi}\left(1+\frac{2\pi}{R\beta}\right)\sqrt{2\delta}}\right)}\right) \tag{2.55}$$

At this point we can choose R and $\delta < \frac{1}{2}$ appropriately. For any $\tau > 0$ we choose $\delta = \beta^{2+\tau}$ and $R = -\log \beta$ we find that for β going to zero

$$
\begin{aligned}
&\frac{2\log(2R) - 2\log\left(1 - \frac{E_\lambda}{R^\lambda}\right) + \frac{2H(g)+4E_\lambda+4C_\lambda}{1-\frac{E_\lambda}{R^\lambda}}}{\log\left(1 + \frac{\log(2R)-\log\left(1-\frac{E_\lambda}{R^\lambda}\right)+\frac{H(g)+2E_\lambda+2C_\lambda}{1-\frac{E_\lambda}{R^\lambda}}}{\frac{1}{2\pi}\left(1+\frac{2\pi}{R\beta}\right)\sqrt{2\delta}}\right)} \\
&\approx \frac{2\log\left(-\log(\beta)\right)\left(1 + O\left(\frac{\log\left(1-\frac{1}{(-\log(\beta))^\lambda}\right)}{\log(-\log(\beta))}\right) + O\left(\frac{1}{\log(-\log(\beta))\left(1-\frac{1}{(-\log(\beta))^\lambda}\right)}\right)\right)}{\log\left(1 + \frac{2\pi\log(-\log(\beta))\left(1+O\left(\frac{\log\left(1-\frac{1}{(-\log(\beta))^\lambda}\right)}{\log(-\log(\beta))}\right)+O\left(\frac{1}{\log(-\log(\beta))\left(1-\frac{1}{(-\log(\beta))^\lambda}\right)}\right)\right)}{\sqrt{2}\beta^{1+\frac{\tau}{2}}-\frac{2\pi\sqrt{2}\beta^{\frac{\tau}{2}}}{\log(\beta)}}\right)} \\
&\approx \frac{2\log\left(-\log(\beta)\right)}{\log\left(1 - \frac{\log(-\log(\beta))\log(\beta)}{\sqrt{2}\beta^{\frac{\tau}{2}}(1+O(\beta\log(\beta)))}\right)} \approx \frac{2\log\left(-\log(\beta)\right)}{\log\left(-\frac{\log(-\log(\beta))\log(\beta)}{\sqrt{2}\beta^{\frac{\tau}{2}}}\right)} \\
&\approx -\frac{4\log\left(-\log(\beta)\right)}{\tau\log(\beta) - 2\log\left(-\log(\beta)\log\left(-\log(\beta)\right)\right)} \underset{\beta\to 0}{\longrightarrow} 0.
\end{aligned}
$$

Thus,

$$|\widehat{g}(\xi)| \leq 1 - \beta^{2+\tau} + \phi_\tau(\beta),$$

where $\frac{\phi_\tau(\beta)}{\beta^{2+\tau}} \underset{\beta\to 0}{\longrightarrow} 0$. □

Before we present the proof for Theorem 2.12, we state the next simple lemma, whose proof we leave to the appendix. A similar argument can be found in [40].

Lemma 2.31. *Let $\widehat{g}$ be the characteristic function of a random real variable X that is in the NDA of $\gamma_{\sigma,\alpha,\beta}$. Then there exists $\beta_0 > 0$ such that for all $|\xi| < \beta_0$ we have that*

$$|\widehat{g}(\xi)| \leq e^{-\frac{\sigma|\xi|^\alpha}{2}}. \tag{2.56}$$

Proof of Theorem 2.12. We start by noticing that

$$\widehat{g_N}(\xi) = \widehat{g}^N\left(\frac{\xi}{N^{\frac{1}{\alpha}}}\right),$$

and from the inversion formula for characteristic functions (see [36]) we have that $\widehat{\gamma}_{\sigma,\alpha,\beta}$ is the characteristic function of $\gamma_{\sigma,\alpha,\beta}$.

Since $g \in L^1(\mathbb{R}) \cap L^p(\mathbb{R})$ we conclude that $g \in L^{p'}(\mathbb{R})$ for any $1 \leq p' \leq p$. Thus, its characteristic function belongs to some $L^q(\mathbb{R})$ for some $q > 1$. One can choose q to be the Hölder conjugate of $\min(2,p)$. For any $N>q$ we have that

$$\int_{\mathbb{R}}|\widehat{g_N}(\xi)|d\xi \leq \|\widehat{g}\|_\infty^{N-q}\int_{\mathbb{R}}|\widehat{g}\left(\frac{\xi}{N^{\frac{1}{\alpha}}}\right)|^q d\xi \leq N^{\frac{1}{\alpha}}\|\widehat{g}\|_{L^q}^q < \infty.$$

This implies that we can use the inversion formula for g: For any $x\in\mathbb{R}$:

$$\begin{aligned} |g_N(x)-\gamma_{\sigma,\alpha,\beta}(x)| &\leq \frac{1}{2\pi}\int_{\mathbb{R}}|\widehat{g}^N\left(\frac{\xi}{N^{\frac{1}{\alpha}}}\right)-\widehat{\gamma}_{\sigma,\alpha,\beta}(\xi)|d\xi \\ &= \frac{1}{2\pi}\int_{\mathbb{R}}|\widehat{g}^N\left(\frac{\xi}{N^{\frac{1}{\alpha}}}\right)-\widehat{\gamma}^N_{\sigma,\alpha,\beta}\left(\frac{\xi}{N^{\frac{1}{\alpha}}}\right)|d\xi \\ &\leq \frac{1}{2\pi}\int_{|\xi|<\beta_N N^{\frac{1}{\alpha}}}|\widehat{g}^N\left(\frac{\xi}{N^{\frac{1}{\alpha}}}\right)-\widehat{\gamma}^N_{\sigma,\alpha,\beta}\left(\frac{\xi}{N^{\frac{1}{\alpha}}}\right)|d\xi \\ &+\frac{1}{2\pi}\int_{|\xi|>\beta_N N^{\frac{1}{\alpha}}}|\widehat{g}^N\left(\frac{\xi}{N^{\frac{1}{\alpha}}}\right)|d\xi+\frac{1}{2\pi}\int_{|\xi|>\beta_N N^{\frac{1}{\alpha}}}|\widehat{\gamma}_{\sigma,\alpha,\beta}(\xi)|d\xi \\ &= I_1+I_2+I_3. \end{aligned} \tag{2.57}$$

The partition in (2.57) corresponds to the near to/far from zero discussed earlier. We wil start with estimating I_1.

Since $\widehat{g}$ is in the NDA of $\gamma_{\sigma,\alpha,\beta}$, Theorem 2.24 assures us that $\widehat{g}$ is in the FDA of $\gamma_{\sigma,\alpha,\beta}$ and there exists a reminder function, η_g, such that

$$|\widehat{g}(\xi)-\widehat{\gamma}_{\sigma,\alpha,\beta}(\xi)| = |\eta_g(\xi)|+|\eta_\gamma(\xi)|, \tag{2.58}$$

with

$$|\eta_\gamma(\xi)| \leq 2\sigma^2|\xi|^{2\alpha}\left(1+\beta^2\tan^2\left(\frac{\pi\alpha}{2}\right)\right) \tag{2.59}$$

when $|\xi|<\beta_1$ for some small $\beta_1>0$. Thus,

$$\sup_{|\zeta|<\beta_N}\frac{|\widehat{g}(\zeta)-\widehat{\gamma}_{\sigma,\alpha,\beta}(\zeta)|}{|\zeta|^\alpha} \leq \omega_g(\beta_N)+2\sigma\beta_N^\alpha\left(1+\beta^2\tan^2\left(\frac{\pi\alpha}{2}\right)\right) \tag{2.60}$$

for N large enough such that $\beta_N<\beta_1$.

Next, we see that

$$\begin{aligned} &|\widehat{g}^N\left(\frac{\xi}{N^{\frac{1}{\alpha}}}\right)-\widehat{\gamma}^N_{\sigma,\alpha,\beta}\left(\frac{\xi}{N^{\frac{1}{\alpha}}}\right)| \\ \leq |\widehat{g}\left(\frac{\xi}{N^{\frac{1}{\alpha}}}\right)-\widehat{\gamma}_{\sigma,\alpha,\beta}\left(\frac{\xi}{N^{\frac{1}{\alpha}}}\right)|&\sum_{k=0}^{N-1}|\widehat{g}\left(\frac{\xi}{N^{\frac{1}{\alpha}}}\right)|^k|\widehat{\gamma}_{\sigma,\alpha,\beta}\left(\frac{\xi}{N^{\frac{1}{\alpha}}}\right)|^{N-1-k}. \end{aligned} \tag{2.61}$$

Picking N such that $\frac{|\xi|}{N^{\frac{1}{\alpha}}}<\beta_N<\beta_0$ from Lemma 2.31 we find that

$$\begin{aligned} \sum_{k=0}^{N-1}|\widehat{g}\left(\frac{\xi}{N^{\frac{1}{\alpha}}}\right)|^k|\widehat{\gamma}_{\sigma,\alpha,\beta}\left(\frac{\xi}{N^{\frac{1}{\alpha}}}\right)|^{N-1-k} &\leq \sum_{k=0}^{N-1}e^{-\frac{\sigma k|\xi|^\alpha}{2N}}\cdot e^{-\frac{\sigma(N-k-1)|\xi|^\alpha}{N}} \\ &\leq Ne^{-\frac{\sigma(N-1)|\xi|^\alpha}{2N}} \leq Ne^{-\frac{\sigma|\xi|^\alpha}{4}}, \end{aligned} \tag{2.62}$$

when $N \geq 2$. Combining (2.60), (2.61) and (2.62) we see that

$$\begin{aligned} I_1 &\leq \frac{\omega_g(\beta_N) + 2\sigma\beta_N^{\alpha}\left(1+\beta^2\tan^2\left(\frac{\pi\alpha}{2}\right)\right)}{2\pi}\int_{|\xi|<\beta_N N^{\frac{1}{\alpha}}} \frac{|\xi|^{\alpha}}{N}\cdot N e^{-\frac{\sigma|\xi|^{\alpha}}{4}} d\xi \\ &\leq C\left(\omega_g(\beta_N) + 2\sigma\beta_N^{\alpha}\left(1+\beta^2\tan^2\left(\frac{\pi\alpha}{2}\right)\right)\right), \end{aligned} \tag{2.63}$$

where $C = \int_{\mathbb{R}} |\xi|^{\alpha} e^{-\frac{\sigma|\xi|^{\alpha}}{4}} d\xi$. Next, we estimate I_2.

The expression I_2 is connected to Theorem 2.29, and as such we need to check that its conditions are satisfied. From the conditions given in the statement of our theorem, we know that there exists $\lambda > 0$ such that $E_{\lambda} < \infty$, following the notations of Theorem 2.29. We only need to show that $H(g) < \infty$. Indeed, since $g \in L^p(\mathbb{R})$ for some $p > 1$ we have that

$$\int_{\mathbb{R}} g(x)|\log g(x)| dx = -\int_{g<1} g(x)\log g(x) dx + \int_{g\geq 1} g(x)\log g(x) dx.$$

We already showed in the proof of Theorem 2.29 that $-\int_{g<1} g(x)\log g(x)dx < \infty$, and since we can always find $C_p > 0$ such that $\log x \leq C_p x^{p-1}$ for $x \geq 1$ we conclude that

$$\int_{g\geq 1} g(x)\log g(x) dx \leq C_p \|g\|_{L^p(\mathbb{R})}^p < \infty,$$

showing that $H(g) < \infty$. Thus, for any $\tau > 0$ and for β small enough we have that

$$|\widehat{g}(\xi)| \leq 1 - \beta^{2+\tau} + \phi_{\tau}(\beta),$$

with $\frac{\phi_{\delta}(\tau)}{\beta^{2+\tau}} \underset{\beta\to 0}{\longrightarrow} 0$.
Using the above, we conclude that

$$I_2 = \frac{N^{\frac{1}{\alpha}}}{2\pi}\int_{|\xi|>\beta_N} |\widehat{g}(\xi)|^N d\xi \leq \frac{N^{\frac{1}{\alpha}}}{2\pi}\left(1 - \beta_N^{2+\tau} + \phi_{\tau}(\beta_N)\right)^{N-q} \|\widehat{g}\|_{L^q(\mathbb{R})}^q. \tag{2.64}$$

Lastly, we need to estimate I_3, which is the simplest of the three integrals. Indeed

$$\begin{aligned} I_3 = \frac{1}{2\pi}\int_{|\xi|>\beta_N N^{\frac{1}{\alpha}}} e^{-\sigma|\xi|^{\alpha}} d\xi &\leq \frac{e^{-\frac{\sigma N\beta_N^{\alpha}}{2}}}{2\pi}\int_{|\xi|>\beta_N N^{\frac{1}{\alpha}}} e^{-\frac{\sigma|\xi|^{\alpha}}{2}} d\xi \\ &\leq D e^{-\frac{\sigma N \beta_N^{\alpha}}{2}}, \end{aligned} \tag{2.65}$$

where $D = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-\frac{\sigma|\xi|^{\alpha}}{2}} d\xi$. Combining (2.63), (2.64) and (2.65) yields the desired result. □

Remark 2.32. It is clear that if $\{\beta_N\}_{N\in\mathbb{N}}$ is chosen such that it goes to zero and

$$\beta_N^{2+\tau} N \underset{N\to\infty}{\longrightarrow} \infty$$

then $\epsilon_\tau(N)$, defined in the above theorem, goes to zero as N goes to infinity, and we have an explicit rate to how fast it does it. A different method to undertake here is to pick β_0 small enough that all the steps of the proof the theorem work, and get that

$$\|g_N - \gamma_{\sigma,\alpha,\beta}\|_\infty \leq C_{g,\alpha}\Bigg(N^{\frac{1}{\alpha}}(1-\beta_0^{2+\tau}+\phi_\tau(\beta_0))^{N-q} + e^{-\frac{\sigma N \beta_0^\alpha}{2}} \\ +\omega_g(\beta_0) + 2\sigma\beta_0^\alpha\left(1+\beta^2\tan^2\left(\frac{\pi\alpha}{2}\right)\right)\Bigg).$$

Thus

$$\limsup_{N\to\infty}\|g_N - \gamma_{\sigma,\alpha,\beta}\|_\infty \leq \lim_{\beta_0\to 0}\left(\omega_g(\beta_0) + 2\sigma\beta_0^\alpha\left(1+\beta^2\tan^2\left(\frac{\pi\alpha}{2}\right)\right)\right) = 0,$$

proving the desired convergence, but losing the explicit N dependency!

An immediate corollary of Theorem 2.12 is the following:

Theorem 2.33. *Let g be the probability density function of a random real variable X. Assume that $g \in L^{p'}(\mathbb{R})$ for some $p' > 1$ and*

(1) $\int |x|g(x)dx < \infty$.

(2) $\mu_g(x) \underset{x\to\infty}{\sim} C_S x^{2-\alpha}$ *for some $C_S > 0$ and $1 < \alpha < 2$ where*

$$\mu_g(x) = \int_{-x}^{x} y^2 g(y)dy.$$

(3)

$$\frac{1-G(x)}{1-G(x)+G(-x)} \underset{x\to\infty}{\longrightarrow} p$$
$$\frac{G(-x)}{1-G(x)+G(-x)} \underset{x\to\infty}{\longrightarrow} q,$$

where $G(x) = \int_{-\infty}^{x} g(y)dy$.

Then, for any positive sequence $\{\beta_N\}_{N\in\mathbb{N}}$ that converges to zero as N goes to infinity and satisfies

$$\beta_N^{2+\tau} N \underset{N\to\infty}{\longrightarrow} \infty, \tag{2.66}$$

for some $\tau > 0$ and for N large enough, we have that

$$\sup_x \left|g^{*N}(x) - \frac{\gamma_{\sigma,\alpha,\beta}\left(\frac{x-NE}{N^{\frac{1}{\alpha}}}\right)}{N^{\frac{1}{\alpha}}}\right| \leq \frac{C_{g,\alpha}}{N^{\frac{1}{\alpha}}}\Bigg(N^{\frac{1}{\alpha}}(1-\beta_N^{2+\tau}+\phi_\tau(\beta_N))^{N-q'} \\ +e^{-\frac{\sigma N\beta_N^\alpha}{2}} + \omega_\eta(\beta_N) + 2\sigma\beta_N^\alpha\left(1+\beta^2\tan^2\left(\frac{\pi\alpha}{2}\right)\right)\Bigg) = \frac{\epsilon_\tau(N)}{N^{\frac{1}{\alpha}}}, \tag{2.67}$$

where

(i) $\sigma = C_S \frac{\Gamma(3-\alpha)}{\alpha(\alpha-1)} \cos\left(\frac{\pi\alpha}{2}\right)$, $\beta = p - q$.

(ii) $E = \int_{\mathbb{R}} x g(x) dx$.

(iii) $C_{g,\alpha} > 0$ *is a constant depending only on* g*, its moments and* α.

(iv) q' *can be chosen to be the Hölder conjugate of* $\min(2, p')$.

(v) ϕ_τ *satisfies*

$$\lim_{x\to 0} \frac{\phi_\tau(x)}{|x|^{2+\tau}} = 0,$$

(vi) $\eta(\xi)$ *is the reminder function of* $e^{-i\xi E}\widehat{g}(\xi)$*, defined in Definition 2.11, and* $\omega_\eta(\beta) = \sup_{|x|\leq\beta} \frac{|\eta(x)|}{|x|^\alpha}$.

Under the condition (2.66) and the conclusions $(i) - (vi)$ *one finds that*

$$\lim_{N\to\infty} \epsilon_\tau(N) = 0.$$

Proof. We start by defining $g_0(x) = g(x+E)$. Clearly $g_0 \in L^{p'}(\mathbb{R})$ and $\int_{\mathbb{R}} |x| g_0(x) dx < \infty$. If we will be able to show that g_0 is in the NDA of $\gamma_{\sigma,\alpha,\beta}$, then, using Theorem 2.12, we can conclude that

$$\sup_x |g^{*N}\left(N^{\frac{1}{\alpha}}x + NE\right) - \frac{\gamma_{\sigma,\alpha,\beta}(x)}{N^{\frac{1}{\alpha}}}| \leq \frac{\epsilon_\tau(N)}{N^{\frac{1}{\alpha}}},$$

as $g_0^{*N}(x) = g^{*N}(x+NE)$, and the desired result follows.

We only have to prove that g_0 is in the appropriate NDA. To do that we will use Theorem 2.26. From its definition we know that g_0 has zero mean. Clearly

$$\frac{1-G_0(x)}{1-G_0(x)+G_0(-x)} \underset{x\to\infty}{\longrightarrow} p$$
$$\frac{G_0(-x)}{1-G_0(x)+G_0(-x)} \underset{x\to\infty}{\longrightarrow} q,$$

with $G_0(x) = \int_{-\infty}^x g_0(y)dy$, as $G_0(x) = G(x+E)$.
Next, we see that

$$\mu_{g_0}(x) = \int_{-x}^{x} y^2 g_0(y) dy = \int_{-x+E}^{x+E} y^2 g(y) dy - 2E \int_{-x+E}^{x+E} y g(y) dy + E^2 \int_{-x+E}^{x+E} g(y) dy.$$

The first term is bounded between $\mu_g(x-E)$ and $\mu_g(x+E)$ and as such behaves like $C_S x^{2-\alpha}$ as x goes to infinity. The rest of the terms have a limit as x goes to infinity, implying that

$$\mu_{g_0}(x) \sim C_S x^{2-\alpha}.$$

All the conditions of Theorem 2.26 are satisfied (see Remark 2.28), with σ and β given by (i), and the proof is complete. □

Before we end this section we'd like to mention that with additional conditions on g, the estimation on ϵ_τ, defined in Theorem 2.33, can become more explicit. This will be done via an explicit estimation for $\omega_\eta(\xi)$. Such estimation can be found in [40], yet the additional conditions are very restrictive. As it is still of interest we will provide some information on the matter in the Appendix.

2.4 Chaoticity and entropic chaoticity for families with unbounded fourth moment

The study of the chaoticity and entropic chaoticity of probability density functions, $\{F_N\}_{N\in\mathbb{N}}$, on Kac's sphere that are obtained by conditioning a tensorisation of a one particle function, f (equation (2.9)), is intimately connected to the asymptotic behaviour of the normalisation function $\mathcal{Z}_N(f,r)$ at all r, and not only its value at $r=\sqrt{N}$. Formula (2.31) for the normalisation function, presented in Section 2.2, and the local central limit theorem we proved in Section 2.3 provide us with the necessary tools to find the desired behaviour.

Theorem 2.34. *Let f be the probability density function of a random real variable V such that $f\in L^p(\mathbb{R})$ for some $p>1$ and let*

$$\nu_f(x)=\int_{-\sqrt{x}}^{\sqrt{x}} y^4 f(y)dy.$$

Assume that

$$\int_{\mathbb{R}} x^2 f(x)dx = E<\infty.$$

and $\nu_f(x) \underset{x\to\infty}{\sim} C_S x^{2-\alpha}$ for some $C_S>0$ and $1<\alpha<2$. Then

$$\sup_x \left|h^{*N}(x)-\frac{\gamma_{\sigma,\alpha,1}\left(\frac{x-NE}{N^{\frac{1}{\alpha}}}\right)}{N^{\frac{1}{\alpha}}}\right| \leq \frac{\epsilon(N)}{N^{\frac{1}{\alpha}}}, \tag{2.68}$$

where $\lim_{N\to\infty}\epsilon(N)=0$, $\sigma=C_S\frac{\Gamma(3-\alpha)}{\alpha(\alpha-1)}\cos\left(\frac{\pi\alpha}{2}\right)$ and h is the probability density function of the random variable V^2. Moreover, $\epsilon(N)$ can be bound by $\epsilon_\tau(N)$, given in Theorem 2.33, with η the reminder function of $e^{-i\xi}\widehat{h}$.

In addition,

$$\mathcal{Z}_N\left(f,\sqrt{r}\right)=\frac{2}{|\mathbb{S}^{N-1}|r^{\frac{N-2}{2}}}\frac{1}{N^{\frac{1}{\alpha}}}\left(\gamma_{\sigma,\alpha,1}\left(\frac{r-NE}{N^{\frac{1}{\alpha}}}\right)+\lambda_N(r)\right), \tag{2.69}$$

where $\sup_u|\lambda_N(u)| \underset{N\to\infty}{\longrightarrow} 0$.

Proof. We start by noticing that (2.69) follows immediately from (2.31) and (2.68). Next, we will show that the conditions of Theorem 2.33 are satisfied by h, concluding inequality (2.68), and the estimation for $\epsilon(N)$.

As was mentioned before, the function h is given by

$$h(x)=\begin{cases}\frac{f(\sqrt{x})+f(-\sqrt{x})}{2\sqrt{x}} & x>0\\ 0 & x\leq 0\end{cases}$$

and $h \in L^{p'}(\mathbb{R})$ for some $p' > 1$ when $f \in L^p(\mathbb{R})$ with $p > 1$ (see Remark 2.20). Moreover, for any $\kappa > 0$

$$\int_{\mathbb{R}} |x|^\kappa h(x)dx = \int_{\mathbb{R}} |x|^{2\kappa} f(x)dx,$$

from which we conclude that

$$\int_{\mathbb{R}} |x|h(x)dx = \int_{\mathbb{R}} xh(x) = E < \infty.$$

By its definition

$$\mu_h(x) = \int_{-x}^{x} y^2 h(y)dy = \nu_f(x) \underset{x\to\infty}{\sim} C_S x^{2-\alpha},$$

and recalling Remark 2.28, we conclude that if H is the probability distribution function of V^2 then for any $x > 0$

$$\frac{1-H(x)}{1-H(x)+H(-x)} = 1$$
$$\frac{H(-x)}{1-H(x)+H(-x)} = 0.$$

Thus, all the condition of Theorem 2.33 are satisfied by h with the appropriate σ, α and $\beta = 1$, and the proof is complete. ☐

Remark 2.35. A couple of remarks:

— As was discussed in the introduction: the finiteness of the fourth moment of f guarantees a *normal* local central limit theorem. When f lacks that condition, a thing that manifests itself via the function $\nu_f(x)$ in the above theorem, there is still something that can be said and our local central limit theorem comes into play by replacing the Gaussian with the stable laws.

— The parameter β represents the skewness of the stable distribution. In general $\beta \in [-1, 1]$ and the closer it is to 1, the more right skewed the distribution is. The closer it gets to -1, the more left skewed the distribution is. Since our probability density function h is supported on the positive real line, it is not surprising that we got that β must be 1!

We are now ready to prove Theorems 2.13 and 2.14.

Proof of Theorem 2.13. Due to the given information on f, we see that it satisfies all the conditions of Theorem 2.34, and as such for any finite $k \in \mathbb{R}$

$$\begin{aligned} &|\mathbb{S}^{N-k-1}| r^{\frac{N-k-2}{2}} \mathcal{Z}_{N-k}\left(f, \sqrt{r}\right) \\ &= \frac{2}{(N-k)^{\frac{1}{\alpha}}} \left(\gamma_{\sigma,\alpha,1}\left(\frac{r-(N-k)}{(N-k)^{\frac{1}{\alpha}}}\right) + \lambda_{N-k}(r) \right), \end{aligned} \tag{2.70}$$

for some $\sigma = C_S \frac{\Gamma(3-\alpha)}{\alpha(\alpha-1)} \cos\left(\frac{\pi\alpha}{2}\right)$ and λ_{N-k} such that

$$\epsilon_{N-k} = \sup_r |\lambda_{N-k}(r)| \underset{N\to\infty}{\longrightarrow} 0.$$

Using Lemma 2.22 with $F_N = \frac{f^{\otimes N}}{\mathcal{Z}_N(f,\sqrt{N})}$ we find that

$$\Pi_k(F_N)(v_1,\dots,v_k) = \frac{|\mathbb{S}^{N-k-1}|\left(N-\sum_{i=1}^k v_i^2\right)_+^{\frac{N-k-2}{2}} \mathcal{Z}_{N-k}\left(f,\sqrt{N-\sum_{i=1}^k v_i^2}\right)}{|\mathbb{S}^{N-1}|N^{\frac{N-2}{2}}\mathcal{Z}_N\left(f,\sqrt{N}\right)} \cdot f^{\otimes k}(v_1,\dots,v_k).$$

Combining this with (2.70) yields

$$\Pi_k(F_N)(v_1,\dots,v_k) = \left(\frac{N}{N-k}\right)^{\frac{1}{\alpha}} \frac{\gamma_{\sigma,\alpha,1}\left(\frac{k-\sum_{i=1}^k v_i^2}{(N-k)^{\frac{1}{\alpha}}}\right)+\lambda_{N-k}\left(N-\sum_{i=1}^k v_i^2\right)}{\gamma_{\sigma,\alpha,1}(0)+\lambda_N(N)} \cdot f^{\otimes k}(v_1,\dots,v_k)\,\chi_{\sum_{i=1}^k v_i^2\le N}(v_1,\dots,v_k), \tag{2.71}$$

where χ_A is the characteristic function of the set A. By its definition, given in (2.18), and the properties of $\widehat{\gamma}_{\sigma,\alpha,\beta}$, we know that $\gamma_{\sigma,\alpha,1}$ is bounded and continuous on $\mathbb{R}$. As such, along with the conditions on λ_{N-k} and λ_N, we conclude that

$$\Pi_k(F_N)(v_1,\dots,v_k) \underset{N\to\infty}{\longrightarrow} f^{\otimes k}(v_1,\dots,v_k),$$

pointwise. Using Lemma 2.23 we obtain that $\{F_N\}_{N\in\mathbb{N}}$ is $f-$chaotic.

Next we turn our attention to the entropic chaos. Using symmetry, (2.70) and (2.71) we find that

$$\begin{aligned} H_N(F_N) &= \frac{1}{\mathcal{Z}_N\left(f,\sqrt{N}\right)}\int_{\mathbb{S}^{N-1}(\sqrt{N})} f^{\otimes N}\log\left(f^{\otimes N}\right)d\sigma^N - \log\left(\mathcal{Z}_N\left(f,\sqrt{N}\right)\right) \\ &= N\int_{\mathbb{R}}\Pi_1(F_N)(v_1)\log\left(f(v_1)\right)dv_1 - \log\left(\frac{2\left(\gamma_{\sigma,\alpha,1}(0)+\lambda_N(N)\right)}{|\mathbb{S}|^{N-1}N^{\frac{N-2}{2}+\frac{1}{\alpha}}}\right) \\ &= N\left(\frac{N}{N-1}\right)^{\frac{1}{\alpha}}\int_{-\sqrt{N}}^{\sqrt{N}} \frac{\gamma_{\sigma,\alpha,1}\left(\frac{1-v_1^2}{(N-1)^{\frac{1}{\alpha}}}\right)+\lambda_{N-1}\left(N-v_1^2\right)}{\gamma_{\sigma,\alpha,1}(0)+\lambda_N(N)} f(v_1)\log f(v_1)dv_1 \\ &\quad -\log\left(2\sqrt{\pi}\left(\gamma_{\sigma,\alpha,1}(0)+\lambda_N(N)\right)\left(1+O\left(\frac{1}{N}\right)\right)\right)+\left(\frac{1}{\alpha}-\frac{1}{2}\right)\log N+\frac{N}{2}\log(2\pi e). \end{aligned}$$

where we have used the fact that $|\mathbb{S}^{N-1}| = \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}$, and an asymptotic approximation for the Gamma function.

We have that

$$\begin{aligned} &\left|\frac{\gamma_{\sigma,\alpha,1}\left(\frac{1-v_1^2}{(N-1)^{\frac{1}{\alpha}}}\right)+\lambda_{N-1}\left(N-v_1^2\right)}{\gamma_{\sigma,\alpha,1}(0)+\lambda_N(N)} f(v_1)\log f(v_1)\right| \\ &\le \frac{\left\|\gamma_{\sigma,\alpha,1}\right\|_\infty+\epsilon_{N-1}}{\gamma_{\sigma,\alpha,1}(0)-\epsilon_N} f(v_1)\left|\log f(v_1)\right| \\ &\le \frac{2\left(\left\|\gamma_{\sigma,\alpha,1}\right\|_\infty+1\right)}{\gamma_{\sigma,\alpha,1}(0)} f(v_1)\left|\log f(v_1)\right| \in L^1(\mathbb{R}), \end{aligned}$$

for N large enough. Combining this with the fact that $\{\Pi_1(F_N)\}_{N\in\mathbb{N}}$ converges to f pointwise, we can use the dominated convergence theorem to conclude that

$$\lim_{N\to\infty}\frac{H_N(F_N)}{N}=\int_{\mathbb{R}} f(v_1)\log f(v_1)dv_1+\frac{\log 2\pi+1}{2}=H(f|\gamma), \tag{2.72}$$

and the proof is complete. □

Proof of Theorem 2.14. It is easy to see that the condition $f(x) \underset{x\to\infty}{\sim} \frac{D}{|x|^{1+2\alpha}}$ for some $1<\alpha<2$ and $D>0$ implies that

$$\nu_f(x) \underset{x\to\infty}{\sim} \frac{D}{2-\alpha}x^{2-\alpha}.$$

Thus, with the added information given in the theorem we know that f satisfies the conditions of Theorem 2.13, and we conclude the desired result. □

Remark 2.36*.* Theorem 2.14 gives rise to many, previously unknown, entropically chaotic families, determined mainly by a simple growth condition. An explicit example to such family is the one generated by the function

$$f(x)=\frac{\sqrt{2}}{\pi\left(1+x^4\right)}.$$

2.5 Lower semi continuity and stability property

As discussed in Section 2.1, the concept of entropic chaoticity is much stronger than that of normal chaoticity. This is due to the inclusion of all correlation information and an appropriate rescaling of the relative entropy. In this section we will show that the rescaled entropy is a good form of distance, one that is stable under certain conditions.

The first step we must make, inspired by [12], is a form of lower semi continuity property for the relative entropy on Kac's sphere, expressed in Theorem 2.15. To begin with, we mention that in [12], the authors proved the following:

Theorem 2.37. *Let f be a probability density function on $\mathbb{R}$ such that $f\in L^p(\mathbb{R})$ for some $p>1$. Assume in addition that*

$$\int_{\mathbb{R}} x^2 f(x)dx=1,\quad \int_{\mathbb{R}} x^4 f(x)dx<\infty.$$

Denote by $d\nu_N=F_N d\sigma^N$, where $F_N=\frac{f^{\otimes N}}{\mathcal{Z}_N\left(f,\sqrt{N}\right)}$, restricted to Kac's sphere and let $\{\mu_N\}_{N\in\mathbb{N}}$ be a family of symmetric probability measures on Kac's sphere such that for some $k\in\mathbb{N}$ we have that

$$\Pi_k(\mu_N) \underset{N\to\infty}{\rightharpoonup} \mu_k.$$

Then

$$\frac{H\left(\mu_k|f^{\otimes k}\right)}{k}\leq \liminf_{N\to\infty}\frac{H_N(\mu_N|\nu_N)}{N}. \tag{2.73}$$

Note that due to the so-called Csiszar-Kullback-Leibler-Pinsker inequality ([68]) one has that

$$\|\mu - \nu\|_{TV} \leq \sqrt{2H(\mu|\nu)}, \tag{2.74}$$

showing that (2.73) gives a stronger result than an L^1 convergence. We will use this theorem as a motivation for our lower semi continuity property, as well as in the particular case of

$$f(x) = \gamma(x), \quad d\nu_N = F_N d\sigma^N = d\sigma^N,$$

where $\gamma(x)$ is the standard Gaussian.

Before we begin the proof of Theorem 2.15 we point out the obvious difference between the $k=1$ and $k>1$ cases. This is due to the fact that the proof relies heavily on our approximation theorem, Theorem 2.34, which is valid *only* in one dimension. The higher dimension case needs to be tackled differently, unlike the proof of Theorem 2.37, where the higher dimension case is proven in a very similar way.

The proof of Theorem 2.15 follows ideas presented in [12], with some modification to our current discussion.

Proof of Theorem 2.15. We start by noticing that since $C_b\left(\mathbb{R}^{k_0}\right)$ can be considered a subspace of $C_b\left(\mathbb{R}^{k}\right)$ whenever $k_0 \leq k$, the weak convergence condition on $\Pi_k(\mu_N)$ implies that

$$\Pi_{k_0}(\mu_N) \underset{N\to\infty}{\rightharpoonup} \mu_{k_0} = \Pi_{k_0}(\mu_k).$$

In particular we find that $\Pi_1(\mu_N)$ converges weakly to $\mu = \Pi_1(\mu_k)$.
Next, we recall a duality formula for the relative entropy (see [53] for instance, for the compact case):

$$H(\mu|\nu) = \sup_{\varphi\in C_b}\left\{\int \varphi d\mu - \log\left(\int e^{\varphi}d\nu\right)\right\}. \tag{2.75}$$

Given $\epsilon > 0$ we can find $\varphi_\epsilon \in C_b(\mathbb{R})$ such that

$$\int_{\mathbb{R}} e^{\varphi_\epsilon(v)} f(v) dv = 1$$

and

$$H(\mu|f) \leq \int_{\mathbb{R}} \varphi_\epsilon(v) d\mu(v) + \frac{\epsilon}{2}. \tag{2.76}$$

We can find a compact set $K_\epsilon \subset \mathbb{R}$ such that

$$\mu\left(K_\epsilon^c\right) \leq \frac{\epsilon}{4\left\|\varphi_\epsilon\right\|_\infty}, \quad \int_{K_\epsilon^c} f(v)dv \leq \frac{\epsilon}{2e^{\|\varphi_\epsilon\|_\infty}}.$$

Let $\eta_\epsilon \in C_c(\mathbb{R})$ be such that

$$0 \leq \eta_\epsilon \leq 1, \quad \eta_\epsilon|_{K_\epsilon} = 1,$$

and define $\varphi(v) = \eta_\epsilon(v)\varphi_\epsilon(v)$. Clearly $\varphi \in C_c(\mathbb{R})$, $|\varphi| \leq |\varphi_\epsilon|$ and

$$H(\mu|f) \leq \int_{\mathbb{R}} \varphi(v)d\mu(v) + 2\left\|\varphi_\epsilon\right\|_\infty \mu\left(K_\epsilon^c\right) + \frac{\epsilon}{2} < \int_{\mathbb{R}} \varphi(v)d\mu(v) + \epsilon. \tag{2.77}$$

Also,

$$|\int_{\mathbb{R}} e^{\varphi(v)}f(v)dv - \int_{\mathbb{R}} e^{\varphi_\epsilon(v)}f(v)dv| \leq 2e^{\|\varphi_\epsilon\|_\infty}\int_{K_\epsilon^c} f(v)dv < \epsilon. \tag{2.78}$$

For any $N \in \mathbb{N}$, define $\phi_N(v_1,\dots,v_N) = \sum_{i=1}^N \varphi(v_i) \in C_b\left(\mathbb{R}^N\right)$. Plugging ϕ_N as a candidate in (2.75), in the setting of Kac's sphere, and using symmetry we find that

$$\begin{aligned}
H_N(\mu_N|\nu_N) &\geq N\int_{\mathbb{R}}\varphi(v_1)d\Pi_1(\mu_N)(v_1) - \log\left(\frac{1}{\mathcal{Z}_N\left(f,\sqrt{N}\right)}\int_{\mathbb{S}^{N-1}(\sqrt{N})}\Pi_{i=1}^N\left(e^{\varphi(v_i)}f(v_i)\right)d\sigma^N\right)\\
&= N\int_{\mathbb{R}}\varphi(v_1)d\Pi_1(\mu_N)(v_1) - \log\left(\frac{\mathcal{Z}_N\left(\frac{e^{\varphi}f}{a},\sqrt{N}\right)}{\mathcal{Z}_N\left(f,\sqrt{N}\right)}\right) - N\log a,
\end{aligned}$$

where $a = \int_{\mathbb{R}} e^{\varphi(v)}f(v)dv$. Since f satisfies the conditions of Theorem 2.34, so does the probability density function $\frac{e^{\varphi}}{a}f$. Denoting by $E = \frac{1}{a}\int_{\mathbb{R}} v^2 e^{\varphi(v)}f(v)dv$ we find that

$$\frac{\mathcal{Z}_N\left(\frac{e^{\varphi}f}{a},\sqrt{N}\right)}{\mathcal{Z}_N(f,\sqrt{N})} = \frac{\gamma_{\sigma_1,\alpha,1}\left(\frac{N-NE}{N^{\frac{1}{\alpha}}}\right)+\epsilon_1(N)}{\gamma_{\sigma,\alpha,1}\left(0\right)+\epsilon_2(N)}, \tag{2.79}$$

for some σ, σ_1, and $\{\epsilon_i(N)\}_{i=1,2}$ that go to zero as N goes to infinity. Since $\gamma_{\sigma_1,\alpha,1}$ is the defined as the inverse Fourier transform of an L^1 function we know that

$$\lim_{|x|\to\infty}\gamma_{\sigma_1,\alpha,1}(x) = 0.$$

Thus,

$$\liminf_{N\to\infty}\left(-\frac{\log\left(\gamma_{\sigma_1,\alpha,1}\left(\frac{N-NE}{N^{\frac{1}{\alpha}}}\right)+\epsilon_1(N)\right)}{N}\right) \geq 0. \tag{2.80}$$

Together with the fact that

$$\lim_{N\to\infty}\left(-\frac{\log\left(\gamma_{\sigma,\alpha,1}(0)+\epsilon_2(N)\right)}{N}\right) = 0,$$

the weak convergence of $\Pi_1(\mu_N)$ and (2.77), we find that

$$\begin{aligned}
\liminf_{N\to\infty}\frac{H_N(\mu_N|\nu_N)}{N} &\geq \int_{\mathbb{R}}\varphi(v)d\mu(v) - \log(1+\epsilon)\\
&\geq H(\mu|f) - \epsilon - \log(1+\epsilon),
\end{aligned} \tag{2.81}$$

where we have used (2.78) to conclude that $|a-1| < \epsilon$. Since ϵ was arbitrary, (i) is proved.

In order to show (ii), we notice that

$$
\begin{aligned}
H_N(\mu_N|\nu_N) &= \int_{\mathbb{S}^{N-1}(\sqrt{N})} \log\left(\frac{d\mu_N}{F_N d\sigma^N}\right) d\mu_N = H_N(\mu_N|\sigma^N) - \int_{\mathbb{S}^{N-1}(\sqrt{N})} \log\left(F_N\right) d\mu_N \\
&= H_N(\mu_N|\sigma^N) - N\int_{\mathbb{R}} \log\left(f(v_1)\right) d\Pi_1(\mu_N) + \log\left(\mathcal{Z}_N\left(f,\sqrt{N}\right)\right).
\end{aligned}
$$

Thus, for any $\delta > 0$,

$$
\begin{aligned}
\liminf_{N\to\infty} \frac{H_N(\mu_N|\nu_N)}{N} &+ \limsup_{N\to\infty}\int_{\mathbb{R}} \log\left(f(v_1)+\delta\right) d\Pi_1(\mu_N) \\
&\geq \liminf_{N\to\infty}\frac{H_N(\mu_N|\sigma^N)}{N} - \frac{\log(2\pi)+1}{2},
\end{aligned} \tag{2.82}
$$

where we have used the fact that $\lim_{N\to\infty}\frac{\log\left(\mathcal{Z}_N\left(f,\sqrt{N}\right)\right)}{N} = -\frac{\log(2\pi)+1}{2}$, shown in the proof of Theorem 2.13. From Theorem 2.37 we know that

$$
\liminf_{N\to\infty}\frac{H_N(\mu_N|\sigma^N)}{N} \geq \frac{H(\mu_k|\gamma^{\otimes k})}{k},
$$

and since

$$
\begin{aligned}
H(\mu_k|f^{\otimes k}) &= H(\mu_k|\gamma^{\otimes k}) + \int_{\mathbb{R}^k}\log\left(\frac{\gamma^{\otimes k}}{f^{\otimes k}}\right) d\mu_k \\
&= H(\mu_k|\gamma^{\otimes k}) - \frac{k\left(\log(2\pi)+\int_{\mathbb{R}} v^2 d\mu(v)\right)}{2} - k\int_{\mathbb{R}}\log\left(f(v)\right) d\mu(v)
\end{aligned}
$$

we get the desired result from (2.82). □

We will now prove our first stability result, Theorem 2.16. Again, the ideas presented here are motivated by [12].

Proof of Theorem 2.16. We start with the simple observation that if $\{\mu_N\}_{N\in\mathbb{N}}$ is a family of symmetric probability measures on Kac's sphere then $\{\Pi_k(\mu_N)\}_{N\in\mathbb{N}}$ is a tight family, for any $k\in\mathbb{N}$. Indeed, given $k\in\mathbb{N}$ we can find $m_N, r_N\in\mathbb{N}$ such that

$$
N = m_N k + r_N,
$$

where $0\leq r_N < k$. We have that

$$
\begin{aligned}
\Pi_k(\mu_N)\left(\left\{\sqrt{\sum_{i=1}^k v_i^2} > R\right\}\right) &\leq \frac{1}{R^2}\int_{\sum_{i=1}^k v_i^2 > R^2}\left(\sum_{i=1}^k v_i^2\right) d\Pi_k(\mu_N) \\
&\leq \frac{1}{m_N R^2}\int_{\mathbb{S}^{N-1}(\sqrt{N})}\left(\sum_{i=1}^{m_N k} v_i^2\right) d\mu_N \leq \frac{N}{m_N R^2} < \frac{2k}{R^2},
\end{aligned}
$$

proving the tightness.
Since $\{\Pi_1\mu_N\}_{N\in\mathbb{N}}$ is tight, we can find a subsequence, $\left\{\Pi_1\left(\mu_{N_{k_j}}\right)\right\}_{j\in\mathbb{N}}$, to any subsequence $\{\Pi_1\left(\mu_{N_k}\right)\}_{k\in\mathbb{N}}$, that converges to a limit. Denote by κ the weak limit of such one subsequence. Using (2.24) we conclude that

$$H(\kappa|f) \leq \liminf_{j\to\infty} \frac{H_{N_{k_j}}\left(\mu_{N_{k_j}}|\nu_{N_{k_j}}\right)}{N_{k_j}} = 0, \tag{2.83}$$

due to condition (2.26). Thus, $\kappa = f(v)dv$, and since κ was an arbitrary weak limit, we conclude that all possible weak limit points must be $f(v)dv$. Since the weak topology on $P(\mathbb{R})$ is metrisable we conclude that

$$\Pi_1(\mu_N) \underset{N\to\infty}{\rightharpoonup} f(v)dv = \mu.$$

We will show that the convergence is actually in L^1 with the weak topology.

As an intermediate step in the proof of Theorem 2.15 we have shown that

$$\begin{aligned} H(\mu_N|\nu_N) = H(\mu_N|\sigma^N) - N\int_{\mathbb{R}} \log\left(f(v_1)\right) d\Pi_1(\mu_N)(v_1) \\ + \log\left(\mathcal{Z}_N\left(f,\sqrt{N}\right)\right). \end{aligned} \tag{2.84}$$

Using condition (2.26), the fact that $\lim_{N\to\infty}\frac{\mathcal{Z}_N\left(f,\sqrt{N}\right)}{N} = -\frac{\log(2\pi)+1}{2}$, and the fact that $f\in L^\infty(\mathbb{R})$ we conclude that there exists $C>0$, independent of N, such that for any $\delta>0$

$$\frac{H(\mu_N|\sigma_N)}{N} \leq C + \log\left(\|f\|_\infty + \delta\right). \tag{2.85}$$

The inequality

$$\frac{H\left(\Pi_k(\mu_N)|\Pi_k(\sigma^N)\right)}{k} \leq 2\frac{H_N(\mu_N|\sigma^N)}{N}$$

proven in [3] and valid for any $k\geq 1$ and $N\geq k$, implies that

$$H\left(\Pi_k(\mu_N)|\Pi_k(\sigma^N)\right) \leq 2k\left(C + \log\left(\|f\|_\infty\right) + \delta\right), \tag{2.86}$$

for all $k\in\mathbb{N}$, $N\geq k$ and $\delta>0$.

Similar to the proof of Theorem 2.15, one can easily see that

$$H\left(\Pi_k(\mu_N)|\gamma^{\otimes k}\right) = H\left(\Pi_k(\mu_N)|\Pi_k(\sigma^N)\right) + \int_{\mathbb{R}^k} \log\left(\frac{\Pi_k(\sigma^N)}{\gamma^{\otimes k}}\right) d\Pi_k(\mu_N) \tag{2.87}$$

where γ is the standard Gaussian. Since $d\sigma^N = \frac{\gamma^{\otimes N}}{\mathcal{Z}_N\left(\gamma,\sqrt{N}\right)}d\sigma^N$, and γ is a probability density with finite fourth moment, one can employ similar theorems to those presented here and find that

$$\frac{\Pi_k(\sigma^N)\left(v_1,\dots,v_k\right)}{\gamma^{\otimes k}\left(v_1,\dots,v_k\right)} = \sqrt{\frac{N}{N-k}}\cdot\frac{\gamma\left(\frac{k-\sum_{i=1}^k v_i^2}{\sqrt{2N}}\right) + \lambda_{N-k}\left(N-k-\sum_{i=1}^k v_I^2\right)}{1+\lambda_N(N)}\chi_{\sum_{i=1}^k v_i^2\leq N},$$

where $\sup_u |\lambda_{N-k}(u)| \underset{N\to\infty}{\longrightarrow} 0$ and $\lambda_N(N) \underset{N\to\infty}{\longrightarrow} 0$ (see [12] for more details). As such,

$$\int_{\mathbb{R}^k} \log\left(\frac{\Pi_k(\sigma^N)}{\gamma^{\otimes k}}\right) d\Pi_k(\mu_N) \leq \log\left(\max_{N>k}\sqrt{\frac{N}{N-k}}\frac{\|\gamma\|_\infty + \sup_N \sup_u |\lambda_{N-k}(u)|}{1+\inf_N \lambda_N(N)}\right),$$

which, together with (2.86) and (2.87) shows that

$$H\left(\Pi_k(\mu_N)|\gamma^{\otimes k}\right) \leq 2k\left(C + \log\left(\|f\|_\infty\right) + \delta\right) + D,$$

for some $C, D > 0$ independent of N, and $\delta > 0$. Thus, $\{\Pi_k \mu_N\}_{N\in\mathbb{N}}$ has bounded relative entropy with respect to $\gamma^{\otimes k}$ and we can apply the Dunford-Pettis compactness theorem and conclude that the densities of $\{\Pi_k(\mu_N)\}_{N\in\mathbb{N}}$ form a relatively compact set in $L^1(\mathbb{R}^k)$ with the weak topology. Since this is true for all k, and we know that $\{\Pi_1(\mu_N)\}_{N\in\mathbb{N}}$ converge weakly (in the measure sense) to μ, with density function $f(v)$, we conclude that for any $\phi \in L^\infty(\mathbb{R})$ we have that

$$\int_{\mathbb{R}} \phi(v) d\Pi_1(\mu_N)(v) \underset{N\to\infty}{\longrightarrow} \int_{\mathbb{R}} \phi(v) f(v) dv. \tag{2.88}$$

In particular, since $f \in L^\infty(\mathbb{R})$ and $f \geq 0$ we have that for any $\delta > 0$

$$\int_{\mathbb{R}} \log\left(f(v) + \delta\right) d\Pi_1(\mu_N)(v) \underset{N\to\infty}{\longrightarrow} \int_{\mathbb{R}} \log\left(f(v)+\delta\right) f(v) dv. \tag{2.89}$$

Combining (2.89), (2.26) with the fact that $\Pi_1(\mu_N)$ converges to $f(v)dv$, we find that if $\left\{\Pi_k\left(\mu_{N_j}\right)\right\}_{j\in\mathbb{N}}$ converges weakly to κ_k, then by (2.25)

$$\frac{H(\kappa_k|f^{\otimes k})}{k} \leq \int_{\mathbb{R}} \log\left(f(v)+\delta\right) f(v) dv - \int_{\mathbb{R}} \log\left(f(v)\right) f(v) dv \tag{2.90}$$

where we have used the fact that $\int_{\mathbb{R}} v^2 d\mu(v) = \int_{\mathbb{R}} v^2 f(v) dv = 1$. Using the dominated convergence theorem to take δ to zero shows that $H(\kappa_k|f^{\otimes k}) = 0$, and so

$$\kappa_k = f^{\otimes k}\left(v_1, \dots, v_k\right) dv_1 \dots dv_k.$$

Much like $\{\Pi_1(\mu_N)\}_{N\in\mathbb{N}}$, since $\{\Pi_k(\mu_N)\}_{N\in\mathbb{N}}$ is tight we can always find weak limits for some subsequences of it. We have just proved that all possible weak limits of subsequences of $\{\Pi_k(\mu_N)\}_{N\in\mathbb{N}}$ are $f^{\otimes k}$, from which we conclude that

$$\Pi_k(\mu_N) \underset{N\to\infty}{\rightharpoonup} f^{\otimes k},$$

showing the chaoticity. It is worth to note that we actually proved more than the above: we have proved convergence in $L^1(\mathbb{R}^k)$ with the weak topology.

Going back to (2.84), and using (2.26), (2.89) and the known limit of $\frac{\log\left(\mathcal{Z}_N\left(f,\sqrt{N}\right)\right)}{N}$ we find that

$$\limsup_{N\to\infty} \frac{H_N(\mu_N|\sigma^N)}{N} \leq \int_{\mathbb{R}} \log\left(f(v)+\delta\right) f(v) dv + \frac{\log(2\pi)+1}{2}. \tag{2.91}$$

Taking δ to zero we conclude that

$$\limsup_{N\to\infty} \frac{H_N(\mu_N|\sigma^N)}{N} \leq H(f|\gamma). \tag{2.91}$$

Since the inequality

$$\liminf_{N\to\infty} \frac{H_N(\mu_N|\sigma^N)}{N} \geq H(f|\gamma)$$

follows from Theorem 2.37, we see that

$$\lim_{N\to\infty} \frac{H_N(\mu_N|\sigma^N)}{N} = H(f|\gamma), \tag{2.90}$$

proving the entropic chaoticity and completing the proof. □

The last proof of this section will involve the second 'closeness' criteria, associated with the Fisher information functional, and given by Theorem 2.18. The proof is similar to those appearing in [46] and [19] with appropriate modifications. The proof will rely heavily on tools from the field of Optimal Transportation.

Proof of Theorem 2.18. The first step of the proof will be to show that conditions (2.29) and (2.30) imply that the marginal limit, f, satisfies the conditions of Theorem 2.13.

We start by showing that $f \in L^p(\mathbb{R})$ for some $p > 1$. In [46] the authors have presented a lower semi continuity result for the relative Fisher Information, from which we conclude that

$$I(f|\gamma) \leq \liminf_{N\to\infty} \frac{I_N(\mu_N|\sigma^N)}{N} \leq C. \tag{2.91}$$

Denoting by

$$I(f) = \int_{\mathbb{R}} \frac{|f'(x)|}{f(x)} dx = 4\int_{\mathbb{R}} |\frac{d}{dx}\sqrt{f(x)}|^2 dx$$

we see that

$$I(f) = I(f|\gamma) + 2 - \int_{\mathbb{R}} v^2 f(v)dv < C + 2 - \int_{\mathbb{R}} v^2 f(v)dv < \infty,$$

as f is a weak limit of $\Pi_1(\mu_N)$, implying that

$$\int_{\mathbb{R}^2} v^2 f(v)dv \leq \liminf_{N\to\infty} \int_{\mathbb{R}} v^2 d\Pi_1(\mu_N)(v) = 1.$$

We conclude that $\sqrt{f} \in H^1(\mathbb{R})$ and using a Sobolev embedding theorem we find that $\sqrt{f} \in L^\infty(\mathbb{R})$. Thus, since f is also in $L^1(\mathbb{R})$, we have that $f \in L^p(\mathbb{R})$ for all $p \geq 1$.

The next step will be to show that condition (2.29) implies a uniform bound for the $1+\alpha$ moment of $\Pi_1(\mu_N)$, i.e.

$$\int_{\mathbb{R}} |v_1|^{1+\alpha} d\Pi_1(\mu_N)(v_1) \leq C, \tag{2.92}$$

for some $C > 0$, independent of N. This will show that

$$\int_{\mathbb{R}} v^2 f(v) dv = \lim_{N\to\infty} \int_{\mathbb{R}} v^2 d\Pi_1(\mu_N)(v) = 1, \tag{2.93}$$

as well as

$$\int_{\mathbb{R}} |v|^{1+\alpha} f(v) dv \leq \liminf_{N\to\infty} \int_{\mathbb{R}} |v|^{1+\alpha} d\Pi_1(\mu_N)(v) \leq C. \tag{2.94}$$

To prove (2.92) we notice that

$$\begin{aligned}
&\int_{\mathbb{R}} |v_1|^{1+\alpha} d\Pi(\mu_N)(v_1) = \frac{3-\alpha}{2^{3-\alpha}-1} \int_{\mathbb{R}} \int_{\frac{|v_1|}{2}}^{|v_1|} x^{\alpha-4} v_1^4 d\Pi_1(\mu_N)(v_1) dx \\
&= \frac{3-\alpha}{2^{3-\alpha}-1} \int_0^\infty x^{\alpha-4} \left(\int_{-2x}^{-x} v_1^4 d\Pi(\mu_N)(v_1) + \int_x^{2x} v_1^4 d\Pi(\mu_N)(v_1) \right) dx \\
&= \frac{3-\alpha}{2^{3-\alpha}-1} \int_0^\infty x^{\alpha-4} \left(\int_{-2x}^{2x} v_1^4 d\Pi(\mu_N)(v_1) - \int_{-x}^{x} v_1^4 d\Pi(\mu_N)(v_1) \right) dx
\end{aligned} \tag{2.95}$$

Using condition (2.29) we know that for any $\epsilon > 0$ we can find $R > 0$, such that for any $|x| > R$ and any $N \in \mathbb{N}$

$$(1-\epsilon) C_S x^{2-\alpha} \leq \int_{-\sqrt{x}}^{\sqrt{x}} v_1^4 d\Pi_1(\mu_N)(v_1) \leq (1+\epsilon) C_S x^{2-\alpha} \tag{2.96}$$

In addition, for any probability measure μ on $\mathbb{R}$ we have that

$$\int_{-x}^{x} v^4 d\mu(v) \leq 2x^4. \tag{2.97}$$

Combining (2.95), (2.96) and (2.97) we conclude that

$$\begin{aligned}
&\int_{\mathbb{R}} |v_1|^{1+\alpha} d\Pi(\mu_N)(v_1) \leq \frac{3-\alpha}{2^{3-\alpha}-1} \Bigg(32 R^{\frac{\alpha+1}{2}} \\
&+ C_S \left((1+\epsilon) 2^{4-2\alpha} - (1-\epsilon) \right) \int_{\sqrt{R}}^\infty \frac{dx}{x^\alpha} \Bigg) = C
\end{aligned} \tag{2.98}$$

for a choice of $0 < \epsilon < 1$.

Lastly, we want to show that ν_f, defined in Theorem 2.13, satisfies the appropriate growth condition.

Since $\Pi_1(\mu_N)$ converges to f weakly, we have that for any lower semi continuous function, ϕ, that is bounded from below,

$$\int_{\mathbb{R}} \phi(v) f(v) dv \leq \liminf_{N\to\infty} \int_{\mathbb{R}} \phi(v_1) d\Pi_1(\mu_N)(v_1). \tag{2.99}$$

Similarly, if ϕ is upper semi continuous and bounded from above then

$$\int_{\mathbb{R}} \phi(v) f(v) dv \geq \limsup_{N\to\infty} \int_{\mathbb{R}} \phi(v_1) d\Pi_1(\mu_N)(v_1). \tag{2.100}$$

Choosing $\phi(v) = v^4\chi_{(-\sqrt{x},\sqrt{x})}(v)$ and $\phi(v) = v^4\chi_{[-\sqrt{x},\sqrt{x}]}(v)$ respectively, and using condition (2.29) proves that

$$\nu_f(x) = \int_{-\sqrt{x}}^{\sqrt{x}} v^4 f(v)dv \underset{x\to\infty}{\sim} C_S x^{2-\alpha},$$

and we can conclude that f satisfies the conditions of Theorem 2.13. This implies that the function $F_N = \frac{f^{\otimes N}}{\mathcal{Z}_N(f,\sqrt{N})}$ is well defined, and as usual we denote $\nu_N = F_N d\sigma^N$.

Next, we will show that $\frac{I_N(\nu_N|\sigma^N)}{N}$ is uniformly bounded in N. Denoting by ∇ the normal gradient on $\mathbb{R}^N$ and by ∇_S its tangential component to Kac's sphere we find that

$$\begin{aligned}\int_{\mathbb{S}^{N-1}(\sqrt{N})} \frac{|\nabla_S F_N|^2}{F_N} d\sigma^N &\leq \frac{1}{\mathcal{Z}_N\left(f,\sqrt{N}\right)} \int_{\mathbb{S}^{N-1}(\sqrt{N})} \frac{|\nabla f^{\otimes N}|^2}{f^{\otimes N}} d\sigma^N \\ &= \sum_{i=1}^N \frac{1}{\mathcal{Z}_N\left(f,\sqrt{N}\right)} \int_{\mathbb{S}^{N-1}(\sqrt{N})} \frac{|f'(v_i)|^2}{f(v_i)} \Pi_{j=1,j\neq i}^N f(v_j) d\sigma^N \\ &= N\int_{\mathbb{R}} \frac{|\mathbb{S}^{N-2}|\left(N-v_1^2\right)_+^{\frac{N-3}{2}}}{|\mathbb{S}^{N-1}|N^{\frac{N-2}{2}}} \frac{\mathcal{Z}_{N-1}\left(f,\sqrt{N-v_1^2}\right)}{\mathcal{Z}_N\left(f,\sqrt{N}\right)} \cdot \frac{|f'(v_1)|^2}{f(v_1)} dv_1,\end{aligned} \tag{2.101}$$

where we have used Lemma 2.21, and the definition of the normalisation function. Using the asymptotic behaviour of $\mathcal{Z}_N\left(f,\sqrt{r}\right)$ from Theorem 2.34 we conclude that

$$\begin{aligned}\frac{I_N(\nu_N|\sigma^N)}{N} &\leq \left(\frac{N}{N-1}\right)^{\frac{1}{\alpha}} \int_{\mathbb{R}} \frac{\gamma_{\sigma,\alpha,1}\left(\frac{1-v_1^2}{N^{\frac{1}{\alpha}}}\right) + \lambda_{N-1}\left(N-v_1^2\right)}{\gamma_{\sigma,\alpha,1}(0) + \lambda_N(N)} \frac{|f'(v_1)|^2}{f(v_1)} dv_1 \\ &\leq CI(f) \leq C_1,\end{aligned} \tag{2.102}$$

for $C_1 > 0$, independently of N.

At this point we'd like to invoke the HWI inequality, a strategy that was first proved to be successful in this context in [46] and [62]. In our settings we find that

$$\begin{aligned}H(\mu_N|\sigma^N) - H(\nu_N|\sigma^N) &\leq \frac{\pi}{2}\sqrt{I_N(\mu_N|\sigma_N)} W_2(\mu_N,\nu_N) \\ H(\nu_N|\sigma^N) - H(\mu_N|\sigma^N) &\leq \frac{\pi}{2}\sqrt{I_N(\nu_N|\sigma_N)} W_2(\mu_N,\nu_N),\end{aligned} \tag{2.103}$$

where W_2 stands for the quadratic Wasserstein distance with distance function induced from the quadratic distance function on $\mathbb{R}^N$:

$$W_2^2(\mu_N,\nu_N) = \inf_{\pi\in\Pi(\mu_N,\nu_N)} \int_{\mathbb{S}^{N-1}\left(\sqrt{N}\right)\times\mathbb{S}^{N-1}\left(\sqrt{N}\right)} |x-y|^2 d\pi(x,y),$$

where $\Pi(\mu_N,\nu_N)$, the space of pairing, is the space of all probability measures on $\mathbb{S}^{N-1}\left(\sqrt{N}\right) \times \mathbb{S}^{N-1}\left(\sqrt{N}\right)$ with marginal μ_N and ν_N respectively.

The reason we are allowed to use the HWI inequality follows from the fact that Kac's sphere has a positive Ricci curvature. Moreover, in the original statement of the HWI inequality, the quadratic Wasserstein distance is taken with the quadratic *geodesic* distance, yet, fortunately for us, it is equivalent to the normal distance on $\mathbb{R}^N$, hence the factor $\frac{\pi}{2}$ that appears in (2.103). For more information about the Wasserstein distance and the HWI inequality, we refer the interested reader to [78].

Combining (2.103) with the boundness of the rescaled relative Fisher information of μ_N and ν_N with respect to σ^N, we conclude that

$$\left|\frac{H(\mu_N|\sigma^N)}{N} - \frac{H(\nu_N|\sigma^N)}{N}\right| \leq C\frac{W_2(\mu_N,\nu_N)}{\sqrt{N}} \tag{2.104}$$

for some $C > 0$.

The next step of the proof is to show that the first marginals of μ_N and ν_N have some joint bounded moment of order $l > 2$, uniformly in N. This will help us give a quantitative estimation to the quadratic Wasserstein distance. Indeed, using several results from [46], one can show the following estimation:

$$\frac{W_2(\kappa_N, f^{\otimes N})}{\sqrt{N}} \leq C_1 B_l^{\frac{1}{l}} \left(W_1\left(\Pi_2(\kappa_N), f^{\otimes 2}\right) + \frac{1}{N^{p_1}}\right)^{\frac{1}{2}-\frac{1}{l}} \tag{2.105}$$

where C_1 and p_1 are positive constants that depends only on $l > 2$, κ_N is a probability measure on Kac's sphere, f is a probability measure on $\mathbb{R}$ and

$$B_l = \int_{\mathbb{R}} |v_1|^l d\Pi_1(\kappa_N)(v_1) + \int_{\mathbb{R}} |v_1|^l f(v_1)dv_1 < \infty.$$

We have already shown that $\{\Pi_1(\mu_N)\}_{N\in\mathbb{N}}$ has a uniformly bounded moment of order $1+\alpha$. Using (2.71) from the proof of Theorem 2.13, we find that

$$\int_{\mathbb{R}} |v_1|^{1+\alpha} d\Pi_1(\nu_N)(v_1) = \left(\frac{N}{N-1}\right)^{\frac{1}{\alpha}} \int_{|v_1|\leq\sqrt{N}} \frac{\gamma_{\sigma,\alpha,1}\left(\frac{1-v_1^2}{N^{\frac{1}{\alpha}}}\right) + \lambda_{N-1}\left(N - v_1^2\right)}{\gamma_{\sigma,\alpha,1}(0) + \lambda_N(N)} |v_1|^{1+\alpha} f(v_1)dv_1$$

for some $\sigma > 0$, $1 < \alpha < 2$ and λ_{N-k}, λ_N with

$$\sup_u |\lambda_{N-1}(u)| \underset{N\to\infty}{\longrightarrow} 0, \quad \lambda_N(N) \underset{N\to\infty}{\longrightarrow} 0.$$

Thus, along with (2.94), we conclude that

$$\int_{\mathbb{R}} |v_1|^{1+\alpha} d\Pi_1(\nu_N)(v_1) \leq C, \tag{2.106}$$

for some $C > 0$.

Defining

$$\begin{aligned} M = &\int_{\mathbb{R}} |v_1|^{1+\alpha} d\Pi_1(\mu_N)(v_1) + \int_{\mathbb{R}} |v_1|^{1+\alpha} d\Pi_1(\nu_N)(v_1) \\ &+ \int_{\mathbb{R}} |v_1|^{1+\alpha} f(v_1)dv_1 < \infty \end{aligned} \tag{2.107}$$

and combining (2.104), (2.105), and the triangle inequality for the Wasserstein distance, leads us to conclude that

$$\begin{aligned}|\frac{H(\mu_N|\sigma^N)}{N} - \frac{H(\nu_N|\sigma^N)}{N}| \leq CM^{\frac{1}{1+\alpha}} \Bigg[\left(W_1\left(\Pi_2(\mu_N), f^{\otimes 2}\right) + \frac{1}{N^{p_1}} \right)^{\frac{1}{2}-\frac{1}{1+\alpha}} \\ + \left(W_1\left(\Pi_2(\nu_N), f^{\otimes 2}\right) + \frac{1}{N^{p_1}} \right)^{\frac{1}{2}-\frac{1}{1+\alpha}} \Bigg].\end{aligned} \tag{2.108}$$

As $\Pi_2(\nu_N), \Pi_2(\nu_N)$ and $f^{\otimes 2}$ all have unit second moment (for any N), the Wasserstein distance is equivalent to weak topology with respect to them. Since $\{\mu_N\}_{N\in\mathbb{N}}$ and $\{\nu_N\}_{N\in\mathbb{N}}$ are $f-$chaotic, we conclude that

$$W_1\left(\Pi_2(\mu_N), f^{\otimes 2}\right) \underset{N\to\infty}{\longrightarrow} 0, \quad W_1\left(\Pi_2(\nu_N), f^{\otimes 2}\right) \underset{N\to\infty}{\longrightarrow} 0,$$

implying that

$$\lim_{N\to\infty} |\frac{H(\mu_N|\sigma^N)}{N} - \frac{H(\nu_N|\sigma^N)}{N}| = 0. \tag{2.109}$$

We are almost ready to conclude the proof. Before we do, we use the lower semi continuity of the entropy, discussed in Theorem 2.37, to see that

$$H(f|\gamma) \leq \liminf_{N\to\infty} \frac{H_N(\mu_N|\sigma^N)}{N} \leq C < \infty.$$

Thus,

$$\begin{aligned}|\frac{H(\mu_N|\sigma^N)}{N} - H(f|\gamma)| \leq |\frac{H(\mu_N|\sigma^N)}{N} - \frac{H(\nu_N|\sigma^N)}{N}| \\ + |\frac{H(\nu_N|\sigma^N)}{N} - H(f|\gamma)| \underset{N\to\infty}{\longrightarrow} 0,\end{aligned} \tag{2.110}$$

where we have used (2.109) and Theorem 2.13, completing proof. □

Remark 2.38. We'd like to point out that following the above proof, one can see that condition (2.29), giving us a uniform asymptotic behaviour for the fourth moments of the first marginals of $\{\mu_N\}_{N\in\mathbb{N}}$, can be replaced with the conditions that f satisfies the conditions of Theorem 2.13, and the first marginals of $\{\mu_N\}_{N\in\mathbb{N}}$ have a uniformly bounded $k-$th moment, for some $k > 2$. This gives us a different approach to the stability problem, expressed with the Fisher information functional, one that assumes less information on the first marginals, but more conditions on the marginal limit.

2.6 Connections to the trend to equilibrium in Kac's model and Cercignani's conjecture

The study of the conditioned tensorisation of a function f is closely related to the problem of finding the rate of convergence to equilibrium in Kac's Model. In this section we will outline some of the history, and recent results, dealing with this subject.

Recall that Kac's model have managed to show validity (in some sense) for the spatially homogenous Boltzmann equation. Kac hoped to use his simple model to infer quantitative rate of convergence to equilibrium in Boltzmann equation, as a limit of his 'master' equation. He started by noticing that his evolution equation, (2.1), is ergodic, with an equilibrium state represented by the constant function 1. As such, for any fixed N, one can easily see that

$$\lim_{t\to\infty} F_N\left(t, v_1, \dots, v_N\right) = 1,$$

for any solution to Kac's equation, $F_N(t, v_1, \dots, v_N)$. The rate of convergence to equilibrium under the L^2 norm is determined by the spectral gap

$$\Delta_N = \inf\left\{ \frac{\langle \varphi, N(I-Q)\varphi\rangle_{L^2\left(\mathbb{S}^{N-1}\left(\sqrt{N}\right)\right)}}{\|\varphi\|^2_{L^2\left(\mathbb{S}^{N-1}\left(\sqrt{N}\right)\right)}} \mid \varphi \text{ is symmetric }, \varphi \in L^2\left(\mathbb{S}^{N-1}\left(\sqrt{N}\right)\right),\ \varphi \perp 1 \right\}.$$

Kac's conjectured that

$$\Delta = \liminf_{N\to\infty} \Delta_N > 0,$$

resulting in an exponential convergence to equilibrium, for any fixed N, with a rate that is independent with the number of particles.

The spectral gap problem remained open until 2000, when a series of papers by authors such as Janversse, Maslen, Carlen, Carvahlo, Loss and Geronimo gave a satisfactory positive answer to the conjecture, even in McKean's model (see [48, 54, 13, 15] for more details). However, the L^2 norm is catastrophic when dealing with chaotic families and in this setting attempts to pass to the limit in the number of particles is futile. Indeed, one can easily find a chaotic family, $\{F_N\}_{N\in\mathbb{N}}$, such that

$$\|F_N\|_{L^2} \geq C^N,$$

where $C > 1$ (for example, the conditioned tensorisation of a function f we discussed so extensively in this paper!).

As was mentioned in the introduction, the next 'distance' to be considered was the entropy

$$H_N(F_N) = \int_{\mathbb{S}^{N-1}\left(\sqrt{N}\right)} F_N \log F_N d\sigma^N.$$

In an attempt to imitate the idea behind the spectral gap method, one can define the *entropy production* to be the minus of the formal derivation of the entropy under Kac's evolution equation

$$D_N(F_N) = -\frac{d}{dt} H_N(F_N) = \langle \log F_N, N(I-Q)F_N\rangle_{L^2\left(\mathbb{S}^{N-1}\left(\sqrt{N}\right)\right)}, \tag{2.111}$$

The appropriate 'spectral gap' will be given by

$$\Gamma_N = \inf_{F_N} \frac{D_N(F_N)}{H_N(F_N)}$$

and the appropriate conjecture is: Can one find a positive constant, $C > 0$, such that

$$\Gamma_N \geq C$$

for all N. This problem is called *Cercignani's many body conjecture*, named after a similar conjecture posed for the real Boltzmann equation in [21]. If there exists such a C, we have that

$$H_N(F_N(t)) \leq e^{-Ct} H_N(F_N(0))$$

then, hoping the entropic chaoticity propagates with time, one can divide by N and take a limit as N to find that

$$H(f_t|\gamma) \leq e^{-Ct} H(f_0|\gamma). \tag{2.112}$$

This, along with a known inequality on $H(f|\gamma)$ gives an exponential rate of decay towards the equilibrium.

Unfortunately, in general, Cercignani's many body conjecture is false. The first hint to it was revealed in [77] where Villani managed to prove that

$$\Gamma_N \geq \frac{2}{N-1}.$$

Villani has conjectured that $\Gamma_N = O\left(\frac{1}{N}\right)$, which was proven to be essentially true by the second author. In [34] (and later on in [35] for McKean's model) the second author extended the *normal* local central limit theorem, Theorem 2.5, to the case where the underlying generating function, f, also varies with N. In particular, the second author showed that:

Theorem 2.39. *Let $0 < \eta < 1$ and $\delta_N = \frac{1}{N^\eta}$. Define*

$$f_N(v) = \delta_N M_{\frac{1}{2\delta_N}}(v) + (1-\delta_N) M_{\frac{1}{2(1-\delta_N)}}(v),$$

where $M_a(v) = \frac{e^{-\frac{v^2}{2a}}}{\sqrt{2\pi a}}$. Then

$$\mathcal{Z}_N(f, \sqrt{u}) = \frac{2}{\sqrt{N}\Sigma_N \left|\mathbb{S}^{N-1}\right| u^{\frac{N-2}{2}}} \left(\frac{e^{-\frac{(u-N)^2}{2N\Sigma_N^2}}}{\sqrt{2\pi}} + \lambda_N(u) \right), \tag{2.113}$$

where $\Sigma^2 = \frac{3}{4\delta_N(1-\delta_N)} - 1$ and $\sup_u |\lambda_N(u)| \underset{N\to\infty}{\longrightarrow} 0$. Moreover, using the same notation as (2.9) with f replaced by f_N, one finds that there exists $C_{\eta'} > 0$, depending only on η' such that

$$\Gamma_N \leq \frac{D_N(F_N)}{H_N(F_N)} < \frac{C_{\eta'}}{N^{\eta'}} \tag{2.114}$$

for $0 < \eta' < \eta$.

The above theorem poses an interesting insight: The family constructed in Theorem 2.39 has two peculiar properties:

(i)

$$\int_{\mathbb{R}} v^4 f_N(v)dv = \frac{3}{4\delta_N(1-\delta_N)} \underset{N\to\infty}{\longrightarrow} \infty$$

so the fourth moment condition unbounded in some sense in this example.

(ii) F_N is $M_{\frac{1}{2}}$−chaotic yet $\lim_{N\to\infty} \frac{H_N(F_N)}{N}$ exists but doesn't equal $H\left(M_{\frac{1}{2}}|\gamma\right)$!

This insinuates that moments of the limit function, as well as entropic chaoticity, may be very important to the validity of Cercignani's many body conjecture. This is one reason that prompted us to try and investigate the conditioned tensorisation of a function f that has an unbounded fourth moment. While we have attained some answers, we believe that there is much more that can be discovered.

2.7 Final remarks

While Kac's model, chaoticity and entropic chaoticity, and Cercignani's many body conjecture are far from being completely understood and resolved, we hope that our paper has shed some light on the interplay between the moments of a generating function and its associated tensorised measure, restricted to Kac's sphere. As an epilogue, we present here a few remarks about our work, along with associated questions we'll be interested in investigating next.

— One fundamental problem we're very interested in is finding conditions under which Cercignani's many body conjecture is valid. While our work showed that the requirement of a bounded fourth moment is not a major issue for chaoticity and even entropic chaoticity, we still believe that the fourth moment plays an important role in the conjecture. At the very least, due to its probabilistic interpretation as a measurement of deviation from the sphere, we believe that the fourth moment will be needed for an initial positive answer to the conjecture.

— The following was communicated to us by Clément Mouhot: Using a Talagrand inequality, one can show that if the family of functions $\{G_N\}_{N\in\mathbb{N}}$, restricted to the sphere, satisfies a Log-Sobolev inequality that is uniform in N, one has that

$$\lim_{N\to\infty} \frac{H(F_N|G_N)}{N} = 0$$

implies that $\lim_{N\to\infty} (\Pi_k(F_N) - \Pi_k(G_N)) = 0$. Our stability result, Theorem 2.16, gives many examples where the function G_N doesn't satisfy any Log-Sobolev inequality (due to how the underlying function behaves), but we still get equality of marginal. Moreover, we actually get that F_N is entropically chaotic! The connection between the limit of the 'distance'

$$d(F_N, G_N) = \frac{H(F_N|G_N)}{N}$$

and the convergence of marginals is still not understood fully.

— We'll be interested to know if one can find an easy criteria for which we can evaluate quantitatively the convergence of h^{*N} (appearing in Theorem 2.34) without relying on the reminder function. This will allow for possibilities to extend the work done by the second author in [34, 35] and allow the underlying generating function, f, to rely on N as well. While we present such quantitative estimation in the Appendix, we found them to be unusable while trying to deal with concrete examples.

2.A Additional proofs

In this section of the appendix we will present several proofs of technical items we thought would only hinder the flow of the paper.

Proof of Lemma 2.25. Assume that the conclusion is false. We can find a sequence $x_n \underset{n\to\infty}{\longrightarrow} 0$, $x_n \neq 0$, and an $\epsilon_0 > 0$ such that

$$|g(x_n)| \geq \epsilon_0.$$

Due to continuity, we can find $d_1 > 0$ such that for any $x \in [x_1, x_1 + d_1]$ we have

$$|g(x)| \geq \frac{\epsilon_0}{2}.$$

Denote $n_1 = 1$, $x_{k_1} = x_1$ and $\xi_1 = n_1 \cdot x_1 = x_1$.
Since x_n converges to zero and is non zero, we can find x_{k_2} such that $0 < x_{k_2} < \frac{\xi_1}{2}$. Let $n_2 = \left[\frac{\xi_1}{x_{k_2}}\right] + 1 \geq 2$, where $[\cdot]$ is the lower integer part function. We may assume that $x_{k_2} < d_1$ and conclude that

$$\xi_1 \leq n_2 x_{k_2} < \xi_1 + x_{k_2} \leq \xi_1 + n_1 d_1.$$

Next, we can find d_2 such that $n_2(x_{k_2} + d_2) \leq \xi_1 + n_1 d_1$. We may also assume that d_2 is small enough so that $x \in [x_{k_2}, x_{k_2} + d_2]$ implies

$$|g(x)| \geq \frac{\epsilon_0}{2}.$$

Denoting by $\xi_2 = n_2 x_{k_2}$, we notice that $[\xi_2, \xi_2 + n_2 d_2] \subset [\xi_1, \xi_1 + n_1 d_1]$ and the closed intervals are non empty.
We continue by induction. Assume we found $n_i, k_i \in \mathbb{N}$, $n_i \geq i$, and $d_i > 0$ for $i = 1, \dots, j$ such that $\xi_i = n_i x_{k_i}$ satisfies

$$[\xi_i, \xi_i + n_i d_i] \subset [\xi_{i-1}, \xi_{i-1} + n_{i-1} d_{i-1}]$$

and for any $x \in [\xi_i, \xi_i + n_i d_i]$ we have that

$$\left|g\left(\frac{x}{n_i}\right)\right| \geq \frac{\epsilon_0}{2}.$$

We find $x_{k_{j+1}}$ such that $x_{k_{j+1}} < \frac{\xi_j}{j+1}$ and define $n_j = \left[\frac{\xi_j}{x_{k_{j+1}}}\right] + 1 \geq j+1$. As such, we have that

$$\xi_j \leq n_{j+1} x_{k_{j+1}} < \xi_j + x_{k_{j+1}} < \xi_j + n_j d_j,$$

where the last inequality is valid since we can pick $x_{k_{j+1}} < n_j d_j$. We can find d_{j+1} such that $n_{j+1}(x_{k_{j+1}} + d_{j+1}) < \xi_j + n_j d_j$ and for any $x \in [x_{k_{j+1}}, x_{k_{j+1}} + d_{j+1}]$

$$|g(x)| \geq \frac{\epsilon_0}{2}.$$

Denoting $\xi_{j+1} = n_{j+1} x_{k_{j+1}}$ gives us the interval with the desired properties.
Since we have a nested sequence of non-empty closed intervals in $\mathbb{R}$ we know that the intersection of all of them must be non-empty. Thus, there exists $x \in [\xi_i, \xi_i + n_i d_i]$ for all $i \in \mathbb{N}$. Moreover, by construction

$$\left| g\left(\frac{x}{n_i}\right) \right| \geq \frac{\epsilon_0}{2}$$

which contradicts the assumption that $\lim_{n\to\infty} g\left(\frac{x}{n}\right) = 0$ for any $x \neq 0$. □

The next result we will prove, is Lemma 2.31:

Proof of Lemma 2.31. Since $\widehat{g}$ is in the NDA of $\gamma_{\sigma,\alpha,\beta}$ we conclude that $\widehat{g}$ is actually in the FDA of $\gamma_{\sigma,\alpha,\beta}$, due to Theorem 2.24. Thus, there exists η_1, with $\frac{\eta_1(\xi)}{|\xi|^\alpha} \in L^\infty(\mathbb{R})$ and

$$\frac{\eta_1(\xi)}{|\xi|^\alpha} \underset{\xi\to 0}{\longrightarrow} 0,$$

such that

$$\begin{aligned}\widehat{g}(\xi) &= 1 - \sigma|\xi|^\alpha \left(1 + i\beta \operatorname{sgn}(\xi) \tan\left(\frac{\pi\alpha}{2}\right)\right) + \eta_1(\xi) \\ &= e^{-\sigma|\xi|^\alpha \left(1 + i\beta \operatorname{sgn}(\xi)\tan\left(\frac{\pi\alpha}{2}\right)\right)} + \eta_2(\xi) + \eta_1(\xi),\end{aligned}$$

where $\eta_2(\xi)$ has the same properties as $\eta_1(\xi)$. We conclude that

$$|\widehat{g}(\xi)| \leq e^{-\sigma|\xi|^\alpha} + |\eta_1(\xi) + \eta_2(\xi)| \leq 1 - \sigma|\xi|^\alpha + |\eta_1(\xi)| + |\eta_2(\xi)| + |\eta_3(\xi)|,$$

where $\eta_3(\xi)$ has the same properties as $\eta_1(\xi)$.
Let $\beta_0 > 0$ be such that if $|\xi| < \beta_0$

$$|\eta_1(\xi)| + |\eta_2(\xi)| + |\eta_3(\xi)| \leq \frac{\sigma|\xi|^\alpha}{2}.$$

For any $|\xi| < \beta_0$ one has that

$$|\widehat{g}(\xi)| \leq 1 - \frac{\sigma|\xi|^\alpha}{2} \leq e^{-\frac{\sigma|\xi|^\alpha}{2}},$$

completing the proof. □

2.B Quantitative Approximation Theorem.

An item of great importance in Kinetic Theory, and our problem in particular, is *quantitative* estimation of errors. Our local Lévy Central Limit Theorem involves such an estimation, yet it is dependent on the function

$$\omega(\beta) = \sup_{|\xi|} \frac{|\eta(\xi)|}{|\xi|^{\alpha}},$$

where η is the reminder function of a probability density function g in the NDA of some $\gamma_{\sigma,\alpha,\beta}$. In some cases one can find explicit estimation for the behaviour of η near zero, and get a better quantitative estimation on the error term $\epsilon(N)$. Such conditions are explored in [40] and we will satisfy ourselves by mentioning them, but providing no proof.

Definition 2.40. Let $\delta > 0$. The Fourier Domain of Attraction *of order* δ of $\gamma_{\sigma,\alpha,\beta}$ is the subset of the FDA of $\gamma_{\sigma,\alpha,\beta}$ such that the reminder function, η, satisfies

$$\frac{|\eta(\xi)|}{|\xi|^{\alpha}} \leq C|\xi|^{\delta},$$

for some $C > 0$.

Clearly the FDAs of order δ are nested sets, all contained in the FDA. Also, if g is in the FDA of order δ of $\gamma_{\sigma,\alpha,\beta}$ then we can replace $\omega(\beta)$, defined in Theorem 2.12 by $C\beta^{\delta}$ and get an explicit estimation to the error term $\epsilon(N)$!.
The following is a variant of a theorem appearing in [40] that gives sufficient conditions to be in the FDA of order δ of some $\gamma_{\sigma,\alpha,\beta}$:

Theorem 2.41. *Let g be a probability density on $\mathbb{R}$ that has zero mean. Let $1 < \alpha < 2$ and $0 < \delta < 2 - \alpha$ be given. Then if*

$$\int_{\mathbb{R}} |x|^{\alpha+\delta} |g(x) - \gamma_{\sigma,\alpha,\beta}(x)| dx < \infty \tag{2.115}$$

for some $\sigma > 0$ and $\beta \in [-1, 1]$, g is in the FDA of order δ of $\gamma_{\sigma,\alpha,\beta}$.

Chapitre 3

Propagation of chaos for the spatially homogeneous Landau equation with Maxwellian molecules

ABSTRACT. We prove a quantitative propagation of chaos, uniformly in time, for the spatially homogeneous Landau equation in the case of Maxwellian molecules. We improve the results of Fontbona, Guérin and Méléard [37] and Fournier [38] where the propagation of chaos is proved for finite time.

3.1 Introduction

An important open problem in kinetic theory is to derive Boltzmann equation from a many-particle system undergoing Newton's laws of dynamics. The correct scaling limit for this is the so-called Boltzmann-Grad or low density limit, see Grad [41]. The best result is Lanford [51] who proved the limit for short times (see also Illner and Pulvirenti [47] and Gallagher, Saint-Raymond and Texier [39]).

Kac [49] proposed to derive the spatially homogeneous Boltzmann equation from a many-particle Markov process, performing a *mean-field limit*. The program set by Kac in [49] was then to investigate the behavior of solutions of the mean-field equation in terms of the behaviour of the solutions of the *master equation*, i.e. the equation for the law of the many-particle process. We refer to Mischler and Mouhot [62] for a detailed introduction on Kac's program and for recent important results.

In the same way, we would like to derive rigorously another equation from kinetic theory, the Landau equation, from a many-particle system described by Newton's laws. It is an open problem, but the correct scalling to this is also known, the weak-coupling limit, and we refer to Bobylev, Pulvirenti and Saffirio [8] and the references therein for more information on this topic and partial results. We do not pursue this problem here.

Instead, in this work, we shall use the approach described above introduced by Kac [49]. Hence, we shall introduce a N-particle Markov process (see section 3.2.3) from which we derive the spatially homogeneous Landau equation in the mean-field limit. The N-particle process used here is obtained by means of the grazing collisions limit applied to the N-particle master equation for the Boltzmann model. We should mention that the N-particle master equation introduced here was, in fact, originally proposed by Balescu and Prigogine in the 1950's (see [50] and references therein); and it is also studied by Kiessling and Lancelloti [50] and Miot, Pulvirenti and Saffirio [58] (both in the Coulomb case).

Let us briefly explain how we can prove the *mean-field limit* with the approach proposed by Kac. Consider the probability density F_t^N associated to the Landau N-particle system and its evolution equation, i.e. the master equation (section 3.2.3). Integrating this equation over all variables but the first, we obtain an evolution equation for the first marginal $\Pi_1(F_t^N)$ that depends on the second marginal $\Pi_2(F_t^N)$. If the second marginal of the probability density was the 2-fold tensorization of a one-particle probability f_t, then f_t would satisfy the Landau equation (section 3.2.2). However, even if at initial time we start with an N-fold tensor probability $F^N(0) = f(0)^{\otimes N}$, this property can not be satisfied at later time because there are interactions between the particles. Kac suggested then that the chaos property (see definition below (3.1)), which is weaker than tensorization, could be propagated in time, which in turns would prove the mean-field limit.

3.1.1 Known results

Before giving our main results let us present known results concerning the propagation of chaos for the Landau equation for maxwellian molecules.

The work of Fontbona, Guérin and Méléard [37] consider nonlinear diffusion processess driven by a white noise that have an interpretation in terms of PDEs corresponding to the Landau equation. They construct an N-particle system that converges, in the limit $N \to \infty$ and in finite time, to the nonlinear process and, moreover, obtain quantitative convergence rates in Wasserstein distance W_2. Then Fournier [38], with the same probabilistic interpretation, improves the rate of convergence of [37].

We should mention that the Landau master equation introduce in this work (section 3.2.3) differs from the master equation associated to the N-particle process in [37, 38], see remark 3.2.

3.1.2 Main results

Consider a Polish space E, we shall denote by $\mathbf{P}(E)$ the space of probability measures on E. We denote by $\mathbf{P}_{\text{sym}}(E^N)$ the space of symmetric probabilities on E^N. We say that a symmetric probablity $F^N \in \mathbf{P}_{\text{sym}}(E^N)$ is f-chaotic (or Kac chaotic), for some probability $f \in \mathbf{P}(E)$, if for all $\ell \in \mathbb{N}^*$ we have

$$F_\ell^N \rightharpoonup f^{\otimes \ell} \quad \text{when} \quad N \to \infty, \tag{3.1}$$

where $F_\ell^N = \Pi_\ell(F^N)$ is the ℓ-th marginal of F^N and the convercence has to be understood in weak sense on $\mathbf{P}(E^\ell)$, i.e. the convergence of integral against continuous and bounded functions $\varphi \in C_b(E^\ell)$. In this paper we are interested in quantitave rates of convergence, more precisely we shall investigate estimates of the type, for any $\varphi \in \mathcal{F}^{\otimes \ell}$ with $\mathcal{F} \subset C_b(E)$ and $\|\varphi\|_{\mathcal{F}^{\otimes \ell}} \le 1$,

$$\left| \left\langle F_\ell^N - f^{\otimes \ell}, \varphi \right\rangle \right| \le C(\ell)\, \varepsilon(N),$$

with a constant $C(\ell)$ possibly depending on ℓ and a function $\varepsilon(N) \to 0$ when $N \to \infty$. Another possibility is to replace the left-hand side of the last equation by some distance on the space of probabilities, as for exemple the Wasserstein distance, $W_1(F_\ell^N, f^{\otimes \ell})$.

The many-particle process can be considered in $\mathbb{R}^{dN}$ and then its law F^N is a symmetric probality measure on $\mathbb{R}^{dN}$, however, thanks to the conservation laws, the process can be restricted to some submanifold of $\mathbb{R}^{dN}$. In our case, the dynamics of the many-particle process conserves momentum and energy (see section 3.2 for details), which implies that the process can be restricted to

$$\mathcal{S}^N(\mathcal{E}, \mathcal{M}) := \left\{ V = (v_1, \dots, v_N) \in \mathbb{R}^{dN} \; ; \; \frac{1}{N} \sum_{i=1}^N |v_i - \mathcal{M}|^2 = \mathcal{E}, \; \frac{1}{N} \sum_{i=1}^N v_i = \mathcal{M} \right\} \tag{3.2}$$

where $\mathcal{E} \ge 0$ and $\mathcal{M} \in \mathbb{R}^d$. We consider through the paper, without loss of generality, the case $\mathcal{M} = 0$, we denote $\mathcal{S}^N(\mathcal{E}) := \mathcal{S}^N(\mathcal{E}, 0)$ and call these submanifolds Boltzmann's spheres.

Initial data

Considering the process in $\mathcal{S}^N(\mathcal{E})$, we shall need an initial data $F_0^N \in \mathbf{P}_{\rm sym}(\mathcal{S}^N(\mathcal{E}))$ that is f_0-chaotic for some $f_0 \in \mathbf{P}(\mathbb{R}^d)$. This problem was studied in [19], where it is proven that for some (regular enough) probability measure $f \in \mathbf{P}(\mathbb{R}^d)$, with zero momentum $\mathcal{M} = \int v f = 0$ and energy $\mathcal{E} = \int |v|^2 f$, we can construct a propability measure $F^N \in \mathbf{P}_{\rm sym}(\mathcal{S}^N(\mathcal{E}))$ that is f-chaotic (and also entropically f-chaotic, see section 3.5 for the definition), by tensorization of f and restriction to the Boltzmann's sphere $\mathcal{S}^N(\mathcal{E})$. We shall denote this probability measure by

$$F^N = \left[f^{\otimes N} \right]_{\mathcal{S}^N(\mathcal{E})} := \frac{f^{\otimes N}}{\int_{\mathcal{S}^N(\mathcal{E})} f^{\otimes N} \, d\gamma^N} \, \gamma^N, \tag{3.3}$$

where γ^N is the uniform probability measure on $\mathcal{S}^N(\mathcal{E})$.

We can now state our main results in a simplified version.

Theorem 3.1. *Consider $f_0 \in \mathbf{P}(\mathbb{R}^d)$, with zero momentun and energy $\mathcal{E}$, and also $F_0^N = [f_0^{\otimes N}]_{\mathcal{S}^N(\mathcal{E})} \in \mathbf{P}_{\rm sym}(\mathcal{S}^N(\mathcal{E}))$. Let f_t be the solution of the Landau equation (see* (3.13)*) with initial data f_0 and F_t^N the solution of the Landau master equation (see* (3.32)*) with initial data F_0^N.*

(1) (Theorems 3.14 and 3.15) Then, for all $N \in \mathbb{N}^*$ we have

$$\sup_{t\geq 0} \frac{W_1\left(F_t^N, f_t^{\otimes N}\right)}{N} \leq \varepsilon_1(N),$$

where ε_1 is a polynomial function and $\varepsilon_1(N) \to 0$ as $N \to \infty$.

(2) (Theorem 3.31) Then for all $N \in \mathbb{N}^*$ we have

$$\sup_{t\geq 0} \left| \frac{1}{N} H(F_t^N|\gamma^N) - H(f_t|\gamma) \right| \leq \varepsilon_2(N),$$

where ε_2 is a polynomial function $\varepsilon_2(N) \to 0$ as $N \to \infty$, $H(f|\gamma)$ denotes the relative entropy of f_t with respect to γ, the centered Gaussian probability measure in $\mathbb{R}^d$ with energy $\mathcal{E}$, and $H(F_t^N|\gamma^N)$ denotes the relative entropy of F_t^N with respect to γ^N (see Section 3.5).

3.1.3 Strategy

The main idea is to use the consistency-stability method developped by Mischler, Mouhot and Wennberg in [62, 63]. Consider the semigroups associated to the evolution of the N-particle system and the limit mean-field equation, this method reduces the problem of propagation of chaos to proving consistency and stability estimates for these semigroups. First of all, we need to introduce the Landau master equation, which is derived by the asymptotics of grazing collisions from the Boltzmann master equation. Then, with the Landau master equation and the limit Landau equation at hand, we can investigate the estimates needed to apply the consistency-stability method and prove the propagation of chaos.

3.1.4 Organization of the paper

Section 3.2 is devoted to deduce a N-particle stochastic process modeling the Landau dynamics and to present the limit Landau equation. In Section 3.3 we present the consistency-satability method, with some adaptations, developped by Mischler, Mouhot and Wennberg in [62, 63]. In Section 3.4 we apply the method presented before to the Landau model in order to prove the propagation of chaos with quantitative rate and uniformly in time. Finally, in Section 3.5 we prove a quantitative propagation of entropic chaos.

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3.2 The Landau model

Our aim in this section is to present the N-particle system and the limit mean-field equation corresponding to Landau model. The limit Landau equation is well known and we shall present it in the Subsection 3.2.2. Furthermore, in Subsection 3.2.3 we deduce a master equation for the N-particle system corresponding to Landau.

Fisrt of all, we present the Boltzmann model, with its master equation and limit equation, which will be very useful in the sequel since Boltzmann and Landau equations are linked by the asymptotics of grazing collision that we shall explain in details later.

3.2.1 The Boltzmann model

We present here the Boltzmann model, with the limit mean field equation and the master equation. The spatially homogeneous Boltzmann equation [76, 62] is given by, for $f = f(t, v)$,

$$\partial_t f = Q(f, f) \tag{3.4}$$

with the collision operator given by

$$Q(g, f) = \int_{\mathbb{R}^d \times \mathbb{S}^{d-1}} B(|v - v_*|, \cos\theta)\Big(g(v'_*)f(v') - g(v_*)f(v)\Big)\, dv_*\, d\sigma, \tag{3.5}$$

and where the post-collisional velocities v' and v'_* are given by

$$v' = \frac{v + v_*}{2} + \frac{|v - v_*|}{2}\sigma, \qquad v'_* = \frac{v + v_*}{2} - \frac{|v - v_*|}{2}\sigma \tag{3.6}$$

and $\cos\theta = \sigma \cdot (v - v_*)/|v - v_*|$.

We assume that the collision kernel B satisfies $B(|z|, \cos\theta) = \Gamma(|z|)b(\cos\theta)$ (for more information on the collision kernel we refer to [76]) for some nonnegative functions Γ and b. When the interaction potential is proportional to r^{-s}, where r denotes the distance between particles, then we have

$$\Gamma(|z|) = |z|^\gamma, \qquad \sin^{d-2}\theta\, b(\cos\theta) \sim C_b\, \theta^{-1-\nu} \text{ when } \theta \sim 0,$$

with $\gamma = (s - 2d + 2)/s$, for some constant $C_b > 0$ and some fixed $\nu \in (0, 2)$. For example, in the 3-dimensional case we have $\nu = 2/s$.

In this work we are concerned with the case of *true Maxwellian molecules* $\gamma = 0$ (which corresponds to $s = 2d - 2$), we shall then consider through the paper the following assumption :

$$\begin{aligned} &B(|v - w|, \cos\theta) = b(\cos\theta), \\ &\int_{\mathbb{S}^{d-1}} b(\cos\theta)(1 - \cos\theta)^{\alpha + 1/4}\, d\sigma < +\infty, \quad \forall\, \alpha > 0. \end{aligned} \tag{3.7}$$

We remark that in this case we have $\int_{\mathbb{S}^{d-1}} b(\cos\theta)\, d\sigma = \infty$ but $\int_{\mathbb{S}^{d-1}} b(\cos\theta)(1 - \cos\theta)\, d\sigma < \infty$.

Another possible way to describe the pre and post-collisional velocities is the ω-representation

$$v' = v - (v - v_*, \omega)\omega, \quad v'_* = v_* + (v - v_*,)\omega, \quad \omega \in \mathbb{S}^{d-1}, \tag{3.8}$$

which gives us

$$Q_B(f,f) = \int_{\mathbb{R}^d\times\mathbb{S}^{d-1}} \widetilde{B}(|v-v_*|,\omega)\Big(f(v'_*)f(v') - f(v_*)f(v)\Big)\,dv_*\,d\omega,$$

with $\widetilde{B}(z,\omega) = |z|^\gamma b_\omega(\alpha)$ and α the angle formed by z and ω, and the following relation holds

$$\sigma = \frac{v-v_*}{|v-v_*|} - 2\left(\omega, \frac{v-v_*}{|v-v_*|}\right)\omega. \tag{3.9}$$

Let us now present the many-particle model [57, 49, 62, 12, 63]. Given a pre-collisional system of velocities $V = (v_1,\ldots,v_N) \in \mathbb{R}^{dN}$ and a collision kernel $B(|z|,\cos\theta) = \Gamma(|z|)b(\cos\theta)$, the process is: for any $i' \neq j'$, pick a random time $T(\Gamma(|v_{i'} - v_{j'}|))$ of collision accordingly to an exponential law of parameter $\Gamma(|v_{i'} - v_{j'}|)$ and choose the minimum time T_1 and the colliding pair (v_i, v_j); draw $\sigma \in \mathbb{S}^{d-1} \subset \mathbb{R}^d$ according to the law $b(\cos\theta_{ij})$, with $\cos\theta_{ij} = \sigma\cdot(v_i - v_j)/|v_i - v_j|$; after collision the new velocities become $V'_{ij} = (v_1,\ldots,v'_i,\ldots,v'_j,\ldots,v_N)$ with

$$v'_i = \frac{v_i+v_j}{2} + \frac{|v_i-v_j|}{2}\sigma, \qquad v'_j = \frac{v_i+v_j}{2} - \frac{|v_i-v_j|}{2}\sigma. \tag{3.10}$$

Iterating this construction we built then the associated Markov process $(\mathcal{V}_t)_{t\geq 0}$ on $\mathbb{R}^{dN}$. As explained in the introduction, we can also consider this process on $\mathcal{S}^N(\mathcal{E})$. Then, after a rescaling of time, the master equation is given in dual form by [62, 63],

$$\partial_t\langle F^N_t, \varphi\rangle = \langle F^N_t, G^N_B\varphi\rangle \tag{3.11}$$

where

$$G^N_B\varphi = \frac{1}{2N}\sum_{i,j=1}^N \Gamma(|v_i - v_j|)\int_{\mathbb{S}^{d-1}} b(\cos\theta_{ij})\left(\varphi'_{ij} - \varphi\right)\,d\sigma \tag{3.12}$$

with the shorthand notation $\varphi'_{ij} = \varphi(V'_{ij})$ and $\varphi = \varphi(V) \in C_b(\mathbb{R}^{dN})$. We shall consider the case of Maxwellian molecules, i.e. $\Gamma(|z|) = 1$ and $b(\cos\theta)$ satisfying (3.7).

3.2.2 Limit equation

We present here the limit spatially homogeneous Landau equation for maxwellian molecules and some of its basic properties, for more information we refer to [76, 74, 73].

The Landau equation is a kinetic model in plasma physics that describes the evolution of the density f of a gas in the phase space of all positions and velocities of particles. Assuming that the density function does not depend on the position, we obtain the *spatially homogeneous Landau equation* in the form

$$\partial_t f = Q_L(f,f) \tag{3.13}$$

where $f = f(t, v)$ is the density of particles with velocity v at time t, $v \in \mathbb{R}^d$ and $t \in \mathbb{R}^+$. The Landau operator is given by

$$Q_L(g, f) = \partial_i \left\{ \int_{\mathbb{R}^d} a_{ij}(v - v_*) \left(g(v_*)\partial_j f(v) - \partial_j g(v_*) f(v) \right) \, dv_* \right\}, \tag{3.14}$$

where here and below we shall use the convention of implicit summation over indices. Moreover, the matrix a is nonnegative, symmetric and depends on the interaction between particles. If two particles interact with a potential proportional to $1/r^s$, where r denotes their distance, then we have

$$a_{ij}(z) = \Lambda |z|^{\gamma+2} \Pi_{ij}(z), \qquad \Pi_{ij}(z) = \delta_{ij} - \frac{z_i z_j}{|z|^2},$$

with $\gamma = (s - 2d + 2)/s$ and some constant $\Lambda > 0$. As for the Boltzmann equation, we consider the case of Maxwellian molecules $\gamma = 0$, i.e.

$$a_{ij}(z) = \Lambda |z|^2 \Pi_{ij}(z). \tag{3.15}$$

We also define

$$b_i(z) = \partial_j a_{ij} = -\Lambda(d-1) z_i, \qquad c(z) = \partial_{ij} a_{ij} = -3\Lambda(d-1), \tag{3.16}$$

and we denote

$$\bar{a}_{ij} = a_{ij} * f, \; \bar{b}_i = b_i * f, \; \bar{c} = c * f.$$

Hence, we can write the Landau equation in another form

$$\partial_t f = \bar{a}_{ij} \partial_{ij} f - \bar{c} f. \tag{3.17}$$

Moreover, let $\varphi(v)$ be a test function, then we have the following weak forms

$$\int Q_L(f, f)\varphi = -\frac{1}{2} \int dv \, dv_* \, f f_* \, a(v - v_*) \left(\frac{\nabla f}{f} - \frac{\nabla_* f_*}{f_*} \right) (\nabla \varphi - \nabla_* \varphi_*) \tag{3.18}$$

or

$$\begin{aligned} \int Q_L(f, f)\varphi =& \frac{1}{2} \int dv \, dv_* \, f f_* \, a_{ij}(v - v_*)(\partial_{ij}\varphi + (\partial_{ij}\varphi)_*) \\ &+ \int dv \, dv_* \, f f_* \, b_i(v - v_*)(\partial_i \varphi - (\partial_i \varphi)_*). \end{aligned} \tag{3.19}$$

This equation satisfies the conservation of mass, momentum and energy. Moreover, the entropy $H(f) = \int f \log f$ is nonincreasing, indeed taking $\varphi = \log f$ we obtain

$$\frac{d}{dt} H(f) = -\frac{1}{2} \int f f_* \, a(v - v_*) \left(\frac{\nabla f}{f} - \frac{\nabla_* f_*}{f_*} \right) \cdot \left(\frac{\nabla f}{f} - \frac{\nabla_* f_*}{f_*} \right) dv \, dv_* \leq 0, \tag{3.20}$$

since a is nonnegative, which is the Landau version of the H-theorem of Boltzmann. For more information we refer to [74].

The Landau equation was introduce by Landau in 1936. Later it was proven that the Landau equation can be obtained as a limit of the Boltzmann equation when grazing collisions prevail (see [26, 1, 25, 73] and the references therein for more details).

3.2.3 Master equation

We derive a master equation for the Landau model. It is based on [73] where they use the grazing collisions limit to pass from Boltzmann to Landau limit equations (see also [26, 1, 25]). Since we know the master equation for the Boltzmann model (3.11), we shall perform the grazing collisions limit to obtain a master equation for Landau model. As explained in the introduction, the master equation we derive here (see (3.32)) is the same introduced by Balescu and Prigorine in the 1950's, and it is also studied in the works [50, 58].

Grazing collisions

We present here the grazing collision limit as in [73]. Consider the true Maxwellian molecules collision kernel b satisfying (3.7). We say that b_ε concentrates in grazing collision if:

$$\begin{cases} \forall\, \theta_0 > 0, \qquad \sup_{\theta > \theta_0} b_\varepsilon(\cos\theta) \xrightarrow[\varepsilon\to 0]{} 0 \\ \Lambda_\varepsilon = \displaystyle\int_{\mathbb{S}^{d-1}} b_\varepsilon(\cos\theta)(1-\cos\theta)\, d\sigma \xrightarrow[\varepsilon\to 0]{} \Lambda > 0. \end{cases} \tag{3.21}$$

For the sake of simplicity, to derive the Landau master equation in this section, we suppose the dimension $d = 3$ to perfom the computations, the other cases being the same.

From (3.9), using a spherical coordinate system (in dimension $d = 3$) with axis $v - v_*$, we have

$$\sigma = \frac{v - v_*}{|v - v_*|} \cos\theta + (\cos\phi\, \vec{h} + \sin\phi\, \vec{i}) \sin\theta.$$

Moreover we have $|(v - v_*, \omega)| = |v - v_*| \sin(\theta/2)$. Finally we can write the operator in the following way (see [73])

$$Q_B(f,f) = \int_0^{2\pi} d\phi \int_0^{\pi/2} d\theta \int_{\mathbb{R}^d} dv_*\, \zeta(\theta)(f' f'_* - f f_*), \tag{3.22}$$

with $\zeta(\theta) = \sin^{d-2}\theta\, b(\cos\theta)$. In this case, we can rewrite (3.21) and say that ζ_ε concentrates in grazing collisions if for all $\theta_0 \geq 0$

$$\begin{cases} \sup_{\theta \geq \theta_0} \zeta_\varepsilon(\theta) \to 0 \quad \text{when} \quad \varepsilon \to 0 \\ \Lambda_\varepsilon := \dfrac{\pi}{2} \displaystyle\int_0^{\pi/2} \sin^2\frac{\theta}{2}\, \zeta_\varepsilon(\theta)\, d\theta \to \Lambda < \infty \quad \text{when} \quad \varepsilon \to 0. \end{cases} \tag{3.23}$$

Let us consider then the Boltzmann master equation (3.11)-(3.12), using the form of (3.22), that is

$$(G_B^N \varphi)(V) = \frac{1}{2N} \sum_{i,j=1}^N \int_0^{2\pi} d\phi \int_0^{\pi/2} d\theta\, \zeta(\theta)(\varphi'_{ij} - \varphi).$$

In this equation, we take a second order Taylor expansion of φ'_{ij} and obtain

$$\begin{aligned}\varphi(V'_{ij}) - \varphi(V) =& D\varphi[V](V'_{ij} - V) + \frac{1}{2}(V'_{ij} - V)^T D^2\varphi[V](V'_{ij} - V) \\ &+ O(|V'_{ij} - V|^3).\end{aligned} \tag{3.24}$$

With the incoming and outgoing velocities V and V'_{ij} (see (3.10)), we have

$$V'_{ij} - V = (0, \dots, 0, v'_i - v_i, 0, \dots, 0, v'_j - v_j, 0, \dots, 0).$$

In (3.24), $D\varphi[V]$ and $D^2\varphi[V]$ are given by

$$D\varphi[V] = (\nabla_i \varphi(V))_{1\le i\le N}\,, \quad D^2\varphi[V] = \left(\nabla^2_{ij}\varphi(V)\right)_{1\le i,j\le N}$$

where $\nabla_i\varphi = \left(\partial_{v_{i,\alpha}}\varphi\right)_{1\le\alpha\le 3}$ and $\nabla^2_{ij}\varphi = \left(\partial_{v_{i,\alpha}}\partial_{v_{j,\beta}}\varphi\right)_{1\le\alpha,\beta\le 3}$. Now we substitute $V'_{ij} - V$ in (3.24) and we get

$$\begin{aligned}\varphi(V'_{ij}) - \varphi(V) =& \nabla_i\varphi(V)(v'_i - v_i) + \nabla_j\varphi(V)(v'_j - v_j) \\ &+ \frac{1}{2}\Big\{\nabla^2_{ii}\varphi(V)(v'_i - v_i)^2 + \nabla^2_{jj}\varphi(V)(v'_j - v_j)^2 \\ &+ \nabla^2_{ij}\varphi(V)(v'_i - v_i)(v'_j - v_j) + \nabla^2_{ji}\varphi(V)(v'_i - v_i)(v'_j - v_j)\Big\} \\ &+ O(|V'_{ij} - V|^3).\end{aligned} \tag{3.25}$$

Finally, using (3.10) and (3.8) with v_i and v_j, one obtains

$$\begin{aligned}\varphi(V'_{ij}) - \varphi(V) =& -(v_i - v_j, \omega)(\nabla_i\varphi - \nabla_j\varphi, \omega) && (= T_1) \\ &+ \frac{1}{2}(v_i - v_j, \omega)^2\left\{\nabla^2_{ii}\varphi + \nabla^2_{jj}\varphi - \nabla^2_{ij}\varphi - \nabla^2_{ji}\varphi\right\}(\omega, \omega) && (= T_2) \\ &+ O\left(|v_i - v_j|^3 \sin^3\frac{\theta}{2}\right).\end{aligned} \tag{3.26}$$

For each pair of particules i and j, in the orthonormal basis $\{\frac{v_i - v_j}{|v_i - v_j|}, \vec{h}, \vec{i}\}$, one has

$$\omega = \frac{v_i - v_j}{|v_i - v_j|}\sin\frac{\theta}{2} + (\cos\phi\,\vec{h} + \sin\phi\,\vec{i})\cos\frac{\theta}{2}$$

and then, using the fact that linear combinations of $\cos\phi$ and $\sin\phi$ vanish when integrated over ϕ, we can compute the contribution of T_1 integrated over ϕ

$$-\int_0^{2\pi} d\phi(v_i - v_j, \omega)(\nabla_i\varphi - \nabla_j\varphi, \omega) = -2\pi\sin^2\frac{\theta}{2}\,(v_i - v_j, \nabla_i\varphi - \nabla_j\varphi)\,.$$

Now we have to compute the integral of T_2 over ϕ, we denote

$$\lambda_{\alpha\beta} = \{\partial_{v_{i,\alpha}}\partial_{v_{i,\beta}}\varphi + \partial_{v_{j,\alpha}}\partial_{v_{j,\beta}}\varphi - \partial_{v_{i,\alpha}}\partial_{v_{j,\beta}}\varphi - \partial_{v_{j,\alpha}}\partial_{v_{i,\beta}}\varphi\}/2$$

and in the same orthonormal basis, we compute

$$A = |v_i - v_j|^2 \sin^2 \frac{\theta}{2} \int_0^{2\pi} d\phi \lambda_{\alpha\beta}\omega_\alpha\omega_\beta.$$

Again, linear combinations of $\cos\phi$ and $\sin\phi$ vanish, which implies

$$\begin{aligned} A &= |v_i - v_j|^2 \sin^2 \frac{\theta}{2} \int_0^{2\pi} d\phi \left(\lambda_{11} \sin^2 \frac{\theta}{2} + \lambda_{22} \cos^2 \phi \cos^2 \frac{\theta}{2} + \lambda_{33} \sin^2 \phi \cos^2 \frac{\theta}{2} \right) \\ &= 2\pi |v_i - v_j|^2 \left(\lambda_{11} \sin^4 \frac{\theta}{2} + \frac{\lambda_{22}}{2} \cos^2 \frac{\theta}{2} \sin^2 \frac{\theta}{2} + \frac{\lambda_{33}}{2} \cos^2 \frac{\theta}{2} \sin^2 \frac{\theta}{2} \right) \end{aligned} \tag{3.27}$$

and we remark that the first coefficient is of order greater than 2 in θ.

We introduce $\Pi_{\alpha\beta}(v_i - v_j)$ the projection over the orthogonal space of $\frac{v_i - v_j}{|v_i - v_j|}$, and the dominat term of (3.27) when $\theta \to 0$ is

$$\pi |v_i - v_j|^2 \sin^2 \frac{\theta}{2} \Pi_{\alpha\beta}(v_i - v_j) \lambda_{\alpha\beta}$$

or in matricial notation

$$\pi |v_i - v_j|^2 \sin^2 \frac{\theta}{2} \Pi(v_i - v_j) : \frac{\left(\nabla^2_{ii}\varphi + \nabla^2_{jj}\varphi - \nabla^2_{ij}\varphi - \nabla^2_{ji}\varphi \right)}{2}.$$

Finally, we obtain

$$\begin{aligned} \frac{1}{2} \int_0^{2\pi} d\phi (\varphi(V'_{ij}) - \varphi(V)) &= -\frac{\pi}{2} \sin^2 \frac{\theta}{2} \left(2(v_i - v_j), \nabla_i\varphi - \nabla_j\varphi\right) \\ + \frac{\pi}{4} |v_i - v_j|^2 \sin^2 \frac{\theta}{2} &\Pi(v_i - v_j) : \left(\nabla^2_{ii}\varphi + \nabla^2_{jj}\varphi - \nabla^2_{ij}\varphi - \nabla^2_{ji}\varphi \right) \\ &+ O\left(|v_i - v_j|^2 \theta^4 \wedge 1 \right) + O\left(|v_i - v_j|^3 \theta^3 \wedge 1 \right). \end{aligned} \tag{3.28}$$

Consider now the Boltzmann master equation with kernel ζ_ε satisfying the grazing collisions (3.23) and plug (3.28) in it, we obtain then

$$\begin{aligned} G^N_B\varphi &= \frac{1}{N} \sum_{i,j=1}^N \int_0^{\pi/2} d\theta\, \zeta_\varepsilon(\theta) \frac{1}{2} \int_0^{2\pi} d\phi (\varphi'_{ij} - \varphi) \\ &= \frac{1}{N} \sum_{i,j=1}^N \int_0^{\pi/2} d\theta\, \zeta_\varepsilon(\theta) \left\{ -\frac{\pi}{2} \sin^2 \frac{\theta}{2} \left(2(v_i - v_j), \nabla_i\varphi - \nabla_j\varphi\right) \right. \\ &+ \frac{\pi}{4} |v_i - v_j|^2 \sin^2 \frac{\theta}{2} \Pi(v_i - v_j) : \left(\nabla^2_{ii}\varphi + \nabla^2_{jj}\varphi - \nabla^2_{ij}\varphi - \nabla^2_{ji}\varphi \right) \\ &\left. + O\left(|v_i - v_j|^2 \theta^4 \wedge 1 \right) + O\left(|v_i - v_j|^3 \theta^3 \wedge 1 \right) \right\}. \end{aligned} \tag{3.29}$$

This can be written in the following way

$$
\begin{aligned}
G_B^N\varphi &= \frac{1}{N}\sum_{i,j=1}^N \frac{\pi}{2}\int_0^{\pi/2} d\theta\, \sin^2\frac{\theta}{2}\,\zeta_\varepsilon(\theta)\,(-2|v_i-v_j|^2\frac{(v_i-v_j)}{|v_i-v_j|^2})\cdot(\nabla_i\varphi-\nabla_j\varphi)\\
&+\frac{1}{2N}\sum_{i,j=1}^N \frac{\pi}{2}\int_0^{\pi/2} d\theta\, \sin^2\frac{\theta}{2}\,\zeta_\varepsilon(\theta)\,|v_i-v_j|^2\Pi(v_i-v_j):\left(\nabla^2_{ii}\varphi+\nabla^2_{jj}\varphi-\nabla^2_{ij}\varphi-\nabla^2_{ji}\varphi\right)\\
&+\frac{1}{N}\sum_{i,j=1}^N \int_0^{\pi/2} d\theta\,\zeta_\varepsilon(\theta)\left(O\left(|v_i-v_j|^2\theta^4\wedge 1\right)+O\left(|v_i-v_j|^3\theta^3\wedge 1\right)\right).
\end{aligned}
\tag{3.30}
$$

As in [73], the last term converges to 0 when $\varepsilon\to 0$. Then we have, using (3.23) and the definition of the functions a (3.15) and b (3.16), when $\varepsilon\to 0$

$$
\begin{aligned}
G_B^N\varphi \longrightarrow &\frac{1}{N}\sum_{i,j=1}^N -2\Lambda|v_i-v_j|^2\frac{(v_i-v_j)}{|v_i-v_j|^2}\cdot(\nabla_i\varphi-\nabla_j\varphi)\\
&+\frac{1}{2N}\sum_{i,j=1}^N \Lambda|v_i-v_j|^2\Pi(v_i-v_j):\left(\nabla^2_{ii}\varphi+\nabla^2_{jj}\varphi-\nabla^2_{ij}\varphi-\nabla^2_{ji}\varphi\right)\\
&=\frac{1}{N}\sum_{i,j=1}^N b(v_i-v_j)\cdot(\nabla_i\varphi-\nabla_j\varphi)\\
&+\frac{1}{2N}\sum_{i,j=1}^N a(v_i-v_j):\left(\nabla^2_{ii}\varphi+\nabla^2_{jj}\varphi-\nabla^2_{ij}\varphi-\nabla^2_{ji}\varphi\right)=:G_L^N\varphi
\end{aligned}
\tag{3.31}
$$

and that defines the Landau generator G_L^N. Finally, we derive the following Landau master equation

$$
\begin{aligned}
\partial_t\langle f_t^N,\varphi\rangle=\left\langle f_t^N, G_L^N\varphi\right\rangle &= \frac{1}{N}\int\sum_{i,j=1}^N b(v_i-v_j)\cdot(\nabla_i\varphi-\nabla_j\varphi)f_t^N(dV)\\
&+\frac{1}{2N}\int\sum_{i,j=1}^N a(v_i-v_j):(\nabla^2_{ii}\varphi+\nabla^2_{jj}\varphi-\nabla^2_{ij}\varphi-\nabla^2_{ji}\varphi)f_t^N(dV).
\end{aligned}
\tag{3.32}
$$

Remark 3.2. In the paper [37], the Landau equation is studied with a probabilistic approach. In particular they prove that the following process associated to a N-particle system, for $i=1,...,N$,

$$
dX_t^i=\frac{\sqrt{2}}{\sqrt{N}}\sum_{k=1}^N\sigma(X_t^i-X_t^k)\,dB_t^{i,k}+\frac{2}{N}\sum_{k=1}^N b(X_t^i-X_t^k)\,dt, \tag{3.33}
$$

where $B^{i,k}$ are N^2 independant $\mathbb{R}^d$-valued Brownian motions, converges to the process

$$
X_t=X_0+\sqrt{2}\int_0^t\int_{\mathbb{R}^d}\sigma(X_s-y)W(dy,ds)+2\int_0^t\int_{\mathbb{R}^d}b(X_s-y)P_s(dy)ds \tag{3.34}
$$

where P_t is the law of V_t and W is a white noise in space-time. Moreover the process (3.34) is associated to the spatially homogeneous Landau equation with the coefficients $a_{\alpha\beta}(z) := (\sigma\sigma^*)_{\alpha\beta}(z)$ and $b_\alpha(z) := \partial_\beta a_{\alpha\beta}(z)$. Then, the Kolmogorov equation of (3.33) is, for a test function $\varphi : \mathbb{R}^{dN} \longrightarrow \mathbb{R}$ and where f_t^N represents the law of X_t,

$$\begin{aligned}\partial_t\langle f_t^N, \varphi\rangle &= \langle f_t^N, G_2^N \varphi\rangle \\ &= \frac{1}{N}\sum_{i,j=1}^N \int_{\mathbb{R}^{dN}} b(v_i - v_j)\cdot(\nabla_i\varphi(V) - \nabla_j\varphi(V))\, f_t^N(dV) \\ &+ \frac{1}{2N}\sum_{i,j=1}^N \int_{\mathbb{R}^{dN}} a(v_i - v_j) : \left(\nabla_{ii}^2\varphi(V) + \nabla_{jj}^2\varphi(V)\right) f_t^N(dV).\end{aligned} \tag{3.35}$$

In [38], instead of the process (3.34) it was used a similar process with only N independent Brownian motion, i.e. replacing $B^{i,k}$ by B^i in (3.34), and it gives the same master equation (3.35). We remark that this equation differs from (3.32) by the terms $\sum_{i,j} a(v_i - v_j) : \left(-\nabla_{ij}^2\varphi - \nabla_{ji}^2\varphi\right)$.

Now, in order to obtain a N-particle SDE associated to (3.31)-(3.32), we shall modify (3.34). Consider then, for $i = 1, \dots, N$, $\mathbb{R}^d$-valued random variables $(X_t^i)_{t\geq 0}$ satisfying the following equation

$$\forall\, i = 1,\dots,N \qquad dX_t^i = \frac{\sqrt{2}}{\sqrt{N}}\sum_{\substack{k=1\\k\neq i}}^N \sigma(X_t^i - X_t^k)\, dZ_t^{i,k} + \frac{2}{N}\sum_{\substack{k=1\\k\neq i}}^N b(X_t^i - X_t^k)\, dt \tag{3.36}$$

where, for all $1 \leq i \leq N$ and $i < k$, $Z_t^{i,k} = B_t^{i,k}$ are $N(N-1)/2$ independent $\mathbb{R}^d$-valued Brownian motions and the other terms are anti-symmetric $Z_t^{k,i} = -B_t^{i,k}$. As in (3.34), we have $a(z) = \sigma(z)\sigma^*(z)$ and σ is symmetric (recall that a also is), i.e. $\sigma(-z) = \sigma(z)$. We can rewrite (3.36), let $X = (X^1, \dots, X^N)$ we have the following equation

$$dX_t = \mu(X_t)\, dt + \Sigma(X_t)\, dW_t \tag{3.37}$$

where $\mu = (\mu_1, \dots, \mu_N)$ is given by

$$\forall\, i = 1,\dots,N, \forall\, X = (X^1,\dots,X^N) \in \mathbb{R}^{dN}, \qquad \mu_i(X) = \frac{2}{N}\sum_{k=1}^N b(X^i - X^k),$$

moreover, W_t is constitued of the $N(N-1)/2$ independent Brownian motion, more precisely

$$W_t = (W_t^1, \dots, W_t^{N(N-1)/2}) = (B_t^{1,2}, \dots, B_t^{1,N}, B_t^{2,3}, \dots, B_t^{2,N}, \dots, B_t^{(N-1),N}),$$

i.e. for $i < k$, $B_t^{i,k} = W_t^\alpha$ with $\alpha = (i-1)(N - i/2) + k - i$. Finally, $\Sigma : \mathbb{R}^{dN} \to \mathcal{M}_{N,N(N-1)/2}(\mathcal{M}_{d,d}(\mathbb{R}^d))$ is given by, for all $1 \leq i \leq N$, $1 \leq \beta \leq N(N-1)/2$ and

$X = (X^1, \dots, X^N) \in \mathbb{R}^{dN}$,

$$(\Sigma(X))_{i\beta} = \begin{cases} -\frac{\sqrt{2}}{\sqrt{N}}\,\sigma(X^i - X^k); & \beta = Nk - k\frac{(k+1)}{2},\ 1 \le k \le i-1, \\ \frac{\sqrt{2}}{\sqrt{N}}\,\sigma(X^i - X^k); & (i-1)\left(N - \frac{i}{2}\right) + 1 \le \beta \le i\left(N - \frac{i+1}{2}\right),\ k = \beta - (i-1)\left(N - \frac{i}{2}\right) + i, \\ 0; & \text{otherwise.} \end{cases}$$

Indeed, multipliying the ith-row of $\Sigma(X)$ by dW_t we obtain

$$-\frac{\sqrt{2}}{\sqrt{N}}\sum_{k=1}^{i-1}\sigma(X^i - X^k)\,dB_t^{i,k} + \frac{\sqrt{2}}{\sqrt{N}}\sum_{k=i+1}^{N}\sigma(X^i - X^k)\,dB_t^{i,k} = \frac{\sqrt{2}}{\sqrt{N}}\sum_{\substack{k=1\\k\neq i}}^{N}\sigma(X^i - X^k)\,dZ_t^{i,k},$$

which corresponds to the first term on the right-hand side of (3.36).

Consider a test function $\phi : \mathbb{R}^{dN} \to \mathbb{R}$ and f_t^N the law of X_t, then the Kolmogorov equation of (3.37) is

$$\begin{aligned} \partial_t \left\langle f_t^N, \phi \right\rangle &= \frac{2}{N}\sum_{i,j=1}^{N}\int_{\mathbb{R}^{dN}} b(v_i - v_j)\cdot(\nabla_i\phi)\, f_t^N(dV) \\ &\quad + \frac{1}{N}\sum_{i,j=1}^{N}\int_{\mathbb{R}^{dN}} a(v_i - v_j) : \left(\nabla_{ii}^2\phi - \nabla_{ij}^2\phi\right)\, f_t^N(dV), \end{aligned}$$

and using the symmetry property of a and anti-symmetry of b we obtain (3.32).

3.3 The consistency-stability method for the Landau equation

In this section we present the method develloped in [62, 63] with some modifications, in order to be able to apply it later to the Landau equation in section 3.4.

3.3.1 Abstract framework

Consider a polish space E and we shall denote by $\mathbf{P}(E)$ the space of probability measures on E. Consider also E^N and the space of symmetric probability measures $\mathbf{P}_{\text{sym}}(E^N)$, more precisely, we say that $F^N \in \mathbf{P}(E^N)$ is symmetric if for all $\varphi \in C_b(E^N)$ we have that

$$\int_{E^N} \varphi\, dF^N = \int_{E^N} \varphi_\sigma\, dF^N,$$

for any permutation σ of $\{1, \dots, N\}$, and where

$$\varphi_\sigma := \varphi(V_\sigma) = \varphi(v_{\sigma(1)}, \dots, v_{\sigma(N)}),$$

for $V = (v_1, \dots, v_N) \in E^N$.

We then consider a N-particle system with initial probability density $F_0^N \in \mathbf{P}_{\rm sym}(E^N)$ and its evolution equation in dual form, for all $\varphi \in C_b(E^N)$,

$$\partial_t \left\langle F_t^N, \varphi \right\rangle = \left\langle F_t^N, G^N \varphi \right\rangle. \tag{3.38}$$

It generates a linear semigroup denoted by $S_t^N : F_0^N \mapsto F_t^N$. Moreover, we define the dual semigroup T_t^N by

$$\forall\, \phi \in C_b(E^N),\ \forall\, F^N \in \mathbf{P}_{\rm sym}(E^N), \qquad \langle T_t^N(\phi), F^N\rangle = \langle \phi, S_t^N(F^N)\rangle \tag{3.39}$$

and its generator G^N by.

$$\forall\, \phi \in C_b(E^N), \qquad \partial_t \phi = G^N \phi. \tag{3.40}$$

At the level of the limit (mean field) equation, we consider a initial probability density $f_0 \in \mathbf{P}(E)$ and the equation

$$\partial_t f_t = Q(f_t). \tag{3.41}$$

We also define its semigroup $S_t^\infty : f_0 \mapsto f_t$. Then, we define the pullback semigroup T_t^∞ by

$$\forall\, \Phi \in C_b(\mathbf{P}(E)),\ \forall\, f \in \mathbf{P}(E), \qquad T_t^N[\Phi](f) := \Phi\left(S_t^\infty(f)\right) \tag{3.42}$$

and its generator G^∞ by

$$\forall\, \Phi \in C_b(\mathbf{P}(E)), \qquad \partial_t \Phi = G^\infty \Phi. \tag{3.43}$$

We define some applications relating this objetcs. The function $\pi_E^N : E^N/\mathfrak{S}_N \to \mathbf{P}(E)$ is defined by, where $\mathfrak{S}_N$ denotes the group of permutations of $\{1, \ldots, N\}$,

$$\pi_E^N(V) := \mu_V^N = \frac{1}{N} \sum_{i=1}^N \delta_{v_i} \tag{3.44}$$

and μ_V^N is called the empirical measure associated to V. The application $\pi_C^N : C_b(\mathbf{P}(E)) \to C_b(E^N)$ is given by

$$\forall\, V \in E^N,\ \forall\, \Phi \in C_b(\mathbf{P}(E)), \qquad \pi_C^N[\Phi](V) := \Phi(\mu_V^N). \tag{3.45}$$

The application $\pi_P^N : \mathbf{P}_{\rm sym}(E^N) \to \mathbf{P}(\mathbf{P}(E))$ is

$$\begin{aligned} &\forall\, F^N \in \mathbf{P}_{\rm sym}(E^N),\ \forall\, \Phi \in C_b(\mathbf{P}(E)), \\ &\left\langle \pi_P^N(F^N), \Phi \right\rangle := \left\langle F^N, \pi_C^N(\Phi) \right\rangle = \int_{E^N} \Phi(\mu_V^N) F^N(dV), \end{aligned} \tag{3.46}$$

where the first bracket is the duality $\mathbf{P}(\mathbf{P}(E)) \leftrightarrow C_b(\mathbf{P}(E))$ and the second one is the duality $\mathbf{P}_{\rm sym}(E^N) \leftrightarrow C_b(E^N)$. Finally, the application $R^N : C_b(E^N) \to C_b(\mathbf{P}(E))$ is defined by

$$\begin{aligned} &\forall\, \varphi \in C_b(E^N),\ \forall\, f \in \mathbf{P}(E), \\ &R^N[\varphi](f) := \left\langle \varphi, f^{\otimes N} \right\rangle = \int_{E^N} \varphi(V) f(dv_1) \cdots f(dv_N), \end{aligned} \tag{3.47}$$

in the sequel we will denote $R^\ell_\varphi := R^\ell[\varphi]$ for $\varphi \in C_b(E^\ell)$. The functions R^ℓ_φ are the "polynomials" on the space $\mathbf{P}(E)$, we will see later (exemple 3.8) that they are continuous in the sense of Definitions 3.5 and 3.6, where we develop a differential calculus on $\mathbf{P}(E)$.

For a given weight function $m : E \to \mathbb{R}_+$ we define the N-particle weight function

$$\forall\, V = (v_1, ..., v_N) \in E^N, \qquad M^N_m(V) := \frac{1}{N}\sum_{i=1}^N m(v_i) = \langle \mu^N_V, m\rangle = M_m(\mu^N_V). \quad (3.48)$$

Definition 3.3. For a given weight function $m_{\mathcal{G}} : E \to \mathbb{R}_+$ we define the subspaces of probabilities

$$\mathbf{P}_{\mathcal{G}} := \{f \in \mathbf{P}(E);\ \langle f, m_{\mathcal{G}}\rangle < \infty\}$$

and the corresponding bounded sets, for $a \in (0,\infty)$,

$$\mathcal{B}\mathbf{P}_{\mathcal{G},a} := \{f \in \mathbf{P}_{\mathcal{G}};\ \langle f, m_{\mathcal{G}}\rangle \le a\}\,.$$

For a given constraint function $\mathbf{m}_{\mathcal{G}} : E \to \mathbb{R}^D$ such that $\langle f, \mathbf{m}_{\mathcal{G}}\rangle$ is well defined for any $f \in P_{\mathcal{G}}$ and a given space of constraints $\mathbf{R}_{\mathcal{G}} \subset \mathbb{R}^D$, we define, for any $\mathbf{r} \in \mathbf{R}_{\mathcal{G}}$, the constrained subsets

$$\mathbf{P}_{\mathcal{G},\mathbf{r}} := \{f \in \mathbf{P}_{\mathcal{G}};\ \langle f, \mathbf{m}_{\mathcal{G}}\rangle = \mathbf{r}\},$$

and the corresponding bounded constrained subsets

$$\mathcal{B}\mathbf{P}_{\mathcal{G},a,\mathbf{r}} := \{f \in \mathcal{B}\mathbf{P}_{\mathcal{G},a}\ ;\ \langle f, \mathbf{m}_{\mathcal{G}}\rangle = \mathbf{r}\}.$$

and the corresponding space of increments

$$\mathcal{I}\mathbf{P}_{\mathcal{G}} := \{g - f;\ \exists\, \mathbf{r} \in \mathbf{R}_{\mathcal{G}} \text{ s.t. } g, f \in \mathbf{P}_{\mathcal{G},\mathbf{r}}\}.$$

We shall consider a distance $\mathrm{dist}_{\mathcal{G}}$ defined on the whole space $\mathbf{P}_{\mathcal{G}}$ or such that there is a Banach space $\mathcal{G} \supset \mathcal{I}\mathbf{P}_{\mathcal{G}}$ endowed with a norm $\|\cdot\|_{\mathcal{G}}$ such that $\mathrm{dist}_{\mathcal{G}}$ is defined for any $\mathbf{r} \in \mathbf{R}_{\mathcal{G}}$ on $\mathbf{P}_{\mathcal{G},\mathbf{r}}$, by for any $f, g \in \mathbf{P}_{\mathcal{G},\mathbf{r}}$

$$\mathrm{dist}_{\mathcal{G}}(g, f) = \|g - f\|_{\mathcal{G}}.$$

Definition 3.4. We say that two spaces $\mathcal{F}$ and $\mathbf{P}_{\mathcal{G}}$, endowed with the norm $\|\cdot\|_{\mathcal{F}}$ and the distance $\mathrm{dist}_{\mathcal{G}}$ inherited from the norm $\|\cdot\|_{\mathcal{G}}$, are in duality if

$$\forall\, f, g \in \mathcal{G}\,,\ \forall\, \varphi \in \mathcal{F} \qquad |\langle g - f, \varphi\rangle| \le \mathrm{dist}_{\mathcal{G}}(g, f)\, \|\varphi\|_{\mathcal{F}}. \quad (3.49)$$

Definition 3.5. Consider two metric spaces $\widetilde{\mathcal{G}}_1$ and $\widetilde{\mathcal{G}}_2$, some weight function $\Lambda : \widetilde{\mathcal{G}}_1 \to \mathbb{R}^*_+$ and $\eta \in (0,1]$. We denote by $C^{0,\eta}_\Lambda(\widetilde{\mathcal{G}}_1; \widetilde{\mathcal{G}}_2)$ the (weighted) space of functions with η-Hölder regularity, that is functinos $\mathcal{S} : \widetilde{\mathcal{G}}_1 \to \widetilde{\mathcal{G}}_2$ such that there exists a constant $C > 0$

$$\forall\, f, g \in \widetilde{\mathcal{G}}_1\,, \quad \mathrm{dist}_{\mathcal{G}_2}(\mathcal{S}(f), \mathcal{S}(g)) \le C\,\Lambda(g, f)\ \mathrm{dist}_{\mathcal{G}_1}(f, g)^\eta. \quad (3.50)$$

where $\Lambda(g, f) = \max\{\Lambda(g), \Lambda(f)\}$.

We define then a higher order differential calculus.

Definition 3.6. Consider two normed spaces $\mathcal{G}_1$ and $\mathcal{G}_2$, two metric spaces $\widetilde{\mathcal{G}}_1$ and $\widetilde{\mathcal{G}}_2$ such that $\widetilde{\mathcal{G}}_i - \widetilde{\mathcal{G}}_i \subset \mathcal{G}_i$, some weight function $\Lambda : \widetilde{\mathcal{G}}_1 \to [1, \infty)$ and $\eta \in (0, 1]$. We denote by $C^{2,\eta}_\Lambda(\widetilde{\mathcal{G}}_1; \widetilde{\mathcal{G}}_2)$ the (weighted) space of functions two times continuously differentiable from $\widetilde{\mathcal{G}}_1$ to $\widetilde{\mathcal{G}}_2$, and such that the 2th derivative satisfies some weighted η-Hölder regularity (in the sense of Definition 3.5).

More precisely, these are functions $\mathcal{S} : \widetilde{\mathcal{G}}_1 \to \widetilde{\mathcal{G}}_2$ continuous, such that there exists maps (for $j = 1, 2$) $D^j\mathcal{S} : \widetilde{\mathcal{G}}_1 \to \mathcal{L}^j(\mathcal{G}_1, \mathcal{G}_2)$, where $\mathcal{L}^j(\mathcal{G}_1, \mathcal{G}_2)$ is the space of j-multilinear applications from $\mathcal{G}_1$ to $\mathcal{G}_2$, and there exists some constants $C_j > 0$, so that we have for all $f, g \in \widetilde{\mathcal{G}}_1$,

$$
\begin{aligned}
\|\mathcal{S}(g) - \mathcal{S}(f)\|_{\mathcal{G}_2} &\leq C_0\, \Lambda(g, f)\, \|g - f\|^{\eta_0}_{\mathcal{G}_1}, \\
\|\langle D\mathcal{S}[f], g - f\rangle\|_{\mathcal{G}_2} &\leq C_1\, \Lambda(g, f)\, \|g - f\|^{\eta_0}_{\mathcal{G}_1}, \\
\|\mathcal{S}(g) - \mathcal{S}(f) - \langle D\mathcal{S}[f], g - f\rangle\|_{\mathcal{G}_2} &\leq C_2\, \Lambda(g, f)\, \|g - f\|^{1+\eta_1}_{\mathcal{G}_1}, \\
\left\|\left\langle D^2\mathcal{S}[f], (g - f)^{\otimes 2}\right\rangle\right\|_{\mathcal{G}_2} &\leq C_3\, \Lambda(g, f)\, \|g - f\|^{1+\eta_1}_{\mathcal{G}_1}, \\
\left\|\mathcal{S}(g) - \mathcal{S}(f) - \sum_{i=1}^{2}\left\langle D^i\mathcal{S}[f], (g - f)^{\otimes i}\right\rangle\right\|_{\mathcal{G}_2} &\leq C_4\, \Lambda(g, f)\, \|g - f\|^{2+\eta}_{\mathcal{G}_1},
\end{aligned}
\tag{3.51}
$$

where $\eta_0, \eta_1 \in [\eta, 1]$.

We define then the seminorms on $C^{2,\eta}_\Lambda(\widetilde{\mathcal{G}}_1; \widetilde{\mathcal{G}}_2)$

$$
[\mathcal{S}]_{C^{1,0}_\Lambda} := \sup_{f\in\widetilde{\mathcal{G}}_1, h\in\mathcal{G}_1} \frac{\|\langle D\mathcal{S}[f], h\rangle\|_{\mathcal{G}_2}}{\Lambda(f)\, \|h\|^{\eta_0}_{\mathcal{G}_1}}, \qquad [\mathcal{S}]_{C^{2,0}_\Lambda} := \sup_{f\in\widetilde{\mathcal{G}}_1, h\in\mathcal{G}_1} \frac{\|\langle D^2\mathcal{S}[f], (h, h)\rangle\|_{\mathcal{G}_2}}{\Lambda(f)\, \|h\|^{1+\eta_1}_{\mathcal{G}_1}}
$$

and

$$
\begin{aligned}
[\mathcal{S}]_{C^{0,\eta_0}_\Lambda} &:= \sup_{f,g\in\widetilde{\mathcal{G}}_1} \frac{\|\mathcal{S}(g) - \mathcal{S}(f)\|_{\mathcal{G}_2}}{\Lambda(g, f)\, \|g - f\|^{\eta_0}_{\mathcal{G}_1}}, \\
[\mathcal{S}]_{C^{1,\eta_1}_\Lambda} &:= \sup_{f,g\in\widetilde{\mathcal{G}}_1} \frac{\|\mathcal{S}(g) - \mathcal{S}(f) - \langle D\mathcal{S}[f], g - f\rangle\|_{\mathcal{G}_2}}{\Lambda(g, f)\, \|g - f\|^{1+\eta_1}_{\mathcal{G}_1}}, \\
[\mathcal{S}]_{C^{2,\eta}_\Lambda} &:= \sup_{f,g\in\widetilde{\mathcal{G}}_1} \frac{\left\|\mathcal{S}(g) - \mathcal{S}(f) - \sum_{i=1}^{2}\langle D^i\mathcal{S}[f], (g - f)^{\otimes i}\rangle\right\|_{\mathcal{G}_2}}{\Lambda(g, f)\, \|g - f\|^{2+\eta}_{\mathcal{G}_1}}.
\end{aligned}
$$

Finally we combine these seminorms into

$$
\|\mathcal{S}\|_{C^{2,\eta}_\Lambda} := [\mathcal{S}]_{C^{0,\eta_0}_\Lambda} + [\mathcal{S}]_{C^{1,\eta_1}_\Lambda} + [\mathcal{S}]_{C^{2,\eta}_\Lambda} + [\mathcal{S}]_{C^{1,0}_\Lambda} + [\mathcal{S}]_{C^{2,0}_\Lambda}.
$$

This differential calculus holds for composition, more precisely for $\mathcal{U} \in C^{k,\eta}_{\Lambda_\mathcal{U}}(\widetilde{\mathcal{G}}_1; \widetilde{\mathcal{G}}_2)$ and $\mathcal{V} \in C^{k,\eta}_{\Lambda_\mathcal{V}}(\widetilde{\mathcal{G}}_2; \widetilde{\mathcal{G}}_3)$ we have $\mathcal{S} = \mathcal{V} \circ \mathcal{U} \in C^{k,\eta_\mathcal{S}}_{\Lambda_\mathcal{S}}(\widetilde{\mathcal{G}}_1; \widetilde{\mathcal{G}}_3)$ for some appropriate weight function $\Lambda_\mathcal{S}$ and exposant $\eta_\mathcal{S}$. We now state the following lemma

Lemma 3.7. *Let $\mathcal{G}_i$ be normed spaces and $\widetilde{\mathcal{G}}_i$ be metric spaces for $i = 1, 2, 3$, such that $\widetilde{\mathcal{G}}_i - \widetilde{\mathcal{G}}_i \subset \mathcal{G}_i$. Consider $\mathcal{U} \in C^{2,\eta}_\Lambda \cap C^{1,(1+2\eta)/3}_\Lambda \cap C^{0,(2+\eta)/3}_\Lambda(\widetilde{\mathcal{G}}_1; \widetilde{\mathcal{G}}_2)$, with $\eta \in (0, 1]$, and $\mathcal{V} \in C^{2,1}(\widetilde{\mathcal{G}}_2; \widetilde{\mathcal{G}}_3)$. Then the composition function $\mathcal{S} = \mathcal{V} \circ \mathcal{U} \in C^{2,\eta}_{\Lambda^3} \cap C^{1,(1+2\eta)/3}_{\Lambda^3} \cap C^{0,(2+\eta)/3}_{\Lambda^3}(\widetilde{\mathcal{G}}_1; \widetilde{\mathcal{G}}_3)$ and we have*

$$
\begin{aligned}
&D\mathcal{S}[f] = D\mathcal{V}[\mathcal{U}(f)] \circ D\mathcal{U}[f],\\
&D^2\mathcal{S}[f] = D^2\mathcal{V}[\mathcal{U}(f)] \circ (D\mathcal{U}[f] \otimes D\mathcal{U}[f]) + D\mathcal{V}[\mathcal{U}(f)] \circ D^2\mathcal{U}[f].
\end{aligned}
$$

More precisely, the following estimates hold

$$
\begin{aligned}
[\mathcal{S}]_{C^{0,(2+\eta)/3}_\Lambda} &\le [\mathcal{V}]_{C^{0,1}}\, [\mathcal{U}]_{C^{0,(2+\eta)/3}_\Lambda}\,,\\
[\mathcal{S}]_{C^{1,0}_\Lambda} &\le [\mathcal{V}]_{C^{1,0}}\, [\mathcal{U}]_{C^{1,0}_\Lambda}\,,\\
[\mathcal{S}]_{C^{1,(1+2\eta)/3}_{\Lambda^2}} &\le [\mathcal{V}]_{C^{1,0}}\; [\mathcal{U}]_{C^{1,(1+2\eta)/3}_\Lambda} + [\mathcal{V}]_{C^{1,1}}\; [\mathcal{U}]^2_{C^{0,(2+\eta)/3}_\Lambda}\,,\\
[\mathcal{S}]_{C^{2,0}_{\Lambda^2}} &\le [\mathcal{V}]_{C^{1,0}}\; [\mathcal{U}]_{C^{2,0}_\Lambda} + [\mathcal{V}]_{C^{2,0}}\; [\mathcal{U}]^2_{C^{1,0}_\Lambda}\,,\\
[\mathcal{S}]_{C^{2,\eta}_{\Lambda^3}} &\le [\mathcal{V}]_{C^{1,0}}\; [\mathcal{U}]_{C^{2,\eta}_\Lambda} + [\mathcal{V}]_{C^{2,0}}\; [\mathcal{U}]^2_{C^{1,(1+2\eta)/3}_\Lambda}\\
&\quad + 2\,[\mathcal{V}]_{C^{2,0}}\; [\mathcal{U}]_{C^{1,0}_\Lambda}\; [\mathcal{U}]_{C^{1,(1+2\eta)/3}_\Lambda} + [\mathcal{V}]_{C^{2,1}}\, [\mathcal{U}]^3_{C^{0,(2+\eta)/3}_\Lambda}\,.
\end{aligned}
$$

Proof of Lemma 3.7. Let $f, g \in \widetilde{\mathcal{G}}_1$ and $\bar f, \bar g \in \widetilde{\mathcal{G}}_2$.

By Definition 3.6 with $\mathcal{U} \in C^{2,\eta}_\Lambda \cap C^{1,(1+2\eta)/3}_\Lambda \cap C^{0,(2+\eta)/3}_\Lambda(\widetilde{\mathcal{G}}_1; \widetilde{\mathcal{G}}_2)$ and $\mathcal{V} \in C^{2,1}(\widetilde{\mathcal{G}}_2; \widetilde{\mathcal{G}}_3)$ we have

$$
\begin{aligned}
&\mathcal{U}(g) - \mathcal{U}(f) = \langle D\mathcal{U}[f], g - f\rangle + R^1_{\mathcal{U}}(g, f)\\
&\mathcal{U}(g) - \mathcal{U}(f) = \langle D\mathcal{U}[f], g - f\rangle + \left\langle D^2\mathcal{U}[f], (g - f)^{\otimes 2}\right\rangle + R^2_{\mathcal{U}}(g, f)
\end{aligned}
\tag{3.52}
$$

with

$$\|\mathcal{U}(g) - \mathcal{U}(f)\|_{\mathcal{G}_2} \le [\mathcal{U}]_{C^{0,\eta_0}_\Lambda}\, \Lambda(g, f)\, \|g - f\|^{\eta_0}_{\mathcal{G}_1}, \tag{3.53}$$

$$\|\langle D\mathcal{U}[f], g - f\rangle\|_{\mathcal{G}_2} \le [\mathcal{U}]_{C^{1,0}_\Lambda}\, \Lambda(g, f)\, \|g - f\|^{\eta_0}_{\mathcal{G}_1} \tag{3.54}$$

$$\|R^1_{\mathcal{U}}(g, f)\|_{\mathcal{G}_2} \le [\mathcal{U}]_{C^{1,\eta_1}_\Lambda}\, \Lambda(g, f)\, \|g - f\|^{1+\eta_1}_{\mathcal{G}_1}, \tag{3.55}$$

$$\left\|\left\langle D^2\mathcal{U}[f], (g - f)^{\otimes 2}\right\rangle\right\|_{\mathcal{G}_2} \le [\mathcal{U}]_{C^{2,0}_\Lambda}\, \Lambda(g, f)\, \|g - f\|^{1+\eta_1}_{\mathcal{G}_1} \tag{3.56}$$

$$\|R^2_{\mathcal{U}}(g, f)\|_{\mathcal{G}_2} \le [\mathcal{U}]_{C^{2,\eta}_\Lambda}\, \Lambda(g, f)\, \|g - f\|^{2+\eta}_{\mathcal{G}_1}, \tag{3.57}$$

where, for simplicity, we denote $\eta_0 = (2 + \eta)/3$ and $\eta_1 = (1 + 2\eta)/3$.

Similarly we have for $\mathcal{V}$,

$$
\begin{aligned}
&\mathcal{V}(\bar g) - \mathcal{V}(\bar f) = \left\langle D\mathcal{V}[\bar f], \bar g - \bar f\right\rangle + R^1_{\mathcal{V}}(\bar g, \bar f)\\
&\mathcal{V}(\bar g) - \mathcal{V}(\bar f) = \left\langle D\mathcal{V}[\bar f], \bar g - \bar f\right\rangle + \left\langle D^2\mathcal{V}[\bar f], (\bar g - \bar f)^{\otimes 2}\right\rangle + R^2_{\mathcal{V}}(\bar g, \bar f)
\end{aligned}
\tag{3.58}
$$

with

$$\|\mathcal{V}(\bar{g}) - \mathcal{V}(\bar{f})\|_{\mathcal{G}_3} \leq [\mathcal{V}]_{C^{0,1}} \|\bar{g} - \bar{f}\|_{\mathcal{G}_2} \tag{3.59}$$

$$\|\left\langle D\mathcal{V}[\bar{f}], \bar{g} - \bar{f} \right\rangle\|_{\mathcal{G}_3} \leq [\mathcal{V}]_{C^{1,0}} \|\bar{g} - \bar{f}\|_{\mathcal{G}_2} \tag{3.60}$$

$$\|R^1_{\mathcal{V}}(\bar{g}, \bar{f})\|_{\mathcal{G}_3} \leq [\mathcal{V}]_{C^{1,1}} \|\bar{g} - \bar{f}\|^2_{\mathcal{G}_2} \tag{3.61}$$

$$\|\left\langle D^2\mathcal{V}[\bar{f}], (\bar{g} - \bar{f})^{\otimes 2} \right\rangle\|_{\mathcal{G}_3} \leq [\mathcal{V}]_{C^{2,0}} \|\bar{g} - \bar{f}\|^2_{\mathcal{G}_2} \tag{3.62}$$

$$\|R^2_{\mathcal{V}}(\bar{g}, \bar{f})\|_{\mathcal{G}_3} \leq [\mathcal{V}]_{C^{2,1}} \|\bar{g} - \bar{f}\|^3_{\mathcal{G}_2}. \tag{3.63}$$

Using these estimates we can compute for $\mathcal{S} = \mathcal{V} \circ \mathcal{U}$, then we obtain first thanks to (3.59) and (3.53)

$$\begin{aligned}\|\mathcal{S}(g) - \mathcal{S}(f)\|_{\mathcal{G}_3} &= \|\mathcal{V}(\mathcal{U}(g)) - \mathcal{V}(\mathcal{U}(f))\|_{\mathcal{G}_3} \\ &\leq [\mathcal{V}]_{C^{0,1}} \|\mathcal{U}(g) - \mathcal{U}(f)\|_{\mathcal{G}_2} \\ &\leq [\mathcal{V}]_{C^{0,1}} [\mathcal{U}]_{C^{0,\eta_0}_{\Lambda}} \Lambda(g,f) \|g - f\|^{\eta_0}_{\mathcal{G}_1}\end{aligned}$$

which implies $[\mathcal{S}]_{C^{0,(2+\eta)/3}_{\Lambda}} \leq [\mathcal{V}]_{C^{0,1}} [\mathcal{U}]_{C^{0,(2+\eta)/3}_{\Lambda}}$.

We also have, using (3.58) and (3.52),

$$\begin{aligned}\mathcal{S}(g) - \mathcal{S}(f) &= \mathcal{V}(\mathcal{U}(g)) - \mathcal{V}(\mathcal{U}(f)) \\ &= \langle D\mathcal{V}[\mathcal{U}(f)], \mathcal{U}(g) - \mathcal{U}(f) \rangle + R^1_{\mathcal{V}}(\mathcal{U}(g), \mathcal{U}(f)) \\ &= \left\langle D\mathcal{V}[\mathcal{U}(f)], \left\{ \langle D\mathcal{U}[f], g - f \rangle + R^1_{\mathcal{U}}(g,f) \right\} \right\rangle \\ &\quad + R^1_{\mathcal{V}}(\mathcal{U}(g), \mathcal{U}(f)),\end{aligned}$$

from which we deduce $\langle D\mathcal{S}[f], g - f \rangle = \langle D\mathcal{V}[\mathcal{U}(f)], (\langle D\mathcal{U}[f], g - f \rangle) \rangle$ and, by (3.60) and (3.54),

$$\begin{aligned}\| \langle D\mathcal{S}[f], g - f \rangle \|_{\mathcal{G}_3} &\leq [\mathcal{V}]_{C^{1,0}} \| \langle D\mathcal{U}[f], g - f \rangle \|_{\mathcal{G}_2} \\ &\leq [\mathcal{V}]_{C^{1,0}} [\mathcal{U}]_{C^{1,0}_{\Lambda}} \Lambda(f) \|g - f\|^{\eta_0}_{\mathcal{G}_1},\end{aligned}$$

which yields $[\mathcal{S}]_{C^{1,0}_{\Lambda}} \leq [\mathcal{V}]_{C^{1,0}} [\mathcal{U}]_{C^{1,0}_{\Lambda}}$.

Therefore, we obtain using (3.60), (3.61), (3.55) and (3.53),

$$\begin{aligned}&\|\mathcal{S}(g) - \mathcal{S}(f) - \langle D\mathcal{S}[f], g - f \rangle\|_{\mathcal{G}_3} \\ &\quad\leq \|\left\langle D\mathcal{V}[\mathcal{U}(f)], R^1_{\mathcal{U}}(g,f) \right\rangle\|_{\mathcal{G}_3} + \|R^1_{\mathcal{V}}(\mathcal{U}(g), \mathcal{U}(f))\|_{\mathcal{G}_3} \\ &\quad\leq [\mathcal{V}]_{C^{1,0}} \|R^1_{\mathcal{U}}(g,f)\|_{\mathcal{G}_2} + [\mathcal{V}]_{C^{1,1}} \|\mathcal{U}(g) - \mathcal{U}(f)\|^2_{\mathcal{G}_2} \\ &\quad\leq [\mathcal{V}]_{C^{1,0}} [\mathcal{U}]_{C^{1,\eta_1}_{\Lambda}} \Lambda(g,f) \|g - f\|^{1+\eta_1}_{\mathcal{G}_1} + [\mathcal{V}]_{C^{1,1}} [\mathcal{U}]^2_{C^{0,\eta_0}_{\Lambda}} \Lambda(g,f)^2 \|g - f\|^{2\eta_0}_{\mathcal{G}_1}.\end{aligned}$$

Since $1 + \eta_1 = 2\eta_0 = 1 + (1 + 2\eta)/3$ and $\Lambda \geq 1$, the last inequality implies

$$[\mathcal{S}]_{C^{1,(1+2\eta)/3}_{\Lambda^2}} \leq [\mathcal{V}]_{C^{1,0}} [\mathcal{U}]_{C^{1,(1+2\eta)/3}_{\Lambda}} + [\mathcal{V}]_{C^{1,1}} [\mathcal{U}]^2_{C^{0,(2+\eta)/3}_{\Lambda}}.$$

Finally, from (3.58) and (3.52), we have

$$
\begin{aligned}
&\mathcal{S}(g) - \mathcal{S}(f) = \mathcal{V}\left(\mathcal{U}(g)\right) - \mathcal{V}\left(\mathcal{U}(f)\right) \\
&\quad = \langle D\mathcal{V}[\mathcal{U}(f)], \mathcal{U}(g) - \mathcal{U}(f)\rangle + \left\langle D^2\mathcal{V}[\mathcal{U}(f)], \left(\mathcal{U}(g) - \mathcal{U}(f)\right)^{\otimes 2}\right\rangle + R^2_{\mathcal{V}}(\mathcal{U}(g), \mathcal{U}(f)) \\
&\quad = \left\langle D\mathcal{V}[\mathcal{U}(f)], \Big(\langle D\mathcal{U}[f], g - f\rangle + \left\langle D^2\mathcal{U}[f], (g-f)^{\otimes 2}\right\rangle + R^2_{\mathcal{U}}(g,f)\Big)\right\rangle \\
&\quad + \left\langle D^2\mathcal{V}[\mathcal{U}(f)], \Big(\langle D\mathcal{U}[f], g - f\rangle + R^1_{\mathcal{U}}(g,f)\Big)^{\otimes 2}\right\rangle \\
&\quad + R^2_{\mathcal{V}}(\mathcal{U}(g), \mathcal{U}(f)),
\end{aligned}
$$

which yields

$$
\begin{aligned}
\left\langle D^2\mathcal{S}[f], (g-f)^{\otimes 2}\right\rangle = &\left\langle D\mathcal{V}[\mathcal{U}(f)], \left(\left\langle D^2\mathcal{U}[f], (g-f)^{\otimes 2}\right\rangle \right)\right\rangle \\
&+ \left\langle D^2\mathcal{V}[\mathcal{U}(f)], \left(\langle D\mathcal{U}[f], g-f\rangle \right)^{\otimes 2}\right\rangle.
\end{aligned}
$$

Hence we obtain, with (3.60), (3.62), (3.56) and (3.54),

$$
\begin{aligned}
&\left\| \left\langle D^2\mathcal{S}[f], (g-f)^{\otimes 2}\right\rangle \right\|_{\mathcal{G}_3} \\
&\qquad \le [\mathcal{V}]_{C^{1,0}} \left\| \left\langle D^2\mathcal{U}[f], (g-f)^{\otimes 2}\right\rangle \right\|_{\mathcal{G}_2} + [\mathcal{V}]_{C^{2,0}} \left\| \langle D\mathcal{U}[f], g-f\rangle \right\|^2_{\mathcal{G}_2} \\
&\qquad \le [\mathcal{V}]_{C^{1,0}} \, [\mathcal{U}]_{C^{2,0}_{\Lambda}} \, \Lambda(f) \, \|g-f\|^{1+\eta_1}_{\mathcal{G}_1} + [\mathcal{V}]_{C^{2,0}} \left([\mathcal{U}]_{C^{1,0}_{\Lambda}} \, \Lambda(f) \, \|g-f\|^{\eta_0}_{\mathcal{G}_1}\right)^2 \\
&\qquad \le \left([\mathcal{V}]_{C^{1,0}} \, [\mathcal{U}]_{C^{2,0}_{\Lambda}} + [\mathcal{V}]_{C^{2,0}} \, [\mathcal{U}]^2_{C^{1,0}_{\Lambda}} \right) \Lambda(f)^2 \, \|g-f\|^{1+(1+2\eta)/3}_{\mathcal{G}_1},
\end{aligned}
$$

which gives $[\mathcal{S}]_{C^{2,0}_{\Lambda^2}} \le [\mathcal{V}]_{C^{1,0}} \, [\mathcal{U}]_{C^{2,0}_{\Lambda}} + [\mathcal{V}]_{C^{2,0}} \, [\mathcal{U}]^2_{C^{1,0}_{\Lambda}}$.

Now, for the last estimate we obtain

$$
\begin{aligned}
&\|\mathcal{S}(g) - \mathcal{S}(f) - \langle D\mathcal{S}[f], g-f\rangle - \left\langle D^2\mathcal{S}[f], (g-f)^{\otimes 2}\right\rangle \|_{\mathcal{G}_3} \\
&\qquad \le \left\| \left\langle D\mathcal{V}[\mathcal{U}(f)], R^2_{\mathcal{U}}(g,f)\right\rangle \right\|_{\mathcal{G}_3} + \left\| \left\langle D^2\mathcal{V}[\mathcal{U}(f)], \left(R^1_{\mathcal{U}}(g,f)\right)^{\otimes 2}\right\rangle \right\|_{\mathcal{G}_3} \\
&\qquad + 2 \left\| \left\langle D^2\mathcal{V}[\mathcal{U}(f)], \Big(\langle D\mathcal{U}[f], g-f\rangle \otimes R^1_{\mathcal{U}}(g,f)\Big)\right\rangle \right\|_{\mathcal{G}_3} \\
&\qquad + \left\| R^2_{\mathcal{V}}(\mathcal{U}(g), \mathcal{U}(f)) \right\|_{\mathcal{G}_3}
\end{aligned}
$$

and using the equations (3.59) to (3.63) and (3.53) to (3.57), it gives

$$
\begin{aligned}
&\|\mathcal{S}(g) - \mathcal{S}(f) - \langle D\mathcal{S}[f], g-f\rangle - \left\langle D^2\mathcal{S}[f], (g-f)^{\otimes 2}\right\rangle\|_{\mathcal{G}_3} \\
&\quad\le [\mathcal{V}]_{C^{1,0}} \left\|R^2_{\mathcal{U}}(g,f)\right\|_{\mathcal{G}_2} + [\mathcal{V}]_{C^{2,0}} \left\|R^1_{\mathcal{U}}(g,f)\right\|^2_{\mathcal{G}_2} \\
&\qquad + 2\,[\mathcal{V}]_{C^{2,0}} \|\langle D\mathcal{U}[f], g-f\rangle\|_{\mathcal{G}_2} \left\|R^1_{\mathcal{U}}(g,f)\right\|_{\mathcal{G}_2} + [\mathcal{V}]_{C^{2,1}} \|\mathcal{U}(g) - \mathcal{U}(f)\|^3_{\mathcal{G}_2} \\
&\quad\le [\mathcal{V}]_{C^{1,0}} [\mathcal{U}]_{C^{2,\eta}_\Lambda} \Lambda(g,f) \|g-f\|^{2+\eta}_{\mathcal{G}_1} \\
&\qquad + [\mathcal{V}]_{C^{2,0}} [\mathcal{U}]^2_{C^{1,\eta_1}_\Lambda} \Lambda(g,f)^2 \|g-f\|^{2+2\eta_1}_{\mathcal{G}_1} \\
&\qquad + 2\,[\mathcal{V}]_{C^{2,0}} [\mathcal{U}]_{C^{1,0}_\Lambda} [\mathcal{U}]_{C^{1,\eta_1}_\Lambda} \Lambda(g,f)^2 \|g-f\|^{1+\eta_1+\eta_0}_{\mathcal{G}_1} \\
&\qquad + [\mathcal{V}]_{C^{2,1}} [\mathcal{U}]^3_{C^{0,\eta_0}_\Lambda} \Lambda(g,f)^3 \|g-f\|^{3\eta_0}_{\mathcal{G}_1}.
\end{aligned}
$$

Since $1+\eta_1+\eta_0 = 3\eta_0 = 2+\eta < 2+2\eta_1$, we deduce

$$
\begin{aligned}
[\mathcal{S}]_{C^{2,\eta}_{\Lambda^3}} &\le [\mathcal{V}]_{C^{1,0}} [\mathcal{U}]_{C^{2,\eta}_\Lambda} + [\mathcal{V}]_{C^{2,0}} [\mathcal{U}]^2_{C^{1,(1+2\eta)/3}_\Lambda} \\
&\quad + 2\,[\mathcal{V}]_{C^{2,0}} [\mathcal{U}]_{C^{1,0}_\Lambda} [\mathcal{U}]_{C^{1,(1+2\eta)/3}_\Lambda} + [\mathcal{V}]_{C^{2,1}} [\mathcal{U}]^3_{C^{0,(2+\eta)/3}_\Lambda}.
\end{aligned}
$$

□

Example 3.8. Consider the pair $\mathcal{F}$ and $\mathbf{P}_\mathcal{G}$ in duality (Definition 3.4) where $\mathcal{F} \subset C_b(E)$, and consider $\varphi = \varphi_1 \times \cdots \times \varphi_\ell \in \mathcal{F}^{\otimes \ell}$. Then the application R^ℓ_φ defined in (3.47) is $C^{2,1}(\mathbf{P}_\mathcal{G}; \mathbb{R})$. Consider $f, g \in \mathbf{P}_\mathcal{G}$, then we have thanks to the multilinearity of R^ℓ_ϕ [62, 63] that

$$
\begin{aligned}
\left|R^\ell_\varphi(g) - R^\ell_\varphi(f)\right| &\le \ell\, \|\varphi\|_{\mathcal{F}\otimes(L^\infty)^{\ell-1}} \|g-f\|_\mathcal{G}, \\
\left|DR^\ell_\varphi[f](g-f)\right| &\le \ell\, \|\varphi\|_{\mathcal{F}\otimes(L^\infty)^{\ell-1}} \|g-f\|_\mathcal{G}, \\
\left|R^\ell_\varphi(g) - R^\ell_\varphi(f) - DR^\ell_\varphi[f](g-f)\right| &\le \frac{\ell(\ell-1)}{2} \|\varphi\|_{\mathcal{F}^{\otimes 2}\otimes(L^\infty)^{\ell-2}} \|g-f\|^2_\mathcal{G} \\
\left|D^2R^\ell_\varphi[f](g-f)^{\otimes 2}\right| &\le \frac{\ell(\ell-1)}{2} \|\varphi\|_{\mathcal{F}^{\otimes 2}\otimes(L^\infty)^{\ell-2}} \|g-f\|^2_\mathcal{G} \\
\left|R^\ell_\varphi(g) - R^\ell_\varphi(f) - DR^\ell_\varphi[f](g-f) - D^2R^\ell_\varphi[f](g-f)^{\otimes 2}\right|& \\
&\le \frac{\ell(\ell-1)(\ell-2)}{6} \|\varphi\|_{\mathcal{F}^{\otimes 3}\otimes(L^\infty)^{\ell-3}} \|g-f\|^3_\mathcal{G},
\end{aligned}
$$

where we define

$$
\|\varphi\|_{\mathcal{F}^{\otimes j}\otimes(L^\infty)^{\ell-j}} := \max_{i_1,\dots,i_j \text{ distincts in } [1,\ell]} \left(\|\varphi_{i_1}\|_\mathcal{F} \cdots \|\varphi_{i_1}\|_\mathcal{F} \prod_{k\neq i_1,\dots,i_j} \|\varphi_k\|_{L^\infty} \right).
$$

Since we shall need endow the subspaces of probability measures in Definition 3.3 with metrics, let us give useful examples. We denote by $\mathbf{P}_p(\mathbb{R}^d)$ the space of probabilities with finite moments up to order p, more precisely $\mathbf{P}_p(\mathbb{R}^d) := \{f \in \mathbf{P}(\mathbb{R}^d)\ ;\ \langle |v|^p, f\rangle < \infty\}$.

Definition 3.9 (Monge-Kantorovich-Wasserstein distance)**.** For $f, g \in \mathbf{P}_p(\mathbb{R}^d)$ we define the distance

$$W_p^p(f,g) := \inf_{\pi\in\Pi(f,g)} \int_{\mathbb{R}^d\times\mathbb{R}^d} \mathrm{dist}(x,y)^p \, \pi(dx,dy)$$

where $\Pi(f,g)$ is the set of probability measures on $\mathbb{R}^d \times \mathbb{R}^d$ with marginals f and g respectively.

In a analogous way, we also define, for $\mu, \nu \in \mathbf{P}(\mathbf{P}(\mathbb{R}^d))$ and a distance D over $\mathbf{P}(\mathbb{R}^d)$, the distance

$$\mathcal{W}_{1,D}(\mu,\nu) := \inf_{\pi\in\Pi(\mu,\nu)} \int_{\mathbf{P}(\mathbb{R}^d)\times\mathbf{P}(\mathbb{R}^d)} D(f,g)\, \pi(df,dg)$$

where $\Pi(\mu,\nu)$ denotes the set of probabillity measures on $\mathbf{P}(\mathbb{R}^d)\times\mathbf{P}(\mathbb{R}^d)$ with marginals μ and ν.

Definition 3.10 (Fourier based distances)**.** For $f, g \in \mathbf{P}_s(\mathbb{R}^d)$ we define the distance

$$|f-g|_s := \sup_{\xi\in\mathbb{R}^d} \frac{|\hat{f}(\xi)-\hat{g}(\xi)|}{|\xi|^s} \tag{3.64}$$

which is well defined if f and g have equal moments up to order $s-1$ if s is a integer or $\lfloor s \rfloor$ if not. We denote by $\mathcal{H}^{-s}$ the space associated to this norm.

Definition 3.11 (General Fourier based distances)**.** Let $k \in \mathbb{N}^*$ and set

$$m_{\mathcal{G}} := |v|^k, \qquad \mathbf{m}_{\mathcal{G}} := (v^\alpha)_{\alpha\mathbb{N}^d}\,,\ |\alpha| \le k-1$$

and

$$v^\alpha = (v_1^{\alpha_1}, \dots, v_d^{\alpha_d})\,, \qquad \alpha = (\alpha_1, \dots, \alpha_d)$$

and we define

$$\forall\, f \in \mathcal{I}P_{\mathcal{G}}, \qquad \|f\|_{\mathcal{G}} = |f|_s := \sup_{\xi\in\mathbb{R}^d} \frac{|\hat{f}(\xi)|}{|\xi|^s},\ s \in (0,k].$$

We extend the above norm to $M_k^1(\mathbb{R}^d)$ in the following way. First we define for $f \in M_{k-1}^1(\mathbb{R}^d)$ and $\alpha \in \mathbb{N}^d$, $|\alpha| \le k-1$ the moment

$$\mathcal{M}_\alpha[f] := \int_{\mathbb{R}^d} v^\alpha f(dv).$$

For a fixed smooth function with compact support $\chi \in C_c^\infty(\mathbb{R}^d)$ such that $\chi = 1$ over $\{\xi \in \mathbb{R}^d, |\xi| \le 1\}$, we define then

$$\hat{\mathcal{M}}_k[f](\xi) := \chi(\xi) \left(\sum_{|\alpha|\le k-1} \mathcal{M}_\alpha[f]\, \frac{\xi^\alpha}{\alpha!}\, (-i)^{|\alpha|} \right).$$

We define then the norm

$$|||f|||_k := |f - \mathcal{M}_k[f]|_k + \sum_{|\alpha|\le k-1} |M_\alpha[f]|,$$

where as above $|h|_k := \sup_\xi \frac{|\hat{h}(\xi)|}{|\xi|^k}$.

3.3.2 Abstract theorem

We state the assumptions of our abstract theorem 3.13.

Assumption (A1) (N-particle system)**.** The semigroup T_t^N and its generator G^N are well defined on $C_b(E^N)$ and are invariant under permutation so that F_t^N is well defined. Moreover, we assume that the following conditions hold:

(i) *Conservation constraint*: There exists a constraint function $\mathbf{m}_{\mathcal{G}_1} : E \to \mathbb{R}^D$ and a subset $\mathbf{R}_{\mathcal{G}_1} \subset \mathbb{R}^D$ such that defining the set

$$\mathbb{E}_N := \{V \in E^N; \langle \mu_V^N, \mathbf{m}_{\mathcal{G}_1} \rangle \in \mathbf{R}_{\mathcal{G}_1}\}$$

there holds

$$\forall\, t \geq 0, \qquad \operatorname{supp} F_t^N \subset \mathbb{E}_N.$$

(ii) *Propagation of integral moment bound*: There exists a weight function $m_{\mathcal{G}_1}$, a time $T \in (0, \infty]$ and a constant $C^T_{m_{\mathcal{G}_1}} > 0$, possibly depending on $m_{\mathcal{G}_1}$, but not on N, such that

$$\forall\, N \geq 1 \qquad \sup_{t \geq 0} \left\langle F_t^N, M^N_{m_{\mathcal{G}_1}} \right\rangle \leq C_{m_{\mathcal{G}_1}}. \tag{3.65}$$

Assumption (A2) (Nonlinear semigroup)**.** Consider a probability space $\mathbf{P}_{\mathcal{G}_1}$ as in Definition 3.3, associated to some function $m_{\mathcal{G}_1}$ and a constraint function $\mathbf{m}_1$, and endowed with the metric induced from $\mathcal{G}_1$. For some $\delta \in (0, 1]$ and some $\bar{a} \in (0, \infty)$, we have for any $a \in (\bar{a}, \infty)$:

(i) The nonlinear semigroup is well defined $S_t^\infty : \mathcal{B}\mathbf{P}_{\mathcal{G}_1,a} \to \mathcal{B}\mathbf{P}_{\mathcal{G}_1,a}$, which is δ-Hölder continuous locally uniformly in time, in the sense that for any $\tau \in (0, \infty)$ there exists $C_\tau > 0$ such that

$$\forall\, f, g \in \mathcal{B}\mathbf{P}_{\mathcal{G}_1,a} \qquad \sup_{0 \leq t \leq \tau} \|S_t^\infty g - S_t^\infty f\|_{\mathcal{G}_1} \leq C_\tau \, \|g - f\|^\delta_{\mathcal{G}_1}.$$

(ii) The application Q is bounded and δ-Hölder continuous from $\mathcal{B}\mathbf{P}_{\mathcal{G}_1,a}$ into $\mathcal{G}_1$.

With **(A2)** we have the following lemma [62, Lemma 2.11] (see also [63, Lemma 2.9])

Lemma 3.12. *Assume* **(A2)**. *For any $a \in (\bar{a}, \infty)$ the pullback semigroup T_t^∞ defined by*

$$\forall\, f \in \mathcal{B}\mathbf{P}_{\mathcal{G},a},\ \Phi \in C_b(\mathcal{B}\mathbf{P}_{\mathcal{G},a}), \quad T_t^\infty[\Phi](f) := \Phi\left(S_t^{NL}(f)\right)$$

is a C_0-semigroup of contractions on the Banach space $C_b(\mathcal{B}\mathbf{P}_{\mathcal{G},a}(E))$.

Its generator G^∞ is an unbounded linear operator on $C_b(\mathcal{B}\mathbf{P}_{\mathcal{G},a})$ with domain $Dom(G^\infty)$ containing $C_b^{1,\eta}(\mathcal{B}\mathbf{P}_{\mathcal{G},a})$. On the latter space, it is defined by the formula

$$\forall\, \Phi \in C_b^{1,\eta}(\mathcal{B}\mathbf{P}_{\mathcal{G},a}),\ \forall\, f \in \mathcal{B}\mathbf{P}_{\mathcal{G},a}, \quad (G^\infty \Phi)(f) := \langle D\Phi[f], Q(f) \rangle. \tag{3.66}$$

Assumption (A3) (Convergence of the generators)**.** Let $\mathbf{P}_{\mathcal{G}_1}, m_{\mathcal{G}_1}, \mathbf{R}_{\mathcal{G}_1}$ be such as introduced in **(A2)**. Define a weight function $1 \le m'_{\mathcal{G}_1} \le C m_{\mathcal{G}_1}$ and then the corresponding weight $\Lambda_1(f) := \left\langle f, m'_{\mathcal{G}_1} \right\rangle$.

We assume that there exist a function $\varepsilon(N)$ going to 0 as $N \to \infty$ and $\eta \in (0,1]$ such that for all $\Phi \in C^{2,\eta}_{\Lambda_1}(\mathbf{P}_{\mathcal{G}_1,\mathbf{r}}; \mathbb{R})$ we have

$$\left\| \left(M^N_{m_{\mathcal{G}_1}} \right)^{-1} \left(G^N \pi_N - \pi_N G^\infty \right) \Phi \right\|_{L^\infty(\mathbb{E}_N)} \le \varepsilon(N) \sup_{\mathbf{r} \in \mathbf{R}_{\mathcal{G}_1}} \left([\Phi]_{C^{1,\eta}_{\Lambda_1}(\mathbf{P}_{\mathcal{G}_1,\mathbf{r}};\mathbb{R})} + [\Phi]_{C^{2,0}_{\Lambda_1}(\mathbf{P}_{\mathcal{G}_1,\mathbf{r}};\mathbb{R})} \right). \tag{3.67}$$

Assumption (A4) (Differential stability)**.** We assume that the flow

$$S^\infty_t \in C^{2,\eta}_{\Lambda_2} \cap C^{1,(1+2\eta)/3}_{\Lambda_2} \cap C^{0,(2+\eta)/3}_{\Lambda_2}(\mathbf{P}_{\mathcal{G}_1,\mathbf{r}}; \mathbf{P}_{\mathcal{G}_2}),$$

for any $\mathbf{r} \in \mathbf{R}_{\mathcal{G}_1}$, and that there exists $C^\infty_4 > 0$ such that

$$\sup_{\mathbf{r} \in \mathbf{R}_{\mathcal{G}_1}} \int_0^\infty \left([S^\infty_t]_{C^{1,(1+2\eta)/3}_{\Lambda_2}} + [S^\infty_t]^2_{C^{0,(2+\eta)/3}_{\Lambda_2}} + [S^\infty_t]_{C^{2,0}_{\Lambda_2}} + [S^\infty_t]^2_{C^{1,0}_{\Lambda_2}} \right) dt \le C^\infty_4, \tag{3.68}$$

where $\mathbf{P}_{\mathcal{G}_2}$ is the same probability space as $\mathbf{P}_{\mathcal{G}_1}$ but endowed with the norm associated to some Banach space $\mathcal{G}_2 \supset \mathcal{G}_1$, with $\Lambda_2 = \Lambda_1^{\frac{1}{3}}$ and η is the same as in **(A3)**.

Assumption (A5) (Weak stability)**.** We assume that, for some probability space $\mathbf{P}_{\mathcal{G}_3}$ associated to a weight function $m_{\mathcal{G}_3}$, a constraint $\mathbf{m}_{\mathcal{G}_3}$, a set of contraints $\mathbf{R}_{\mathcal{G}_3}$ and a distance $\mathrm{dist}_{\mathcal{G}_3}$, for any $a > 0$ there exists a constant $C^\infty_5 > 0$ such that for any $\mathbf{r} \in \mathbf{R}_{\mathcal{G}_3}$,

$$\forall\, f, g \in \mathcal{B}\mathbf{P}_{\mathcal{G}_3,a,\mathbf{r}}, \quad \sup_{t \ge 0} \mathrm{dist}_{\mathcal{G}_3} \left(S^\infty_t(f), S^\infty_t(g) \right) \le C^\infty_5 \, \mathrm{dist}_{\mathcal{G}_3} \left(f, g \right). \tag{3.69}$$

Theorem 3.13 (Abstract theorem)**.** *Let us consider a family of N-particle initial conditions $F^N_0 \in \mathbf{P}_{\mathrm{sym}}(E^N)$ and the solution associated $F^N_t = S^N_t(F^N_0)$. Consider also a one-particle initial condition $f_0 \in \mathbf{P}(E)$ and the solution associated $f_t = S^\infty_t(f_0)$. Assume that* **(A1)-(A2)-(A3)-(A4)-(A5)** *hold for some spaces $\mathbf{P}_{\mathcal{G}_i}$, $\mathcal{G}_i$ and $\mathcal{F}_i$, $i = 1, 2, 3$, and where $\mathcal{G}_i$ and $\mathcal{F}_i$ are in duality.*

Then there exists a constant $C \in (0, \infty)$ such that for any $N, \ell \in \mathbb{N}$, with $N \ge 2\ell$, and for any $\varphi = \varphi_1 \otimes \cdots \otimes \varphi_\ell \in \mathcal{F}^{\otimes \ell}$, $\mathcal{F} := \mathcal{F}_1 \cap \mathcal{F}_2 \cap \mathcal{F}_3$ we have

$$\begin{aligned} &\sup_{t \ge 0} \left| \left\langle S^N_t(F^N_0) - (S^\infty_t(f_0))^{\otimes N}, \varphi \otimes \mathbf{1}^{N-\ell} \right\rangle \right| \\ &\le C \left[\frac{\ell^2}{N} \|\varphi\|_{L^\infty} + C_{m_{\mathcal{G}_1}} C^\infty_4 \, \varepsilon(N) \, \ell^3 \, \|\varphi\|_{\mathcal{F}_2^{\otimes 3} \otimes (L^\infty)^{\ell-3}} + C^\infty_5 \, \ell \, \|\varphi\|_{\mathcal{F}_3 \otimes (L^\infty)^{\ell-1}} \mathcal{W}_{1,\mathcal{G}_3}(\pi^N_P F^N_0, \delta_{f_0}) \right]. \end{aligned} \tag{3.70}$$

As a consequence, if F^N_0 is f_0-chaotic the propagation of chaos holds.

Proof of Theorem 3.13. We split the term (3.70) in three parts :

$$\begin{aligned}
&\left|\left\langle S_t^N(F_0^N) - (S_t^\infty(f_0))^{\otimes N}, \varphi \otimes \mathbf{1}^{N-\ell}\right\rangle\right| \\
&\quad \leq \left|\left\langle S_t^N(F_0^N), \varphi \otimes \mathbf{1}^{N-\ell}\right\rangle - \left\langle S_t^N(F_0^N), R_\varphi^\ell \circ \mu_V^N\right\rangle\right| \qquad (=: T_1) \\
&\quad + \left|\left\langle F_0^N, T_t^N(R_\varphi^\ell \circ \mu_V^N)\right\rangle - \left\langle F_0^N, (T_t^\infty R_\varphi^\ell) \circ \mu_V^N\right\rangle\right| \qquad (=: T_2) \\
&\quad + \left|\left\langle F_0^N, (T_t^\infty R_\varphi^\ell) \circ \mu_V^N\right\rangle - \left\langle (S_t^\infty(f_0))^{\otimes \ell}, \varphi\right\rangle\right| \qquad (=: T_3)
\end{aligned}$$

and we evaluate each of them.

Step 1. For the first term T_1, a classical combinatorial trick (see [70] and [63, Lemma 2.14]) implies

$$T_1 \leq \frac{2\ell^2 \|\varphi\|_{L^\infty(E^\ell)}}{N}.$$

Step 2. For the term T_2, we have the following identity

$$\begin{aligned}
T_t^N \pi_N - \pi_N T_t^\infty &= -\int_0^t \frac{d}{dt}[T_{t-s}^N \, \pi_N \, T_s^\infty] ds \\
&= \int_0^t T_{t-s}^N \left(G^N \pi_N - \pi_N G^\infty\right) T_s^\infty ds
\end{aligned}$$

and then for any $t \geq 0$

$$\begin{aligned}
T_2 &\leq \int_0^\infty \left|\left\langle (M_{m_{\mathcal{G}_1}}^N) S_{t-s}^N(F_0^N), (M_{m_{\mathcal{G}_1}}^N)^{-1}[G^N \pi_N - \pi_N G^\infty] \, T_s^\infty R_\varphi^\ell \right\rangle\right| ds \\
&\leq \sup_{t \geq 0} \left\langle M_{m_{\mathcal{G}_1}}^N, S_t^N(F_0^N)\right\rangle \int_0^\infty \left\|(M_{m_{\mathcal{G}_1}}^N)^{-1}[G^N \pi_N - \pi_N G^\infty] \, T_s^\infty R_\varphi^\ell\right\|_{L^\infty(\mathbb{E}_N)} ds \qquad (3.71) \\
&\leq C_{m_{\mathcal{G}_1}} \, \varepsilon(N) \sup_{\mathbf{r} \in \mathbf{R}_{\mathcal{G}_1}} \int_0^\infty \left(\left[T_s^\infty R_\varphi^\ell\right]_{C_{\Lambda_1}^{1,\eta}(\mathbf{P}_{\mathcal{G}_1,\mathbf{r}};\mathbb{R})} + \left[T_s^\infty R_\varphi^\ell\right]_{C_{\Lambda_1}^{2,0}(\mathbf{P}_{\mathcal{G}_1,\mathbf{r}};\mathbb{R})} \right) ds,
\end{aligned}$$

thanks to **(A1)-(i)** and **(A3)**.

Furthermore we know that the application $T_s^\infty(R_\varphi^\ell) = R_\varphi^\ell \circ S_s^\infty$ and by assumption **(A4)** the semigroup

$$S_s^\infty \in C_{\Lambda_2}^{2,\eta} \cap C_{\Lambda_2}^{1,(1+2\eta)/3} \cap C_{\Lambda_2}^{0,(2+\eta)/3}(\mathbf{P}_{\mathcal{G}_1,\mathbf{r}}; \mathbf{P}_{\mathcal{G}_2}).$$

Moreover, with $\varphi \in \mathcal{F}_2^{\otimes \ell}$ we have $R_\varphi^\ell \in C^{2,1}(\mathbf{P}_{\mathcal{G}_2}; \mathbb{R})$ (see example 3.8). Finally, by Lemma 3.7 we obtain that

$$T_s^\infty(R_\varphi^\ell) \in C_{\Lambda_2^3}^{2,\eta} \cap C_{\Lambda_2^3}^{1,(1+2\eta)/3} \cap C_{\Lambda_2^3}^{0,(2+\eta)/3}(\mathbf{P}_{\mathcal{G}_1,\mathbf{r}}; \mathbb{R})$$

with

$$\left[T_s^\infty R_\varphi^\ell\right]_{C_{\Lambda_2^3}^{1,(1+2\eta)}(\mathbf{P}_{\mathcal{G}_1,\mathbf{r}};\mathbb{R})} \leq \|R_\varphi^\ell\|_{C^{2,1}(\mathbf{P}_{\mathcal{G}_2};\mathbb{R})} \left([S_t^\infty]_{C_{\Lambda_2}^{1,(1+2\eta)/3}(\mathbf{P}_{\mathcal{G}_1,\mathbf{r}};\mathbf{P}_{\mathcal{G}_2})} + [S_t^\infty]^2_{C_{\Lambda_2}^{0,(2+\eta)/3}(\mathbf{P}_{\mathcal{G}_1,\mathbf{r}};\mathbf{P}_{\mathcal{G}_2})} \right),$$

$$\left[T_s^\infty R_\varphi^\ell\right]_{C_{\Lambda_2^3}^{2,0}(\mathbf{P}_{\mathcal{G}_1,\mathbf{r}};\mathbb{R})} \leq \|R_\varphi^\ell\|_{C^{2,1}(\mathbf{P}_{\mathcal{G}_2};\mathbb{R})} \left([S_t^\infty]_{C_{\Lambda_2}^{2,0}(\mathbf{P}_{\mathcal{G}_1,\mathbf{r}};\mathbf{P}_{\mathcal{G}_2})} + [S_t^\infty]^2_{C_{\Lambda_2}^{1,0}(\mathbf{P}_{\mathcal{G}_1,\mathbf{r}};\mathbf{P}_{\mathcal{G}_2})} \right).$$

From assumption **(A4)**, $\Lambda_2 = \Lambda_1^{1/(3)}$ and the estimate of $\|R_\varphi^\ell\|_{C^{2,1}}$ in example 3.8, we can deduce, plugging the last estimate on (3.71),

$$T_2 \le C_{m_{\mathcal{G}_1}}\, C_4^\infty\, \varepsilon(N)\, \ell^3\, \|\varphi\|_{\mathcal{F}_2^{\otimes 3}\otimes (L^\infty)^{\ell-3}}. \tag{3.72}$$

Step 3. For the third term T_3 we deduce, thanks to the assumption **(A5)** and the fact that $R_\varphi^\ell \in C^{0,1}(\mathbf{P}_{\mathcal{G}_3};\mathbb{R})$ for $\varphi \in \mathcal{F}_3^{\otimes \ell}$,

$$\begin{aligned} T_3 &= \left|\left\langle F_0^N, R_\varphi^\ell(S_t^\infty(\mu_V^N)) - R_\varphi^\ell(S_t^\infty(f_0))\right\rangle\right| \\ &\le \|R_\varphi^\ell\|_{C^{0,1}(\mathbf{P}_{\mathcal{G}_3};\mathbb{R})} \left\langle F_0^N, \mathrm{dist}_{\mathcal{G}_3}(S_t^\infty(\mu_V^N), S_t^\infty(f_0))\right\rangle \\ &\le \ell\, \|\varphi\|_{\mathcal{F}_3\otimes (L^\infty)^{\ell-1}}\, C_5^\infty \left\langle F_0^N, \mathrm{dist}_{\mathcal{G}_3}(\mu_V^N, f_0)\right\rangle. \end{aligned} \tag{3.73}$$

By the definition of the Wasserstein distance

$$\forall\, \alpha,\beta \in \mathbf{P}(\mathbf{P}_{\mathcal{G}_3}) \qquad \mathcal{W}_{1,\mathcal{G}_3}(\alpha,\beta) := \inf_{\pi\in\Pi(\alpha,\beta)} \int_{\mathbf{P}_{\mathcal{G}_3}\times\mathbf{P}_{\mathcal{G}_3}} \mathrm{dist}_{\mathcal{G}_3}(f,g)\, \pi(df,dg)$$

with $\Pi(\alpha,\beta)$ the set of probability measures over $\mathbf{P}_{\mathcal{G}_3}\times\mathbf{P}_{\mathcal{G}_3}$ with marginals α and β respectively. Taking $\beta = \delta_{f_0}$, the set $\Pi(\alpha,\delta_{f_0}) = \{\alpha\otimes\delta_{f_0}\}$ has only one element and we obtain, with $\alpha = \pi_P^N F_0^N$,

$$\begin{aligned} \mathcal{W}_{1,\mathcal{G}_3}(\pi_P^N F_0^N, \delta_{f_0}) &:= \inf_{\pi\in\Pi(\pi_P^N F_0^N,\delta_{f_0})} \iint_{\mathbf{P}_{\mathcal{G}_3}\times\mathbf{P}_{\mathcal{G}_3}} \mathrm{dist}_{\mathcal{G}_3}(f,g)\pi(df,dg) \\ &= \iint_{\mathbf{P}_{\mathcal{G}_3}\times\mathbf{P}_{\mathcal{G}_3}} \mathrm{dist}_{\mathcal{G}_3}(f,g)\, \pi_P^N F_0^N(df)\, \delta_{f_0}(dg) \\ &= \int_{\mathbf{P}_{\mathcal{G}_3}} \mathrm{dist}_{\mathcal{G}_3}(f,f_0)\, \pi_P^N F_0^N(df) \\ &= \int_{E^N} \mathrm{dist}_{\mathcal{G}_3}(\mu_V^N, f_0)\, F_0^N(dV). \end{aligned} \tag{3.74}$$

where we have used the definition of π_P^N (see subsection 3.3.1) in the last equality. We conclude then plugging this estimate on (3.73).

□

3.4 Application to the Landau equation

In this section we will use the consistency-stability method presented in the Section 3.3 to show the propagation of chaos for the Landau equation for maxwellian molecules. To prove some estimates for the Landau equation we will prove first the same estimates for the Boltzmann equation (as in [62]) with a collisional kernel satisfying the grazing collisions (3.21). Then passing to the limit we will recover the same results for Landau.

Our main theorems are:

Theorem 3.14. *Consider a N-particle initial condition $F_0^N \in \mathbf{P}_{\mathrm{sym}}(\mathbb{R}^{dN})$ and, for all $t \geq 0$, the associated solution of the N-particle Landau dynamics $F_t^N = S_t^N(F_0^N)$. Consider also a one-particle initial condition $f_0 \in \mathbf{P}_2(\mathbb{R}^d)$, with zero momentum $\int v f_0 = 0$ and energy $\int |v|^2 f_0 =: \mathcal{E} \in (0,\infty)$, and the associated solution of the limit (mean-field) Landau equation $f_t = S_t^\infty(f_0)$. Suppose further that there exists $\mathcal{E}_0 \in (0,\infty)$ such that*

$$\operatorname{supp} F_0^N \subset \left\{ V \in \mathbb{R}^{dN},\ \frac{1}{N}\sum_{i=1}^N |v_i|^2 \leq \mathcal{E}_0 \right\}. \tag{3.75}$$

Then, for $\ell \in \mathbb{N}^*$, for all

$$\varphi = \varphi_1 \otimes \cdots \otimes \varphi_\ell \in \mathcal{F}^{\otimes \ell}, \quad \mathcal{F} := \left\{ \varphi : \mathbb{R}^d \to \mathbb{R}^d;\ \|\varphi\|_{\mathcal{F}} = \int_{\mathbb{R}^d} (1+|\xi|^6)|\hat\varphi(\xi)|\, d\xi < \infty \right\}$$

we have, for any $N \geq 2\ell$,

$$\begin{aligned}\sup_{t\geq 0} &\left| \left\langle S_t^N(F_0^N) - (S_t^\infty(f_0))^{\otimes N}, \varphi \right\rangle \right| \\ &\leq C \left[\frac{\ell^2}{N} \|\varphi\|_{L^\infty} + C_4^\infty \frac{\ell^3}{N} \|\varphi\|_{\mathcal{F}^3 \otimes (L^\infty)^{\ell-3}} + C_5^\infty\, \ell\, \|\varphi\|_{W^{1,\infty}\otimes (L^\infty)^{\ell-1}}\, \mathcal{W}_{1,W_2}(\pi_P^N F_0^N, \delta_{f_0}) \right].\end{aligned}$$

As a consequence, if F_0^N is f_0-chaotic the third term of the right-hand side goes to 0 when $N \to \infty$, which implies the propagation of chaos.

Theorem 3.15. *Consider the same framework of Theorem 3.14. Assume moreover that $f_0 \in \mathbf{P}_6(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$ for some $p > 1$ and let $F_0^N := [f_0^{\otimes N}]_{\mathcal{S}^N(\mathcal{E})} \in \mathbf{P}_{\mathrm{sym}}(\mathcal{S}^N(\mathcal{E}))$ (observe that (3.75) is satisfied for this choice of initial data with $\mathcal{E}_0 = \mathcal{E}$). Then it holds:*

(1) For all $0 < \epsilon < 9[(7d+6)^2(d+9)]^{-1}$, there exists a constant $C_\epsilon > 0$ such that

$$\sup_{t\geq 0} \frac{W_1(F_t^N, f_t^{\otimes N})}{N} \leq C_\epsilon\, N^{-\epsilon}.$$

(2) For all $t \geq 0$, for all $N \in \mathbb{N}^*$

$$\frac{W_1(F_t^N, \gamma^N)}{N} \leq p(t),$$

for a polynomial rate $p(t) \to 0$ as $t \to \infty$ and where γ^N is the uniform probability measure on $\mathcal{S}^N(\mathcal{E})$.

Remark 3.16. This theorem also holds (with different quantitative rates) for other choices of initial data F_0^N that are f_0-chaotic. In particular, if we consider $f_0 \in \mathbf{P}_2(\mathbb{R}^d)$ with compact support and $F_0^N = f_0^{\otimes N} \in \mathbf{P}_{\mathrm{sym}}(\mathbb{R}^{dN})$.

The proof of Theorem 3.14 relies on the proof of assumptions **(A1)-(A2)-(A3)-(A4)-(A5)**, with a suitable choice of spaces, and then on the application of Theorem 3.13. Furthermore, we shall prove Theorem 3.15 using Theorem 3.14 and some results from [46, 62, 19].

3.4.1 Proof of assumption A1

Consider the N-particle SDE (3.36)-(3.37). Since b and σ are Lipschitz, existence and uniqueness hold by standard arguments (see [69, Chapter 5]). Hence it defines a semigroup T_t^N, we can then define its generator $G^N = G_L^N$ (given by (3.31)-(3.32)) and its dual semigroup S_t^N, as explained in section 3.3.

We have the following lemma.

Lemma 3.17. *The dynamics of the N-particle system* (3.32) *conserves momentum and energy, more precisely there holds, for all $t \geq 0$,*

$$\int_{\mathbb{R}^{dN}} \varphi\left(\sum_{i=1}^{N} v_{i,\alpha}\right) F_t^N(dV) = \int_{\mathbb{R}^{dN}} \varphi\left(\sum_{i=1}^{N} v_{i,\alpha}\right) F_0^N(dV), \qquad \alpha \in \{1,\ldots,d\}$$

and

$$\int_{\mathbb{R}^{dN}} \varphi\left(|V|^2\right) F_t^N(dV) = \int_{\mathbb{R}^{dN}} \varphi\left(|V|^2\right) F_0^N(dV).$$

Remark 3.18. We can easily observe during the proof that if we consider the N-particle system of Remark 3.2 with generator G_2^N (3.35), which is different from the system we considered here (3.32), we have conservation of energy

$$\partial_t \left\langle f_t^N, \sum_{i=1}^{N} |v_i|^2 \right\rangle = \left\langle f_t^N, G_2^N \sum_{i=1}^{N} |v_i|^2 \right\rangle = 0,$$

however this is not true for all functions $\varphi = \varphi(|V|^2)$. Then, Lemma 3.19, which is a consequence os this lemma, does not hold for the N-particle system of Remark 3.2.

Proof of Lemma 3.17. Let us prove the second equality (energy conservation), the proof of the first one (momentun conservation) being similar. Consider the Landau master equation (3.32) and $\varphi(V) = \varphi(|V|^2)$ smooth enough, we have then

$$\nabla_i(\varphi(|V|^2)) = \left(\partial_{v_{i,\alpha}}\varphi(|V|^2)\right)_{1\leq\alpha\leq d} = 2\varphi'(|V|^2)v_i$$

and, for $i \neq j$,

$$\begin{aligned}\partial_{v_{i,\alpha}}\partial_{v_{j,\beta}}\varphi(|V|^2) &= 4\varphi''(|V|^2)v_{i,\alpha}v_{j,\beta}\\ \partial_{v_{i,\alpha}}\partial_{v_{i,\beta}}\varphi(|V|^2) &= 4\varphi''(|V|^2)v_{i,\alpha}v_{i,\beta} + 2\varphi'(|V|^2)\delta_{\alpha\beta}.\end{aligned}$$

Denoting $\varphi' = \varphi'(|V|^2)$ and $\varphi'' = \varphi''(|V|^2)$ for simplicity, we obtain

$$\begin{aligned}(\nabla_{ii}^2\varphi + \nabla_{jj}^2\varphi - \nabla_{ij}^2\varphi - \nabla_{ji}^2\varphi)_{\alpha\beta} &= 4\varphi'\delta_{\alpha\beta} + 4\varphi''\left(v_{i,\alpha}v_{i,\beta} + v_{j,\alpha}v_{j,\beta} - v_{i,\alpha}v_{j,\beta} - v_{j,\alpha}v_{i,\beta}\right)\\ &= 4\varphi'\delta_{\alpha\beta} + 4\varphi''(v_i - v_j)_\alpha(v_i - v_j)_\beta.\end{aligned}$$

Therefore we have

$$\begin{aligned}b(v_i - v_j)(\nabla_i\varphi(|V|^2) - \nabla_j\varphi(|V|^2)) &= -2|v_i - v_j|^\gamma(v_i - v_j)\cdot 2\varphi'(|V|^2)(v_i - v_j)\\ &= -4\varphi'(|V|^2)|v_i - v_j|^{\gamma+2}\end{aligned} \tag{3.76}$$

and

$$
\begin{aligned}
&a(v_i - v_j) : (\nabla^2_{ii}\varphi + \nabla^2_{jj}\varphi - \nabla^2_{ij}\varphi - \nabla^2_{ji}\phi) = \\
&= |v_i - v_j|^\gamma \sum_{\alpha,\beta=1}^{d} \Bigg\{ \Big[|v_i - v_j|^2 \delta_{\alpha\beta} - (v_i - v_j)_\alpha (v_i - v_j)_\beta \Big] 4\varphi' \delta_{\alpha\beta} \\
&+ \Big[|v_i - v_j|^2 \delta_{\alpha\beta} - (v_i - v_j)_\alpha (v_i - v_j)_\beta \Big] 4\varphi'' (v_i - v_j)_\alpha (v_i - v_j)_\beta \Bigg\} \\
&=: |v_i - v_j|^\gamma \{T_1 + T_2\}.
\end{aligned}
$$

Computing T_1 we have

$$
\begin{aligned}
T_1 &= 4\varphi' \sum_{\alpha,\beta=1}^{d} \Big[|v_i - v_j|^2 \delta_{\alpha\beta} - (v_i - v_j)_\alpha (v_i - v_j)_\beta \Big] \delta_{\alpha\beta} \\
&= 8\varphi' |v_i - v_j|^2,
\end{aligned}
\tag{3.77}
$$

and computing T_2

$$
\begin{aligned}
T_2 &= 4\varphi'' \sum_{\alpha,\beta=1}^{d} \Big[|v_i - v_j|^2 (v_i - v_j)_\alpha (v_i - v_j)_\beta \delta_{\alpha\beta} - (v_i - v_j)^2_\alpha (v_i - v_j)^2_\beta \Big] \\
&= 4\varphi'' \left\{ |v_i - v_j|^4 - \left[\sum_{\alpha=1}^{d} (v_i - v_j)^2_\alpha \right]^2 \right\} \\
&= 0.
\end{aligned}
\tag{3.78}
$$

Gathering (3.76), (3.77) and (3.78) we obtain

$$
\begin{aligned}
&b(v_i - v_j)(\nabla_i \varphi - \nabla_j \varphi) + \frac{1}{2} a(v_i - v_j) : (\nabla^2_{ii}\varphi + \nabla^2_{jj}\varphi - \nabla^2_{ij}\varphi - \nabla^2_{ji}\varphi) \\
&\quad = -4\varphi' \, |v_i - v_j|^{\gamma+2} + \frac{1}{2} 8\varphi' \, |v_i - v_j|^{\gamma+2} \\
&\quad = 0,
\end{aligned}
\tag{3.79}
$$

which implies, for all $t \geq 0$,

$$
\int \varphi(|V|^2) \, F_t^N(dV) = \int \varphi(|V|^2) \, F_0^N(dV). \tag{3.80}
$$

□

Lemma 3.19. *Consider F_0^N such that*

$$
\operatorname{supp} F_0^N \subset \{ V \in \mathbb{R}^{dN};\ M_2^N(V) = \frac{1}{N} \sum_{i=1}^{N} |v_i|^2 \leq \mathcal{E}_0 \}.
$$

Then there holds

$$
\forall\, t > 0, \quad \operatorname{supp} F_t^N \subset \{ V \in \mathbb{R}^{dN};\ M_2^N(V) \leq \mathcal{E}_0 \}.
$$

Proof of Lemma 3.19. It is a consequence of Lemma 3.17, with $\varphi(|V|^2) = \mathbf{1}_{|V|^2 > N\mathcal{E}_0}$. Consider a mollifier ρ_η for $\eta > 0$, i.e. $\rho_\eta(x) = \eta^{-1}\rho(\eta^{-1}x)$, with $\rho \in C_c^\infty(\mathbb{R})$, $\rho \geq 0$ and $\operatorname{supp}\rho \subset B_1$, and define $\varphi_\eta = \rho_\eta * \varphi$. Using Lemma 3.17 we have, for all η and for all $t \geq 0$,

$$\int_{\mathbb{R}^{dN}} \varphi_\eta \, F_t^N(dV) = \int_{\mathbb{R}^{dN}} \varphi_\eta \, F_0^N(dV).$$

Passing to the limit $\eta \to 0$ we obtain

$$\int_{\mathbb{R}^{dN}} \mathbf{1}_{|V|^2 > N\mathcal{E}_0} \, F_t^N(dV) = \int_{\mathbb{R}^{dN}} \mathbf{1}_{|V|^2 > N\mathcal{E}_0} \, F_0^N(dV) = 0.$$

□

Lemma 3.20. *Consider F_0^N such that $\left\langle F_0^N, M_k^N \right\rangle \leq C_k$ for $k > 2$. Then there holds*

$$\sup_{t \geq 0} \left\langle F_t^N, M_k^N \right\rangle \leq C_k.$$

Proof of Lemma 3.20. Consider $F_t^{N,\varepsilon}$ the solution of the Boltzmann N-particle system (3.11)-(3.12) with grazing collisions (3.21). Then from [62, Lemma 5.3], we obtain the desired result for $F_t^{N,\varepsilon}$ with a constant independent of ε. We conclude passing to the grazing collisions limit $\varepsilon \to 0$. □

Consider the constraint function $\mathbf{m}_{\mathcal{G}_1} : \mathbb{R}^d \to \mathbb{R}_+ \times \mathbb{R}^d$, $\mathbf{m}_{\mathcal{G}_1}(v) = (|v|^2, v)$ with the set of constraints $\mathbf{R}_{\mathcal{G}_1} : \{(r, \bar{r}) \in \mathbb{R}_+ \times \mathbb{R}^d;\ |\bar{r}|^2 \leq r \leq \mathcal{E}_0\}$. Then Lemma 3.19 proves **(A1i)**. Moreover, Lemma 3.20 proves **(A1ii)** with the weight function $m_{\mathcal{G}_1}(v) := \langle v \rangle^6 = (1 + |v|^2)^3$ for all $v \in \mathbb{R}^d$.

3.4.2 Proof of assumption A2

Let us define the spaces of probabilities (and the corresponding bounded, constrained and increments subsets, see Definition 3.3)

$$\mathbf{P}_{\mathcal{G}_1} := \{f \in \mathbf{P}(\mathbb{R}^d);\ M_6(f) < \infty\},$$

and, for $\mathbf{r} \in \mathbf{R}_{\mathcal{G}_1}$, more precisely $\mathbf{r} = (r, \bar{r}) = (r, r_1, \ldots, r_d)$, the constrained space

$$\mathbf{P}_{\mathcal{G}_1,\mathbf{r}} := \{f \in \mathbf{P}_{\mathcal{G}_1};\ \langle f, |v|^2 \rangle = r,\ \langle f, v_i \rangle = r_i \text{ for } i = 1, \ldots, d\}.$$

We define then the for some $a \in (0, \infty)$ the bounded set

$$\mathcal{B}\mathbf{P}_{\mathcal{G}_1,a} := \{f \in \mathbf{P}_{\mathcal{G}_1};\ M_6(f) \leq a\},$$

and, for any $\mathbf{r} \in \mathbf{R}_{\mathcal{G}_1}$, the bounded constrained set

$$\mathcal{B}\mathbf{P}_{\mathcal{G}_1,a,\mathbf{r}} := \{f \in \mathcal{B}\mathbf{P}_{\mathcal{G}_1,a};\ \langle f, |v|^2 \rangle = r,\ \langle f, v_i \rangle = r_i \text{ for } i = 1, \ldots, d\}.$$

We endow these spaces with the distance $\operatorname{dist}_{\mathcal{G}_1}$ associated with the norm $\|\cdot\|_{\mathcal{G}_1} = |\cdot|_2$ (see definitions 3.10 and 3.11).

We have the following lemma.

Lemma 3.21. *Let $f_0, g_0 \in \mathbf{P}_2(\mathbb{R}^d)$ with same momentum, i.e. $\langle f_0, v\rangle = \langle g_0, v\rangle$, and consider the solutions f_t, g_t of Landau equation* (3.13)-(3.14). *Then*

$$\sup_{t\geq 0} |f_t - g_t|_2 \leq |f_0 - g_0|_2. \tag{3.81}$$

Remark 3.22. Let us mention that this result can be found in [74] proving uniqueness for the Landau equation for maxwellian molecules. There the author indicates that we can prove it using the known result for the Boltzmann equation for maxwellian molecules from [72] and then passing to the limit of grazing collisions.

Proof of Lemma 3.21. Let us split the prove in two steps. First we prove the lemma for the Boltzmann equation then we recover the result for Landau equation passing to the limit of grazing collisions.

Step 1. We shall prove the deasired result for the Boltzmann equation with true Maxwellian molecules. This result is prove in [72, 62], but we write it for completeness.

Consider the solutions f_t^ε and g_t^ε of Boltzmann equation (3.4)-(3.5) with initial data f_0 and g_0 respectively. Denote $d^\varepsilon := g^\varepsilon - f^\varepsilon$ and $s^\varepsilon := g^\varepsilon + f^\varepsilon$, then the equation satisfied for d is

$$\partial_t d^\varepsilon = \frac{1}{2}\Big[Q_{B,\varepsilon}(s^\varepsilon, d^\varepsilon) + Q_{B,\varepsilon}(d^\varepsilon, s^\varepsilon)\Big].$$

Performing the Fourier transform ([7]) and denoting $D^\varepsilon = \hat{d}^\varepsilon$, $S^\varepsilon = \hat{s}^\varepsilon$, we have

$$\partial_t D^\varepsilon(\xi) = \int_{\mathbb{S}^{d-1}} b_\varepsilon(\sigma\cdot\hat{\xi})\left[\frac{D^\varepsilon(\xi^+)S^\varepsilon(\xi^-)}{2} + \frac{D^\varepsilon(\xi^-)S^\varepsilon(\xi^+)}{2} - D^\varepsilon(\xi)\right] d\sigma$$

where $\xi^+ = \frac{\xi+|\xi|\sigma}{2}$, $\xi^- = \frac{\xi-|\xi|\sigma}{2}$ and $\hat{\xi} = \xi/|\xi|$.

We recall that b_ε is not integrable so we perform the following cut-off, which will be relaxed in the end,

$$\int_{\mathbb{S}^{d-1}} b_\varepsilon^K(\sigma\cdot\hat{\xi})d\sigma = K, \qquad b_\varepsilon^K = b_\varepsilon \mathbf{1}_{|\theta|\geq\delta(K)}, \tag{3.82}$$

for some function δ such that $\delta(K)\to 0$ as $K\to+\infty$, so that $b_\varepsilon = b_\varepsilon^K + b_\varepsilon^C$. In [72, 62], we observe that the remainder term

$$R_\varepsilon^K(\xi) := \int_{\mathbb{S}^{d-1}} b_\varepsilon^C(\sigma\cdot\hat{\xi})\left[\frac{D^\varepsilon(\xi^+)S^\varepsilon(\xi^-)}{2} + \frac{D^\varepsilon(\xi^-)S^\varepsilon(\xi^+)}{2} - D^\varepsilon(\xi)\right] d\sigma$$

verifies, for any $\xi\in\mathbb{R}^d$, $|R_\varepsilon^K(\xi)| \leq r_\varepsilon^K|\xi|^2$, where $r_\varepsilon^K \to 0$ as $K\to\infty$, and r_ε^K depends on the second order moments of d and s. Indeed, using that $D(0) = \partial_{\xi_i}D(0) = 0$ for all i, $S(0) = 2$, $\sup_{|\eta|\leq|\xi|}|\partial_{\xi_i}\partial_j D(\eta)|$ and $\sup_{|\eta|\leq|\xi|}|\partial_{\xi_i}\partial_j D(\eta)|$ are bounded thanks to the bounds on the second order moments of d and s, there holds

$$\begin{aligned}
&|D^\varepsilon(\xi^+)S^\varepsilon(\xi^-) + D^\varepsilon(\xi^-)S^\varepsilon(\xi^+) - 2D^\varepsilon(\xi)| \\
&\quad\leq |S^\varepsilon(\xi^-)||D^\varepsilon(\xi^+) - D^\varepsilon(\xi)| + |D^\varepsilon(\xi)||S^\varepsilon(\xi^-) - S^\varepsilon(0)| + |D^\varepsilon(\xi^-)||S^\varepsilon(\xi^+)| \\
&\quad\leq C|\xi|^2(1-\cos\theta)^{1/2},
\end{aligned}$$

and we conclude since $b_\varepsilon^C(\cos\theta)(1-\cos\theta)^{1/2}$ is integrable.

Using that $\|S^\varepsilon\|_\infty \le 2$, we have

$$\frac{d}{dt}\frac{|D^\varepsilon(\xi)|}{|\xi|^2} + K\frac{|D^\varepsilon(\xi)|}{|\xi|^2} \le \sup_{\xi\in\mathbb{R}^d}\frac{|D^\varepsilon(\xi)|}{|\xi|^2}\left(\sup_{\xi\in\mathbb{R}^d}\int_{\mathbb{S}^{d-1}} b_\varepsilon^K(\sigma\cdot\hat\xi)\left(|\hat\xi^+|^2+|\hat\xi^-|^2\right)d\sigma\right) + r_\varepsilon^K$$

with

$$|\hat\xi^+|^2 = \frac{1}{2}\left(1+\sigma\cdot\hat\xi\right), \qquad |\hat\xi^-|^2 = \frac{1}{2}\left(1-\sigma\cdot\hat\xi\right).$$

One obtains

$$\frac{d}{dt}\frac{|D^\varepsilon(\xi)|}{|\xi|^2} + K\frac{|D^\varepsilon(\xi)|}{|\xi|^2} \le K\sup_{\xi\in\mathbb{R}^d}\frac{|D^\varepsilon(\xi)|}{|\xi|^2} + r_\varepsilon^K,$$

and by a Gronwall's lemma one deduces

$$\sup_{\xi\in\mathbb{R}^d}\frac{|D_t^\varepsilon(\xi)|}{|\xi|^2} \le \sup_{\xi\in\mathbb{R}^d}\frac{|D_0^\varepsilon(\xi)|}{|\xi|^2} + t\,r_\varepsilon^K.$$

Relaxing the cut-off $K\to\infty$ one proves

$$|f_t^\varepsilon - g_t^\varepsilon|_2 \le |f_0 - g_0|. \tag{3.83}$$

Step 2. Since (3.83) does not depend on ε and the solution of the Boltzmann equation f_t^ε converges towards the solution of the Landau equation f_t (see [73]) when $\varepsilon\to 0$, we obtain the desired result.

□

Therefore we have that the Landau semigroup S_t^∞ is $C^{0,1}(\mathbf{P}_{\mathcal{G}_1};\mathbf{P}_{\mathcal{G}_1})$ and **(A2)-i** is proved.

To prove **(A2)-ii** we use [62, Lemma 5.5], valid for the Boltzmann operator with grazing collisions $Q_{B,\varepsilon}$, which says that there exists $C>0$ and $\delta\in(0,1]$ such that for any $f,g\in\mathcal{B}\mathbf{P}_{\mathcal{G}_1,a,\mathbf{r}}$ we have

$$|Q_{B,\varepsilon}(f,f) - Q_{B,\varepsilon}(g,g)|_2 \le C|f-g|_2^\delta,$$

with a constant C that does not depend on ε. Finally, passing to the limit of grazing collisions $\epsilon\to 0$, we have that $Q_{B,\varepsilon}\ \to\ Q_L$. We prove then **(A2)-ii** also for the Landau equation.

3.4.3 Proof of assumption A3

Let $\Lambda_1(f) := \langle f, m'_{\mathcal{G}_1}\rangle$ with the weight $m'_{\mathcal{G}_1}(v) := \langle v\rangle^4$, where we recall that $m_{\mathcal{G}_1} = \langle v\rangle^6$, and then consider the generator G^N of the Landau master equation (3.32).

Then we have the following lemma, which proves **(A3)**.

Lemma 3.23. *For all $\Phi \in \bigcap_{\mathbf{r}\in\mathbf{R}_{\mathcal{G}_1}} C^{2,\eta}_{\Lambda_1}(\mathbf{P}_{\mathcal{G}_1,\mathbf{r}};\mathbb{R})$ there exists $C>0$ such that*

$$\left\| \left(M^N_{m_{\mathcal{G}_1}}\right)^{-1} \left(G^N \pi_N - \pi_N G^\infty\right) \Phi \right\|_{L^\infty(\mathbb{E}_N)} \leq \frac{C}{N} \sup_{\mathbf{r}\in\mathbf{R}_{\mathcal{G}_1}} [\Phi]_{C^{2,0}_{\Lambda_1}(\mathbf{P}_{\mathcal{G}_1,\mathbf{r}};\mathbb{R})} . \tag{3.84}$$

Proof of Lemma 3.23. The application $\mathbb{R}^{dN} \to \mathbf{P}_{\mathcal{G}_1}$, $V \mapsto \mu^N_V$ is of class $C^{2,1}$ with (see [63, Lemma 5.4])

$$\begin{aligned} \partial_{i\alpha}(\mu^N_V) &= \frac{1}{N}\partial_\alpha \delta_{V_i} \\ \partial^2_{i\alpha,i\beta}(\mu^N_V) &= \frac{1}{N}\partial^2_{\alpha\beta}\delta_{V_i} \end{aligned} \tag{3.85}$$

and for $i \neq j$, $\partial^2_{i\alpha,j\beta}(\mu^N_V) = 0$.

Let $\Phi \in C^{2,\eta}_{\Lambda_1}(\mathbf{P}_{\mathcal{G}_1};\mathbb{R})$, so $\mathbb{R}^{dN} \to \mathbb{R}$, $V \mapsto \Phi(\mu^N_V)$ is also $C^{2,\eta}$. Indeed, let $\phi = D\Phi[\mu^N_V]$ and we have

$$\begin{aligned} \partial_{v_{i,\alpha}}(\Phi(\mu^N_V)) &= \langle D\Phi[\mu^N_V], \partial_{v_{i,\alpha}}\mu^N_V\rangle = \langle D\Phi[\mu^N_V], \frac{1}{N}\partial_{v_{i,\alpha}}\delta_{v_i}\rangle = \frac{1}{N}\partial_\alpha\phi(v_i) \\ \partial^2_{v_{i,\alpha},v_{i,\beta}}\Phi(\mu^N_V) &= \langle D\Phi[\mu^N_V], \frac{1}{N}\partial^2_{v_{i,\alpha},v_{i,\beta}}\delta_{v_i}\rangle + D^2\Phi[\mu^N_V]\left(\frac{1}{N}\partial_{v_{i,\alpha}}\delta_{v_i}, \frac{1}{N}\partial_{v_{i,\beta}}\delta_{v_i}\right) \\ &= \frac{1}{N}\partial^2_{\alpha,\beta}\phi(v_i) + \frac{1}{N^2}D^2\Phi[\mu^N_V]\left(\partial_{v_{i,\alpha}}\delta_{v_i}, \partial_{v_{i,\beta}}\delta_{v_i}\right). \end{aligned}$$

We compute

$$\begin{aligned} (G^N\pi^N\Phi)(V) &= G^N\Phi(\mu^N_V) \\ &= \frac{1}{N}\sum_{i,j=1}^N\sum_{\alpha=1}^d b_\alpha(v_i-v_j)\left[\partial_{v_{i,\alpha}}(\Phi(\mu^N_V)) - \partial_{v_{j,\alpha}}(\Phi(\mu^N_V))\right] \\ &+ \frac{1}{2N}\sum_{i,j=1}^N\sum_{\alpha,\beta=1}^d a_{\alpha\beta}(v_i-v_j)\left[\partial^2_{v_{i,\alpha},v_{i,\beta}}\Phi(\mu^N_V) + \partial^2_{v_{j,\alpha},v_{j,\beta}}\Phi(\mu^N_V)\right. \\ &\qquad \left. -\partial^2_{v_{i,\alpha},v_{j,\beta}}\Phi(\mu^N_V) - \partial^2_{v_{j,\alpha},v_{i,\beta}}\Phi(\mu^N_V)\right] \\ &= \frac{1}{N}\sum_{i,j=1}^N\sum_{\alpha=1}^d b_\alpha(v_i-v_j)\left[\frac{1}{N}\partial_\alpha\phi(v_i) - \frac{1}{N}\partial_\alpha\phi(v_j)\right] \\ &+ \frac{1}{2N}\sum_{i,j=1}^N\sum_{\alpha,\beta=1}^d a_{\alpha\beta}(v_i-v_j)\left[\frac{1}{N}\partial^2_{\alpha,\beta}\phi(v_i) + \frac{1}{N}\partial^2_{\alpha,\beta}\phi(v_j)\right] \quad (=: I_1) \\ &+ \frac{1}{2N}\sum_{i,j=1}^N\sum_{\alpha,\beta=1}^d a_{\alpha\beta}(v_i-v_j)\left[\frac{1}{N^2}D^2\Phi[\mu^N_V]\left(\partial_{v_{i,\alpha}}\delta_{v_i}, \partial_{v_{i,\beta}}\delta_{v_i}\right)\right. \\ &+ \left.\frac{1}{N^2}D^2\Phi[\mu^N_V]\left(\partial_{v_{j,\alpha}}\delta_{v_j}, \partial_{v_{j,\beta}}\delta_{v_j}\right) - 2\frac{1}{N^2}D^2\Phi[\mu^N_V]\left(\partial_{v_{i,\alpha}}\delta_{v_i}, \partial_{v_{j,\beta}}\delta_{v_j}\right)\right] \quad (=: I_2). \end{aligned}$$

For the first term we have

$$\begin{aligned}
I_1 = & \iint \sum_{\alpha=1}^{d} b_\alpha(v - v_*) \left[\partial_\alpha \phi(v) - \partial_\alpha \phi(v_*)\right] \mu_V^N(dv)\mu_V^N(dv_*) \\
& + \frac{1}{2} \iint \sum_{\alpha,\beta=1}^{d} a_{\alpha\beta}(v - v_*) \left[\partial^2_{\alpha,\beta}\phi(v) + \partial^2_{\alpha,\beta}\phi(v_*)\right] \mu_V^N(dv)\mu_V^N(dv_*) \\
= & \langle Q_L(\mu_V^N, \mu_V^N), \phi\rangle = \langle Q_L(\mu_V^N, \mu_V^N), D\Phi[\mu_V^N]\rangle = (\pi^N G^\infty \Phi)(V),
\end{aligned}$$

thanks to Lemma 3.12. For the second one, we deduce that there exists $C > 0$ depending on d such that

$$|I_2| \leq \frac{C}{N} \left| D^2\Phi[\mu_V^N] \left(\partial_{v_{i,\alpha}} \delta_{v_i}, \partial_{v_{i,\beta}} \delta_{v_i}\right) \right| \frac{1}{N^2} \sum_{i,j=1}^{N} |v_i - v_j|^2,$$

since $|a_{\alpha\beta}(v_i - v_j)| \leq |v_i - v_j|^2$. We conclude then

$$\begin{aligned}
|I_2| & \leq \frac{C}{N} \, [\Phi]_{C^{2,0}_{\Lambda_1}(\mathbf{P}_{\mathcal{G}_1,\mathbf{r}};\mathbb{R})} \, \Lambda_1(\mu_V^N) \frac{1}{N^2} \sum_{i,j=1}^{N} |v_i - v_j|^2 \\
& \leq \frac{C\mathcal{E}}{N} \, [\Phi]_{C^{2,0}_{\Lambda_1}(\mathbf{P}_{\mathcal{G}_1,\mathbf{r}};\mathbb{R})} \, M^N_{m_{\mathcal{G}_1}}(V)
\end{aligned}$$

and therefore we prove (3.84).

□

3.4.4 Proof of assumption A4

In the same way of the subsection 3.4.2, we will use here the Boltzmann equation and then perform the asymptotics of grazing collisions to prove the results for the Landau equation.

We define the following equations, denoting by Q the symmetrized version of the Landau operator Q_L (3.14), i.e. $Q(f,g) = [Q_L(f,g) + Q_L(g,f)]/2$,

$$\left\{ \begin{array}{ll}
\partial_t f = Q(f,f), & f|_{t=0} = f_0, \\
\partial_t g = Q(g,g), & g|_{t=0} = g_0, \\
\partial_t h = 2Q(f,h), & h|_{t=0} = g_0 - f_0, \\
\partial_t u = 2Q(f,u) + Q(h,h), & u|_{t=0} = 0,
\end{array} \right. \tag{3.86}$$

and the new variables

$$d := g - f, \quad s := g + f, \quad \omega := g - f - h, \quad \psi := g - f - h - u,$$

which satisfy

$$\left\{ \begin{array}{ll}
\partial_t d = Q(s,d), & d|_{t=0} = g_0 - f_0, \\
\partial_t \omega = Q(s,w) + Q(h,d), & \omega|_{t=0} = 0, \\
\partial_t \psi = Q(s,\psi) + Q(h,w) + Q(u,d), & \psi|_{t=0} = 0.
\end{array} \right. \tag{3.87}$$

Lemma 3.24. *Consider $f_0, g_0 \in \mathbf{P}_{\mathcal{G}_1,\mathbf{r}}$, $\mathbf{r} \in \mathbf{R}_{\mathcal{G}_1}$, and the solutions f_t, g_t, h_t of* (3.86)-(3.87)*. There exists $\lambda_1 \in (0,\infty)$ that for any $\eta \in [2/3, 1]$, there exists C_η such that we have*

$$\begin{aligned} |g_t - f_t|_2 &\leq C_\eta\, e^{-(1-\eta)\lambda_1 t}\, M_4(f_0+g_0)^{1/3}\, |g_0 - f_0|_2^\eta, \\ |h_t|_2 &\leq C_\eta\, e^{-(1-\eta)\lambda_1 t}\, M_4(f_0+g_0)^{1/3}\, |g_0 - f_0|_2^\eta. \end{aligned} \tag{3.88}$$

Proof of Lemma 3.24. We split the proof into two steps. Again, we shall first prove the lemma for Boltzmann equation with a kernel satisfying the grazing collisions, which is proved in [62], and then passing to the limit of grazing collisions we prove the same result for the Landau equation.

Step 1. Let us denote by Q_ε the symmetrized version of the Boltmann operator $Q_{B,\varepsilon}$ (3.5) with kernel b_ε satisfying (3.21), i.e. $Q_\varepsilon(f,g) = [Q_{B,\varepsilon}(f,g) + Q_{B,\varepsilon}(g,f)]/2$.

Consider the solutions f_t^ε, g_t^ε and h_t^ε of

$$\begin{cases} \partial_t f^\varepsilon = Q_\varepsilon(f^\varepsilon, f^\varepsilon), & f^\varepsilon|_{t=0} = f_0, \\ \partial_t g^\varepsilon = Q_\varepsilon(g^\varepsilon, g^\varepsilon), & g^\varepsilon|_{t=0} = g_0, \\ \partial_t h^\varepsilon = 2Q_\varepsilon(f^\varepsilon, h^\varepsilon), & h^\varepsilon|_{t=0} = g_0 - f_0, \end{cases} \tag{3.89}$$

and define $d^\varepsilon := g^\varepsilon - f^\varepsilon$ which satisfies (where $s^\varepsilon := g^\varepsilon + f^\varepsilon$)

$$\partial_t d^\varepsilon = Q_\varepsilon(s^\varepsilon, d^\varepsilon), \qquad d^\varepsilon|_{t=0} = g_0 - f_0.$$

As in Lemma 3.21 we denote $D^\varepsilon = \hat{d}^\varepsilon$ and $S^\varepsilon = \hat{s}^\varepsilon$. Define $\bar{D}^\varepsilon = D^\varepsilon - \hat{\mathcal{M}}_4[d^\varepsilon]$ (see definition 3.11). Then the equation safisfied for $\bar{D}^\varepsilon$ is

$$\partial_t \bar{D}^\varepsilon = \hat{Q}_\varepsilon(D^\varepsilon, S^\varepsilon) - \partial_t \hat{\mathcal{M}}_4[d^\varepsilon] = \hat{Q}_\varepsilon(\bar{D}^\varepsilon, S^\varepsilon) + \hat{Q}_\varepsilon(\hat{\mathcal{M}}_4[d^\varepsilon], S^\varepsilon) - \hat{\mathcal{M}}_4[Q_\varepsilon(d^\varepsilon, s^\varepsilon)].$$

From [62, Lemma 5.6] we know that, for any $\xi \in \mathbb{R}^d$,

$$\left| \hat{Q}_\varepsilon(\hat{\mathcal{M}}_4[d^\varepsilon], S^\varepsilon) - \hat{\mathcal{M}}_4[Q_\varepsilon(d^\varepsilon, s^\varepsilon)] \right| \leq C|\xi|^4 \sum_{|\alpha|\leq 3} |M_\alpha[f^\varepsilon - g^\varepsilon]|,$$

and also, from [71, Theorem 8.1], that there are constants $C, \delta > 0$ such that for all $t \geq 0$

$$\sum_{|\alpha|\leq 3} |M_\alpha[f_t^\varepsilon - g_t^\varepsilon]| \leq C e^{-\delta t} \sum_{|\alpha|\leq 3} |M_\alpha[f_0 - g_0]|.$$

Then, following [62] and performing the same cut-off as in the proof of lemma 3.21, we have that

$$\begin{aligned} \frac{d}{dt}\frac{|\bar{D}^\varepsilon|}{|\xi|^4} + K\frac{|\bar{D}^\varepsilon|}{|\xi|^4} \leq \left(\sup_{\xi\in\mathbb{R}^d} \frac{|\bar{D}^\varepsilon|}{|\xi|^4}\right) \left(\sup_{\xi\in\mathbb{R}^d} \int_{\mathbb{S}^{d-1}} b_\varepsilon^K(\sigma\cdot\hat{\xi}) \left(|\hat{\xi}^+|^4 + |\hat{\xi}^-|^4\right) d\sigma\right) \\ + C\, e^{-\delta t} \left(\sum_{|\alpha|\leq 3} |M_\alpha[f_0 - g_0]|\right) + \frac{|R_\varepsilon^K|}{|\xi|^4}. \end{aligned} \tag{3.90}$$

where the remainder term

$$R_\varepsilon^K(\xi) := \int_{\mathbb{S}^{d-1}} b_\varepsilon^C(\sigma \cdot \hat{\xi}) \left[\frac{1}{2} \bar{D}^\varepsilon(\xi^+) S^\varepsilon(\xi^-) + \frac{1}{2} \bar{D}^\varepsilon(\xi^-) S^\varepsilon(\xi^+) - \bar{D}^\varepsilon(\xi) \right] d\sigma$$

satisfies, for any $\xi \in \mathbb{R}^d$, $|R_\varepsilon^K(\xi)| \leq r_\varepsilon^K |\xi|^4$, with $r_\varepsilon^K \to 0$ as $K \to \infty$, and r_ε^K depends on the fourth order moments of d and s. Indeed, we have

$$\begin{aligned}
&|\bar{D}^\varepsilon(\xi^+) S^\varepsilon(\xi^-) + \bar{D}^\varepsilon(\xi^-) S^\varepsilon(\xi^+) - 2D(\xi)| \\
&\quad \leq |S^\varepsilon(\xi^-)| |\bar{D}^\varepsilon(\xi^+) - \bar{D}^\varepsilon(\xi^-)| + |\bar{D}^\varepsilon(\xi)| |S^\varepsilon(\xi^-) - S^\varepsilon(0)| + |\bar{D}^\varepsilon(\xi^-)| |S^\varepsilon(\xi^+)| \\
&\quad \leq C |\xi|^4 (1 - \cos\theta)^{1/2},
\end{aligned}$$

where we use that $\nabla_\xi^\alpha \bar{D}(0) = 0$ for all multi-index $|\alpha| \leq 3$ and, for $|\alpha| = 4$, $\sup_{|\eta| \leq |\xi|} \nabla_\xi^\alpha \bar{D}(\eta)$ and $\sup_{|\eta| \leq |\xi|} \nabla_\xi^\alpha S(\eta)$ are bounded thanks to the bounds on the fourth moment of d and s. As in lemma 3.21, the claim follows since $b_\varepsilon^C(\cos\theta)(1 - \cos\theta)^{1/2}$ is integrable.

We denote

$$\lambda_K := \int_{\mathbb{S}^{d-1}} b_\varepsilon^K(\sigma \cdot \hat{\xi}) \left(|\hat{\xi}^+|^4 + |\hat{\xi}^-|^4 \right) d\sigma = \int_{\mathbb{S}^{d-1}} b_\varepsilon^K(\sigma \cdot \hat{\xi}) \frac{1}{2} \left(1 + (\sigma \cdot \hat{\xi})^2 \right) d\sigma$$

and we compute

$$\begin{aligned}
\lambda_K - K &= -\frac{1}{2} \int_{\mathbb{S}^{d-1}} b_\varepsilon^K(\sigma \cdot \hat{\xi}) \left(1 - (\sigma \cdot \hat{\xi})^2 \right) d\sigma \\
&\xrightarrow[K \to \infty]{} -\frac{1}{2} \int_{\mathbb{S}^{d-1}} b_\varepsilon(\sigma \cdot \hat{\xi}) \left(1 - (\sigma \cdot \hat{\xi})^2 \right) d\sigma =: -\bar{\lambda}_\varepsilon \in (-\infty, 0) \\
&\xrightarrow[\varepsilon \to 0]{} -\bar{\lambda} \in (-\infty, 0).
\end{aligned}$$

One can now apply Gronwall's lemma to obtain

$$\begin{aligned}
\sup_{\xi \in \mathbb{R}^d} \frac{|\bar{D}_t^\varepsilon|}{|\xi|^4} \leq e^{(\lambda_K - K)t} \sup_{\xi \in \mathbb{R}^d} \frac{|\bar{D}_0^\varepsilon|}{|\xi|^4} + C \left(\sum_{|\alpha| \leq 3} |M_\alpha[f_0 - g_0]| \right) \left(\frac{e^{-\delta t} - e^{(\lambda_K - K)t}}{K - \lambda_K - \delta} \right) \\
+ r_\varepsilon^K \left(\frac{1 - e^{(\lambda_K - K)t}}{K - \lambda_K} \right).
\end{aligned}$$

Then relaxing the cut-off $K \to \infty$ and choosing $0 < \lambda < \min(\delta, \bar{\lambda}_\varepsilon)$ one has (remark that λ depends on ε)

$$\sup_{\xi \in \mathbb{R}^d} \frac{|\bar{D}_t^\varepsilon|}{|\xi|^4} \leq C\, e^{-\lambda t} \left(\sup_{\xi \in \mathbb{R}^d} \frac{|\bar{D}_0^\varepsilon|}{|\xi|^4} + \sum_{|\alpha| \leq 3} |M_\alpha[f_0 - g_0]| \right). \tag{3.91}$$

Using a standard interpolation argument [62], one obtains

$$
\begin{aligned}
|g-f|_2 &\leq |g-f-\mathcal{M}_4[g-f]|_2 + C\left(\sum_{|\alpha|\leq 3} |M_\alpha[g-f]|\right) \\
&\leq \|g-f-\mathcal{M}_4[f-g]\|_{M_1}^{1/2} |g-f-\mathcal{M}_4[g-f]|_4^{1/2} + C\left(\sum_{|\alpha|\leq 3} |M_\alpha[g-f]|\right) \\
&\leq C\, M_4(f_0+g_0)\, e^{-(\lambda/2)t}.
\end{aligned}
\tag{3.92}
$$

Finally one concludes by writing

$$
\begin{aligned}
|g_t^\varepsilon - f_t^\varepsilon|_2 &\leq |g_t^\varepsilon - f_t^\varepsilon|_2^\eta\, |g_t^\varepsilon - f_t^\varepsilon|_2^{1-\eta} \\
&\leq C_\eta\, e^{-(1-\eta)\lambda t}\, M_4(f_0+g_0)^{1/3}\, |g_0-f_0|_2^\eta
\end{aligned}
\tag{3.93}
$$

where we have used the last estimate (3.92), lemma 3.21 and the fact that $M_4(f_0 + g_0)^{1-\eta} \leq M_4(f_0+g_0)^{1/3}$ for $\eta \in [2/3, 1]$. For h_t one has the same computations.

Step 2. Let us now deduce the result for solutions f_t and g_t of the Landau equation. Coming back to (3.91) and choosing $0 < \lambda_1 < \min(\delta, \bar\lambda)$, where $\bar\lambda_\varepsilon \to \bar\lambda \in (0,\infty)$ as $\varepsilon \to 0$, we recover (3.93) with the exponent λ_1 with does not depend on ε. Hence, passing to the limit $\varepsilon \to 0$, we have $g^\varepsilon - f^\varepsilon \to g - f$ and then

$$
|g_t - f_t|_2 \leq C_\eta\, e^{-(1-\eta)\lambda_1 t}\, M_4(f_0+g_0)^{1/3}\, |g_0-f_0|_2^\eta.
$$

Rigorously, we write

$$
|g_t - f_t|_2 \leq |g_t - g_t^\varepsilon|_2 + |f_t - f_t^\varepsilon|_2 + |g_t^\varepsilon - f_t^\varepsilon|_2,
$$

then for the third term on the right-hand side we use (3.93) with exponent λ_1 that does not depend on ε, and for the other two terms we use that g_t^ε weakly converges towards g_t in L^1 (see Villani [73]), hence $|g_t - g_t^\varepsilon|_2 \to 0$ when $\varepsilon \to 0$ and we deduce

$$
|g_t - f_t|_2 \leq C_\eta\, e^{-(1-\eta)\lambda_1 t}\, M_4(f_0+g_0)^{1/3}\, |g_0-f_0|_2^\eta.
$$

□

Lemma 3.25. *Consider $f_0, g_0 \in \mathbf{P}_{\mathcal{G}_1,\mathbf{r}}$, $\mathbf{r} \in \mathbf{R}_{\mathcal{G}_1}$, and the solutions f_t, g_t, h_t, ω_t and u_t of* (3.86) *and* (3.87)*. There exists $\lambda_1 \in (0,\infty)$ that for any $\eta \in [2/3,1]$, there exists C_η such that we have*

$$
\begin{aligned}
|g_t - f_t - h_t|_4 &\leq C_\eta\, e^{-(1-\eta)\lambda_1 t}\, M_4(f_0+g_0)^{1/3}\, |g_0-f_0|_2^{1+\eta} \\
|u_t|_4 &\leq C_\eta\, e^{-(1-\eta)\lambda_1 t}\, M_4(f_0+g_0)^{1/3}\, |g_0-f_0|_2^{1+\eta}
\end{aligned}
\tag{3.94}
$$

Proof of Lemma 3.25. Let us split the proof into two steps.

Step 1. As in Lemma 3.24, we consider Q_ε the symmetrized version of the Boltmann operator $Q_{B,\varepsilon}$ and the solutions f_t^ε, g_t^ε and h_t^ε of (3.89).

Consider also u_t^ε solution of

$$\partial_t u^\varepsilon = 2Q_\varepsilon(f^\varepsilon, u^\varepsilon) + Q_\varepsilon(h^\varepsilon, h^\varepsilon), \qquad u^\varepsilon|_{t=0} = 0, \tag{3.95}$$

and define $\omega^\varepsilon := g^\varepsilon - f^\varepsilon - h^\varepsilon$ which satisfies

$$\partial_t \omega^\varepsilon = Q_\varepsilon(s^\varepsilon, w^\varepsilon) + Q_\varepsilon(h^\varepsilon, d^\varepsilon), \qquad \omega^\varepsilon|_{t=0} = 0.$$

First of all, we remark that ω_t^ε has moments equals to zero up to order 3. Indeed, let us prove that, for $\alpha \in \mathbb{N}^d$,

$$\forall\, |\alpha| \le 3, \quad M_\alpha(\omega_t^\varepsilon) := \int_{\mathbb{R}^d} v^\alpha\, \omega_t^\varepsilon(v)\, dv = 0. \tag{3.96}$$

Following [62, Lemma 5.8] we know that for maxwellian molecules the α-moment of the Boltzmann operator $Q_{B,\varepsilon}(g,h)$ is a sum of terms given by the product of moments of g and h, then we obtain

$$\forall\, |\alpha| \le 3, \quad \frac{d}{dt} M_\alpha(\omega_t^\varepsilon) = \sum_{\beta \le \alpha} a_{\alpha,\beta}\, M_\beta(\omega_t^\varepsilon) M_{\alpha-\beta}(s_t^\varepsilon) + \sum_{\beta \le \alpha} a_{\alpha,\beta}\, M_\beta(h_t^\varepsilon) M_{\alpha-\beta}(d_t^\varepsilon) \tag{3.97}$$

and we deduce that

$$\forall\, |\alpha| \le 3, \quad \frac{d}{dt} M_\alpha(\omega_t^\varepsilon) = \sum_{\beta \le \alpha} a_{\alpha,\beta}\, M_\beta(\omega_t^\varepsilon) M_{\alpha-\beta}(s_t^\varepsilon) \tag{3.98}$$

because for all $|\alpha| \le 1$ we have $M_\alpha(h_t^\varepsilon) = M_\alpha(d_t^\varepsilon) = 0$. We conclude (3.96) by the fact that $\omega_0 = 0$. Therefore $|\omega|_4$ is well defined and we do not need to "take-off the moments of ω".

Let us denote $\Omega^\varepsilon = \hat{\omega}^\varepsilon$ and $H^\varepsilon = \hat{h}^\varepsilon$. We perform then the same cut-off as in Lemmas 3.21 and 3.24 and we have the following equation for ω_t^ε

$$\begin{aligned}
&\frac{d}{dt}\frac{|\Omega^\varepsilon(\xi)|}{|\xi|^4} + K\frac{|\Omega^\varepsilon(\xi)|}{|\xi|^4} \\
&\quad \le \frac{1}{2}\sup_{\xi\in\mathbb{R}^d}\int_{\mathbb{S}^{d-1}} b_\varepsilon^K(\sigma\cdot\hat{\xi})\left(\frac{|\Omega^\varepsilon(\xi^+)||S^\varepsilon(\xi^-)|}{|\xi|^4} + \frac{|\Omega^\varepsilon(\xi^-)||S^\varepsilon(\xi^+)|}{|\xi|^4}\right) d\sigma \qquad (=: T_1) \\
&\quad + \frac{1}{2}\sup_{\xi\in\mathbb{R}^d}\int_{\mathbb{S}^{d-1}} b_\varepsilon^K(\sigma\cdot\hat{\xi})\left(\frac{|H^\varepsilon(\xi^+)||D^\varepsilon(\xi^-)|}{|\xi|^4} + \frac{|H^\varepsilon(\xi^-)||D^\varepsilon(\xi^+)|}{|\xi|^4}\right) d\sigma \qquad (=: T_2) \\
&\quad + \frac{|R_\varepsilon^K|}{|\xi|^4},
\end{aligned} \tag{3.99}$$

where the remainder term

$$R_\varepsilon^K(\xi) := \frac{1}{2}\int_{\mathbb{S}^{d-1}} b_\varepsilon^C(\sigma\cdot\hat{\xi})\left[\Omega^\varepsilon(\xi^+)S^\varepsilon(\xi^-) + \Omega^\varepsilon(\xi^-)S^\varepsilon(\xi^+) + H^\varepsilon(\xi^+)D^\varepsilon(\xi^-) + H^\varepsilon(\xi^-)D^\varepsilon(\xi^+)\right] d\sigma$$

satisfies, for any $\xi \in \mathbb{R}^d$, $|R^K_\varepsilon(\xi)| \le r^K_\varepsilon |\xi|^4$, with $r^K_\varepsilon \to 0$ as $K \to \infty$, and r^K_ε depends on moments of order 4 of d, s, h and w. To see this, we argue as in lemma 3.24, using that ω^ε has vanishing moments up to order 3, see (3.96).

We compute first T_1 using the fact that $\|S^\varepsilon\|_\infty \le 2$

$$\begin{aligned} T_1 &\le \left(\sup_{\xi\in\mathbb{R}^d} \frac{|\Omega^\varepsilon(\xi)|}{|\xi|^4} \right) \sup_{\xi\in\mathbb{R}^d} \int_{\mathbb{S}^{d-1}} b^K_\varepsilon(\sigma\cdot\hat{\xi}) \left(|\hat{\xi}^+|^4 + |\hat{\xi}^-|^4 \right) d\sigma \\ &\le \lambda_K \sup_{\xi\in\mathbb{R}^d} \frac{|\Omega^\varepsilon(\xi)|}{|\xi|^4}. \end{aligned}$$

where λ_K is the same that in the proof of lemma 3.24. Next, we compute T_2

$$\begin{aligned} T_2 &\le \left(\sup_{\xi\in\mathbb{R}^d} \frac{|H^\varepsilon(\xi)|}{|\xi|^2} \right) \left(\sup_{\xi\in\mathbb{R}^d} \frac{|D^\varepsilon(\xi)|}{|\xi|^2} \right) \sup_{\xi\in\mathbb{R}^d} \int_{\mathbb{S}^{d-1}} b^K_\varepsilon(\sigma\cdot\hat{\xi}) \left(\frac{|\xi^+|^2|\xi^-|^2}{|\xi|^4} \right) d\sigma \\ &\le |h^\varepsilon_t|_2 |d^\varepsilon_t|_2 \sup_{\xi\in\mathbb{R}^d} \int_{\mathbb{S}^{d-1}} b^K_\varepsilon(\sigma\cdot\hat{\xi}) \left(1 - \sigma\cdot\hat{\xi} \right) d\sigma \\ &\le \Lambda_\varepsilon\, e^{-(1-\eta)\lambda t}\, M_4(f_0+g_0)^{1-\eta}\, |h_0|_2 |d_0|_2^\eta \end{aligned}$$

where we have used the estimates of lemmas 3.21 and 3.24, and Λ_ε is defined in (3.21).

After these computations we obtain

$$\frac{d}{dt}\frac{|\Omega^\varepsilon(\xi)|}{|\xi|^4} + K\frac{|\Omega^\varepsilon(\xi)|}{|\xi|^4} \le \lambda_K \sup_{\xi\in\mathbb{R}^d} \frac{|\Omega^\varepsilon(\xi)|}{|\xi|^4} + \Lambda_\varepsilon\, e^{-(1-\eta)\lambda t}\, M_4(f_0+g_0)^{1-\eta}\, |d_0|_2^{1+\eta} + r^K_\varepsilon$$

and by Gronwall's lemma

$$\begin{aligned} \sup_{\xi\in\mathbb{R}^d} \frac{|\Omega^\varepsilon_t(\xi)|}{|\xi|^4} \le \Lambda_\varepsilon\, M_4(f_0+g_0)^{1-\eta}\, |d_0|_2^{1+\eta} &\left(\frac{e^{-(1-\eta)\lambda t} - e^{(\lambda_K - K)t}}{K - \lambda_K - (1-\eta)\lambda} \right) \\ &+ r^K_\varepsilon \left(\frac{1 - e^{(\lambda_K - K)t}}{K - \lambda_K} \right). \end{aligned} \tag{3.100}$$

Finally, we conclude by relaxing the cut-off parameter $K \to \infty$ and choosing $(1-\eta)\lambda \in (0, \bar{\lambda}_\varepsilon)$ where λ_ε is the same that in lemma 3.24, therefore we have

$$|\omega^\varepsilon_t|_4 \le C_\eta\, \Lambda_\varepsilon\, e^{-(1-\eta)\lambda t}\, M_4(f_0+g_0)^{1-\eta}\, |g_0 - f_0|_2^{1+\eta}. \tag{3.101}$$

We obtain the same estimation for u^ε_t.

Step 2. Consider the solutions f, g and h of (3.86) .

Let us choose λ_1 such that $0 < (1-\eta)\lambda_1 < \bar{\lambda}$, where $\bar{\lambda}_\varepsilon \to \bar{\lambda} \in (0,\infty)$ as $\varepsilon \to 0$. Then we recover (3.101) with the exponent λ_1 with does not depend on ε. Hence, passing to the limit $\varepsilon \to 0$, we have $g^\varepsilon - f^\varepsilon - h^\varepsilon \to g - f - h$ (grazing collisions limit), and in the right-hand side of (3.101) we have $\Lambda_\varepsilon \to \Lambda > 0$ (see (3.21)). Then

$$|g_t - f_t - h_t|_4 \le C_\eta\, \Lambda\, e^{-(1-\eta)\lambda_1 t}\, M_4(f_0+g_0)^{1/3}\, |g_0 - f_0|_2^{1+\eta}.$$

$\square$

Lemma 3.26. *Consider $f_0, g_0 \in \mathbf{P}_{\mathcal{G}_1,\mathbf{r}}$, $\mathbf{r} \in \mathbf{R}_{\mathcal{G}_1}$, and the solution ψ_t of* (3.87)*. There exists $\lambda_1 \in (0,\infty)$ such that for any $\eta \in [2/3, 1]$, there exists C_η such that we have*

$$|g_t - f_t - h_t - r_t|_6 \le C_\eta \, e^{-(1-\eta)\lambda_1 t} \, M_4(f_0 + g_0)^{1/3} \, |g_0 - f_0|_2^{2+\eta} \tag{3.102}$$

Proof of Lemma 3.26. We prove the lemma in two steps.

Step 1. Consider the solutions g_t^ε, f_t^ε and h_t^ε of (3.89) and u_t^ε solution of (3.95). Define $\psi_t^\varepsilon := g_t^\varepsilon - f_t^\varepsilon - h_t^\varepsilon - u_t^\varepsilon$ that satisfies

$$\partial_t \psi^\varepsilon = Q_\varepsilon(s^\varepsilon, \psi^\varepsilon) + Q_\varepsilon(h^\varepsilon, w^\varepsilon) + Q_\varepsilon(u^\varepsilon, d^\varepsilon), \qquad \psi^\varepsilon|_{t=0} = 0.$$

First of all, let us prove that ψ_t^ε has moments equals to zero up to order 5, more precisely, for $\alpha \in \mathbb{N}^d$,

$$\forall\, |\alpha| \le 5, \quad M_\alpha(\psi_t^\varepsilon) := \int_{\mathbb{R}^d} v^\alpha \, \psi_t^\varepsilon(v) \, dv = 0. \tag{3.103}$$

In fact, as in the proof of Lemma 3.25, we can compute the α-moment of ψ

$$\begin{aligned} \forall\, |\alpha| \le 5, \quad \frac{d}{dt} M_\alpha(\psi_t^\varepsilon) = \sum_{\beta \le \alpha} a_{\alpha,\beta} \, M_\beta(\psi_t^\varepsilon) M_{\alpha-\beta}(s_t^\varepsilon) + \sum_{\beta \le \alpha} a_{\alpha,\beta} \, M_\beta(h_t^\varepsilon) M_{\alpha-\beta}(\omega_t^\varepsilon) \\ + \sum_{\beta \le \alpha} a_{\alpha,\beta} \, M_\beta(r_t^\varepsilon) M_{\alpha-\beta}(d_t^\varepsilon). \end{aligned} \tag{3.104}$$

Since

$$\begin{aligned} &\forall\, |\alpha| \le 2,\ M_\alpha(h_t^\varepsilon) = M_\alpha(d_t^\varepsilon) = 0, \\ &\forall\, |\alpha| \le 3,\ M_\alpha(\omega_t^\varepsilon) = M_\alpha(r_t^\varepsilon) = 0, \end{aligned} \tag{3.105}$$

we deduce that

$$\forall\, |\alpha| \le 5,\ \frac{d}{dt} M_\alpha(\psi_t^\varepsilon) = \sum_{\beta \le \alpha} a_{\alpha,\beta} \, M_\beta(\psi_t^\varepsilon) M_{\alpha-\beta}(s_t^\varepsilon) \tag{3.106}$$

and we conclude thanks to $\psi_0 = 0$. Then $|\psi|_6$ is well defined.

Denoting $\Psi^\varepsilon = \hat{\psi}^\varepsilon$ and $U^\varepsilon = \hat{u}^\varepsilon$, we perform the same cut-off as in lemmas 3.21, 3.24 and 3.25, and it gives the following equation for ψ_t

$$\begin{aligned} &\frac{d}{dt} \frac{|\Psi^\varepsilon(\xi)|}{|\xi|^6} + K \frac{|\Psi^\varepsilon(\xi)|}{|\xi|^4} \\ &\quad \le \frac{1}{2} \sup_{\xi \in \mathbb{R}^d} \int_{\mathbb{S}^{d-1}} b_\varepsilon^K(\sigma \cdot \hat{\xi}) \left(\frac{|\Psi^\varepsilon(\xi^+)||S^\varepsilon(\xi^-)|}{|\xi|^6} + \frac{|\Psi^\varepsilon(\xi^-)||S^\varepsilon(\xi^+)|}{|\xi|^6} \right) d\sigma \qquad (=: T_1) \\ &\quad + \frac{1}{2} \sup_{\xi \in \mathbb{R}^d} \int_{\mathbb{S}^{d-1}} b_\varepsilon^K(\sigma \cdot \hat{\xi}) \left(\frac{|H^\varepsilon(\xi^+)||\Omega^\varepsilon(\xi^-)|}{|\xi|^6} + \frac{|H^\varepsilon(\xi^-)||\Omega^\varepsilon(\xi^+)|}{|\xi|^6} \right) d\sigma \qquad (=: T_2) \\ &\quad + \frac{1}{2} \sup_{\xi \in \mathbb{R}^d} \int_{\mathbb{S}^{d-1}} b_\varepsilon^K(\sigma \cdot \hat{\xi}) \left(\frac{|U^\varepsilon(\xi^+)||D^\varepsilon(\xi^-)|}{|\xi|^6} + \frac{|U^\varepsilon(\xi^-)||D^\varepsilon(\xi^+)|}{|\xi|^6} \right) d\sigma \qquad (=: T_3) \\ &\quad + \frac{|R_\varepsilon^K|}{|\xi|^6}, \end{aligned} \tag{3.107}$$

where the remainder term

$$R_\varepsilon^K(\xi) := \frac{1}{2}\int_{\mathbb{S}^{d-1}} b_\varepsilon^C(\sigma\cdot\hat{\xi})\Big[\Psi^\varepsilon(\xi^+)S^\varepsilon(\xi^-) + \Psi^\varepsilon(\xi^-)S^\varepsilon(\xi^+) + H^\varepsilon(\xi^+)\Omega^\varepsilon(\xi^-) + H^\varepsilon(\xi^-)\Omega^\varepsilon(\xi^+) + U^\varepsilon(\xi^+)D^\varepsilon(\xi^-) + U^\varepsilon(\xi^-)D^\varepsilon(\xi^+)\Big]d\sigma \quad (3.108)$$

satisfies, for any $\xi\in\mathbb{R}^d$, $|R_\varepsilon^K(\xi)|\le r_\varepsilon^K|\xi|^6$, with $r_\varepsilon^K\to 0$ as $K\to\infty$, and r_ε^K depends on moments of order 6 of d, s, h, w, u and ψ. It easily follows arguing as in lemmas 3.24 and 3.26, using (3.106) and the bounds of moments of order 6.

We compute first T_1 using the fact that $\|S^\varepsilon\|_\infty\le 2$

$$\begin{aligned}T_1 &\le \sup_{\xi\in\mathbb{R}^d}\frac{|\Psi^\varepsilon(\xi)|}{|\xi|^6}\sup_{\xi\in\mathbb{R}^d}\int_{\mathbb{S}^{d-1}} b_\varepsilon^K(\sigma\cdot\hat{\xi})\left(|\hat{\xi}^+|^6+|\hat{\xi}^-|^6\right)d\sigma\\ &\le \alpha_K\sup_{\xi\in\mathbb{R}^d}\frac{|\Psi^\varepsilon(\xi)|}{|\xi|^6}.\end{aligned}$$

Let us analyze α_K,

$$\alpha_K = \int_{\mathbb{S}^{d-1}} b_\varepsilon^K(\sigma\cdot\hat{\xi})\left(|\hat{\xi}^+|^6+|\hat{\xi}^-|^6\right)d\sigma = \int_{\mathbb{S}^{d-1}} b_\varepsilon^K(\sigma\cdot\hat{\xi})\frac{1}{4}\left(1+3(\sigma\cdot\hat{\xi})^2\right)d\sigma$$

and we compute

$$\begin{aligned}\alpha_K - K &= -\int_{\mathbb{S}^{d-1}} b_\varepsilon^K(\sigma\cdot\hat{\xi})\frac{1}{(4/3)}\left(1-(\sigma\cdot\hat{\xi})^2\right)d\sigma\\ &\xrightarrow[K\to\infty]{} -\int_{\mathbb{S}^{d-1}} b_\varepsilon(\sigma\cdot\hat{\xi})\frac{1}{(4/3)}\left(1-(\sigma\cdot\hat{\xi})^2\right)d\sigma =: -\bar{\alpha}_\varepsilon\in(-\infty,0)\\ &\xrightarrow[\varepsilon\to 0]{} -\bar{\alpha}\in(-\infty,0).\end{aligned}$$

Next, we compute T_2

$$\begin{aligned}T_2 &\le \left(\sup_{\xi\in\mathbb{R}^d}\frac{|H^\varepsilon(\xi)|}{|\xi|^2}\right)\left(\sup_{\xi\in\mathbb{R}^d}\frac{|\Omega^\varepsilon(\xi)|}{|\xi|^4}\right)\sup_{\xi\in\mathbb{R}^d}\int_{\mathbb{S}^{d-1}} b_\varepsilon^K(\sigma\cdot\hat{\xi})\frac{1}{2}\left(\frac{|\xi^+|^2|\xi^-|^4}{|\xi|^2|\xi|^4}+\frac{|\xi^+|^4|\xi^-|^2}{|\xi|^4|\xi|^2}\right)d\sigma\\ &\le |h_t^\varepsilon|_2|d_t^\varepsilon|_2\sup_{\xi\in\mathbb{R}^d}\int_{\mathbb{S}^{d-1}} b_\varepsilon^K(\sigma\cdot\hat{\xi})\frac{1}{2}\left(|\hat{\xi}^-|^4+|\hat{\xi}^-|^2\right)d\sigma\\ &\le \beta_K e^{-(1-\eta)\lambda t}M_4(f_0+g_0)^{1-\eta}\,|h_0|_2|d_0|_2^{1+\eta}\end{aligned}$$

where we have used the estimates of lemmas 3.21 and 3.25. We compute β_K

$$\begin{aligned}\beta_K &= \int_{\mathbb{S}^{d-1}} b_\varepsilon^K(\sigma\cdot\hat{\xi})\frac{1}{2}\left(|\hat{\xi}^-|^4+|\hat{\xi}^-|^2\right)d\sigma\\ &= \int_{\mathbb{S}^{d-1}} b_\varepsilon^K(\sigma\cdot\hat{\xi})\frac{1}{2}\left(1-\sigma\cdot\hat{\xi}\right)d\sigma - \int_{\mathbb{S}^{d-1}} b_\varepsilon^K(\sigma\cdot\hat{\xi})\frac{1}{8}\left(1-(\sigma\cdot\hat{\xi})^2\right)d\sigma\\ &\xrightarrow[K\to\infty]{} \bar{\Lambda}_\varepsilon := \frac{\Lambda_\varepsilon}{2}-\frac{\bar{\lambda}_\varepsilon}{4}\\ &\xrightarrow[\varepsilon\to 0]{} \bar{\Lambda} := \frac{\Lambda}{2}-\frac{\bar{\lambda}}{4}\end{aligned}$$

with $\Lambda > 0$ (see (3.21)), $\bar{\lambda} > 0$ and we have the same estimate for T_3.

After these computations we obtain

$$\frac{d}{dt}\frac{|\Psi^\varepsilon(\xi)|}{|\xi|^6} + K\frac{|\Psi^\varepsilon(\xi)|}{|\xi|^6} \leq \alpha_K \sup_{\xi\in\mathbb{R}^d} \frac{|\Psi^\varepsilon(\xi)|}{|\xi|^6} + 2\beta_K e^{-(1-\eta)\lambda t}\, M_4(f_0+g_0)^{1-\eta}\, |d_0|_2^{2+\eta} + r_\varepsilon^K$$

and by Gronwall's lemma

$$\sup_{\xi\in\mathbb{R}^d} \frac{|\hat{\psi}_t(\xi)|}{|\xi|^6} \leq 2\beta_K\, M_4(f_0+g_0)^{1-\eta}\, |d_0|_2^{2+\eta} \left(\frac{e^{-(1-\eta)\lambda t} - e^{(\alpha_K-K)t}}{K-\alpha_K-(1-\eta)\lambda}\right) + r_\varepsilon^K \left(\frac{1-e^{(\alpha_K-K)t}}{K-\alpha_K}\right).$$

We conclude by relaxing the cut-off parameter $K\to\infty$ and choosing $(1-\eta)\lambda \in (0,\bar{\alpha}_\varepsilon)$, therefore we have

$$|\psi_t^\varepsilon|_6 \leq C_\eta\, \bar{\Lambda}_\varepsilon\, e^{-(1-\eta)\lambda t}\, M_4(f_0+g_0)^{1-\eta}\, |d_0|_2^{2+\eta}. \tag{3.109}$$

Step 2. Consider the solutions f, g, h and r of (3.86) .

Let us choose λ_1 such that $0 < (1-\eta)\lambda_1 < \bar{\alpha}$, where $\bar{\alpha}_\varepsilon \to \bar{\alpha} \in (0,\infty)$ as $\varepsilon \to 0$. Then we recover (3.109) with the exponent λ_1 with does not depend on ε. Hence, passing to the limit $\varepsilon\to 0$, we have $g^\varepsilon - f^\varepsilon - h^\varepsilon - u^\varepsilon \to g - f - h - u$ (grazing collisions limit), and in the right-hand side of (3.109) we have $\bar{\Lambda}_\varepsilon \to \bar{\Lambda}$. Then

$$|g_t - f_t - h_t - r_t|_6 \leq C_\eta\, \bar{\Lambda}\, e^{-(1-\eta)\lambda_1 t}\, M_4(f_0+g_0)^{1/3}\, |g_0-f_0|_2^{1+\eta}.$$

□

Therefore the semigroup of the Landau equation

$$S_t^\infty \in C_{\Lambda_2}^{2,\eta} \cap C_{\Lambda_2}^{1,(1+2\eta)/3} \cap C_{\Lambda_2}^{0,(2+\eta)/3}(\mathbf{P}_{\mathcal{G}_1,\mathbf{r}};\mathbf{P}_{\mathcal{G}_2}),$$

where $\mathbf{P}_{\mathcal{G}_2}$ is defined as $\mathbf{P}_{\mathcal{G}_1}$ but endowed with the distance associated to the norm $\|\cdot\|_{\mathcal{G}_2} = |\cdot|_6$, with $\Lambda_2(f) := M_4(f)^{1/3} = \Lambda_1(f)^{1/3}$. Moreover there exists a constant $C_4 > 0$ such that one has

$$\sup_{\mathbf{r}\in\mathbf{R}_{\mathcal{G}_1}} \int_0^\infty \left([S_t^\infty]_{C_{\Lambda_2}^{2,0}} + [S_t^\infty]^2_{C_{\Lambda_2}^{1,0}}\right) dt \leq C_4, \tag{3.110}$$

which proves **(A4)**.

Remark 3.27. In fact, we can deduce that

$$\sup_{\mathbf{r}\in\mathbf{R}_{\mathcal{G}_1}} \int_0^\infty \left([S_t^\infty]_{C_{\Lambda_2}^{1,(1+2\eta)/3}} + [S_t^\infty]^2_{C_{\Lambda_2}^{0,(2+\eta)/3}} + [S_t^\infty]_{C_{\Lambda_2}^{2,0}} + [S_t^\infty]^2_{C_{\Lambda_2}^{1,0}}\right) dt \leq C_4.$$

However, coming back to the proof of Theorem 3.13 and from the proof of **(A3)** (3.84), where we need only $[\Phi]_{C^{2,0}}$ instead of $[\Phi]_{C^{1,\eta}} + [\Phi]_{C^{2,0}}$, we remark that (3.110) is sufficient.

3.4.5 Proof of assumption A5

We define the space of probability measures $\mathbf{P}_{\mathcal{G}_3} := \mathbf{P}_2(\mathbb{R}^d) = \{f \in \mathbf{P}(\mathbb{R}^d);\ M_2(f) < \infty\}$, the constraint function $\mathbf{m}_{\mathcal{G}_3}(v) = (|v|^2, v)$ and $\mathbf{R}_{\mathcal{G}_3} = \{(r, \bar{r}) \in \mathbb{R}_+ \times \mathbb{R}^d;\ r \leq \mathcal{E}_0\}$. Then we define for $a \in (0, \infty)$, the bounded set

$$\mathcal{B}\mathbf{P}_{\mathcal{G}_3,a} := \{f \in \mathbf{P}_{\mathcal{G}_3};\ M_2(f) \leq a\},$$

the constrained bounded set

$$\mathcal{B}\mathbf{P}_{\mathcal{G}_3,a,\mathbf{r}} := \{f \in \mathcal{B}\mathbf{P}_{\mathcal{G}_3,a};\ \langle f, |v|^2\rangle = r,\ \langle f, v_i\rangle \text{ for } i = 1, \dots d\},$$

endowed with the distance $\mathrm{dist}_{\mathcal{G}_3} = W_2$. The following lemma proves **(A5)** with $\mathcal{F}_3 = \mathrm{Lip}(\mathbb{R}^d)$.

Lemma 3.28. *Let f_0, g_0 have the same momentum and energy, and consider $f_t = S_t^\infty(f_0)$, $g_t = S_t^\infty(g_0)$ theirs respective solutions of the Landau Maxwell equation. Then*

$$\sup_{t\geq 0} W_2(f_t, g_t) \leq W_2(f_0, g_0). \tag{3.111}$$

Proof of Lemma 3.28. Consider $f_t^\varepsilon, g_t^\varepsilon$ the solutions of the Boltzmann equation with kernel b_ε satisfying the grazing collisions (3.21) and with initial data f_0 and g_0, respectively. We know from [71] that

$$\sup_{t\geq 0} W_2(f_t^\varepsilon, g_t^\varepsilon) \leq W_2(f_0, g_0)$$

We know also that [73] f_t^ε converges weakly in L^1 to a weak solution f_t of the Landau equation (grazing collisions limit). Moreover, both equations conserve energy so we have, for all ε

$$\int |v|^2 f_t^\varepsilon(v)\, dv = \int |v|^2 f_t(v)\, dv = \int |v|^2 f_0(v)\, dv.$$

Using the fact that the Wasserstein distance W_2 is equivalent to the weak convergence in $\mathbf{P}(\mathbb{R}^d)$ plus the second order momentum convergence [78], we obtain

$$W_2(f_t, g_t) \leq W_2(f_t, f_t^\varepsilon) + W_2(f_t^\varepsilon, g_t^\varepsilon) + W_2(g_t, g_t^\varepsilon)$$

and then, passing to the limit $\varepsilon \to 0$, we have $W_2(f_t, f_t^\varepsilon) \to 0$, $W_2(g_t, g_t^\varepsilon) \to 0$ and

$$\sup_{t\geq 0} W_2(f_t, g_t) \leq W_2(f_0, g_0).$$

□

3.4.6 Proof of Theorem 3.15

The proof is a consequence of Theorem 3.14, some results on differents forms of measuring chaos from [46], quantitative estimates on the chaoticity of initial data from [19].

Proof of Theorem 3.15 (1). Thanks to Theorem 3.14, taking $\ell = 2$, we have for all $\phi = \phi_1 \otimes \phi_2 \in \mathcal{F}^{\otimes 2}$ that

$$\sup_{t\geq 0} \frac{|\left\langle \Pi_2(F_t^N) - f_t^{\otimes 2}, \phi \right\rangle|}{\|\phi\|_{\mathcal{F}}} \leq C \left(\mathcal{W}_{W_2} \left(\pi_P^N F_0^N, \delta_{f_0} \right) + \frac{1}{N} \right),$$

where we recall that $\|\phi\|_{\mathcal{F}} = \int (1 + |\xi|^6)|\hat{\phi}(\xi)|$. Then we observe that, for $r > 0$, applying Cauchy-Schwarz inequality,

$$\begin{aligned}
\|\phi_1\|_{\mathcal{F}} &= \int (1 + |\xi|^6)(1 + |\xi|^2)^{r/2} \, |\hat{\phi}_1(\xi)|(1 + |\xi|^2)^{-r/2} \, d\xi \\
&\leq C \left(\int (1 + |\xi|^2)^{6+r} \, |\hat{\phi}_1(\xi)|^2 \right)^{1/2} \left(\int (1 + |\xi|^2)^{-r} \right)^{1/2}.
\end{aligned}$$

The first integral in the right-rand side is the norm $\|\phi_1\|_{H^{6+r}}$ and the second one is finite if $2r > d$. We have then $H^s \subset \mathcal{F}$ for $s > 6 + d/2$ which implies

$$\sup_{t\geq 0} \left\| \Pi_2(F_t^N) - f_t^{\otimes 2} \right\|_{H^{-s}} \leq C \left(\mathcal{W}_{W_2} \left(\pi_P^N F_0^N, \delta_{f_0} \right) + \frac{1}{N} \right). \tag{3.112}$$

Let us denote $M_k = M_k(\Pi_2(F_t^N)) + M_k(f_t^{\otimes 2})$. Thanks to [46], for any $0 < \alpha < k(dk + d + k)^{-1}$ there exists $C := C(\alpha, d, s, M_k)$ such that

$$\frac{W_1(F_t^N, f_t^{\otimes N})}{N} \leq C \left(\left\| \Pi_2(F_t^N) - f_t^{\otimes 2} \right\|_{H^{-s}}^{\frac{\alpha k}{d+ks}} + N^{-\frac{\alpha}{2}} \right),$$

which implies with (3.112)

$$\frac{W_1(F_t^N, f_t^{\otimes N})}{N} \leq C \left(\mathcal{W}_{1,W_2}(\pi_p^N F_0^N, f_0^{\otimes N})^{\frac{\alpha k}{d+ks}} + N^{-\frac{\alpha}{2}} \right). \tag{3.113}$$

Now, we have just to estimate the first term of the right-hand side of (3.112).

We have from [19, Proof of Theorem 8] that for any $0 < \beta < (7d + 6)^{-1}$ there exists $C = C(\beta)$ such that

$$\mathcal{W}_{1,W_2}(\pi_P^N F_0^N, \delta_{f_0}) \leq C \, N^{-\beta}.$$

We assumed that $M_6(f_0)$ is finite, which implies by contruction that $M_6(\Pi_2(F_0^N))$ is also finite. Then, for all $t \geq 0$ we have $M_6(f_t)$ finite (see [74]) and $M_6(\Pi_2(F_t^N))$ also finite (see Lemma 3.20). We can conclude gathering the last equation with (3.113), $k = 6$ and $s > 6 + d/2$. □

Using this result, we can prove the second part of the theorem following [62].

Proof of Theorem 3.15 (2). We split the expression into

$$\frac{W_1(F_t^N, \gamma^N)}{N} = \frac{W_1(F_t^N, f_t^{\otimes N})}{N} + \frac{W_1(\gamma^{\otimes N}, \gamma^N)}{N} + W_1(f_t, \gamma),$$

where γ is the equilibrium Gaussian probability with zero momentum and energy $\mathcal{E} = \int |v|^2 d\gamma$. For the first term we have from point *(1)* that for all $\epsilon < 9[(7d+6)^2(d+9)]^{-1}$ there exists C_ϵ such that

$$\frac{W_1(F_t^N, f_t^{\otimes N})}{N} \leq C_\epsilon \, N^{-\epsilon}$$

The second term can be estimated by [19, Theorem 18]

$$\frac{W_1(\gamma^{\otimes N}, \gamma^N)}{N} \leq C N^{-\theta},$$

for some $\theta > \varepsilon$. For the third term, thanks to [74, Theorem 6] we have

$$W_1(f_t, \gamma) \leq \|(f_t - \gamma)\langle v\rangle\|_{L^1} \leq C e^{-\lambda t}.$$

for contants $C > 0$ and $\lambda > 0$. Finally, putting togheter these estimates it follows

$$\frac{W_1(F_t^N, \gamma^N)}{N} \leq C'_\varepsilon(N^{-\epsilon} + e^{-\lambda t}). \tag{3.114}$$

Moreover, consider h_t^N the Radon-Nicodym derivative of F_t^N with respect to γ^N, i.e. $h_t^N = dF_t^N/d\gamma^N$. Thanks to [50], for all $N \in \mathbb{N}^*$ and $t \geq 0$, it holds

$$\|h_t^N - 1\|_{L^2(\mathcal{S}^N(\mathcal{E}), d\gamma^N)} \leq e^{-\lambda_1 t} \|h_0^N - 1\|_{L^2(\mathcal{S}^N(\mathcal{E}), d\gamma^N)},$$

where $\lambda_1 > 0$. Since $F_0^N = [f_0^{\otimes N}]_{\mathcal{S}^N(\mathcal{E})}$ and $f_0 \in \mathbf{P}_6(\mathbb{R}^d)$, it is possible to bound the right-hand side by

$$\|h_0^N - 1\|_{L^2(\mathcal{S}^N(\mathbb{E}), d\gamma^N)} \leq A^N,$$

with $A > 1$ that depends on f_0. Hence we deduce, with $\phi : \mathbb{R}^{dN} \to \mathbb{R}$,

$$\begin{aligned} W_1(F_t^N, \gamma^N) &= \sup_{\|\phi\|_{C^{0,1}} \leq 1} \int_{\mathbb{R}^{dN}} \phi(dF^N - d\gamma^N) \\ &\leq \int_{\mathbb{R}^{dN}} \sum_{j=1}^N |v_j| \left| dF^N - d\gamma^N \right| \\ &\leq N \mathcal{E}^{1/2} \|h_t^N - 1\|_{L^1(\mathcal{S}^N(\mathcal{E}), d\gamma^N)} \\ &\leq N \mathcal{E}^{1/2} \|h_t^N - 1\|_{L^2(\mathcal{S}^N(\mathcal{E}), d\gamma^N)}, \end{aligned}$$

which implies

$$\frac{W_1(F_t^N, \gamma^N)}{N} \leq A^N e^{-\lambda_1 t}. \tag{3.115}$$

Define $N(t)$ by $N(t) := \lambda_1 t \, (2 \log A)^{-1}$ for some $\delta > 0$. Then, choosing (3.114) for $N > N(t)$ and (3.115) for $N \leq N(t)$ it yields, for all $N \in \mathbb{N}^*$ and $t \geq 0$,

$$\frac{W_1(F_t^N, \gamma^N)}{N} \leq p(t) := \min \left\{ C'_\varepsilon \left(N(t)^{-\varepsilon} + e^{-\lambda t} \right), e^{-\frac{\lambda_1}{2} t} \right\},$$

with a polynomial funtion $p(t) \to 0$ as $t \to \infty$.

$\square$

3.5 Entropic chaos

We can define the master equation (3.32) on $\mathbb{R}^{dN}$ or $\mathcal{S}^N(\mathcal{E})$ thanks to the conservation of momentum and energy, hence for $g^N \in \mathbf{P}_{\mathrm{sym}}(\mathbb{R}^{dN})$ and $f^N \in \mathbf{P}_{\mathrm{sym}}(\mathcal{S}^N(\mathcal{E}))$ we have

$$\partial_t \left\langle g^N, \psi \right\rangle = \left\langle g^N, G^N \psi \right\rangle, \qquad \forall\, \psi \in C_b^2(\mathbb{R}^{dN}) \tag{3.116}$$

$$\partial_t \left\langle f^N, \phi \right\rangle = \left\langle f^N, G^N \phi \right\rangle, \qquad \forall\, \phi \in C_b^2(\mathcal{S}^N(\mathcal{E})), \tag{3.117}$$

where G^N is given by (3.32).

Suppose that g^N is abolutely continous with respect to the Lebesgue measure (and we still denote by g^N its Radon-Nikodym derivative). Taking $\psi = \log g^N$ in (3.116), we obtain an equation for the entropy of g^N, i.e. $H(g^N) := \int_{\mathbb{R}^{dN}} g^N \log g^N \, dV$,

$$\frac{d}{dt} \int_{\mathbb{R}^{dN}} g^N \log g^N \, dV = -\frac{1}{2N} \sum_{i,j} \int_{\mathbb{R}^{dN}} a(v_i - v_j) \left(\frac{\nabla_i g^N}{g^N} - \frac{\nabla_j g^N}{g^N} \right) \cdot \left(\frac{\nabla_i g^N}{g^N} - \frac{\nabla_j g^N}{g^N} \right) g^N \, dV \le 0, \tag{3.118}$$

since a is nonnegative.

Considering now f^N absolutely continuous with respect to γ^N, the uniform probability measure on $\mathcal{S}^N(\mathcal{E})$, and denoting by $h^N := df^N/d\gamma^N$ its derivative, we want to obtain the equation satisfied for the relative entropy of f^N with respect to γ^N, given by

$$H(f^N|\gamma^N) := \int_{\mathcal{S}^N} h^N \log h^N \, d\gamma^N. \tag{3.119}$$

For this purpose we could take $\phi = \log h^N$ in (3.117), but we have to give a meaning to $\nabla_i h^N$ for a function h^N defined on $\mathcal{S}^N$.

Let us consider h a function on $\mathcal{S}^N(\mathcal{E})$ and we define $\widetilde{h}$ on $\mathbb{R}^{dN}$ by

$$\widetilde{h}(V) = \rho(\mathcal{E}(V))\, \eta(\mathcal{M}(V))\, h\left(\mathcal{E}\, \frac{V - \mathcal{M}(V)}{\mathcal{E}(V)} \right), \qquad \forall\, V \in \mathbb{R}^{dN}. \tag{3.120}$$

where $\mathcal{E}(V) = N^{-1} \sum_{i=1}^N |v_i - \mathcal{M}(V)|^2$, $\mathcal{M}(V) = N^{-1} \sum_{i=1}^N v_i$ and the functions ρ and η are smooth.

Denoting by $\nabla_{\mathcal{S}^N}$ the gradient with respect to $\mathcal{S}^N(\mathcal{E})$ and by $\nabla_\perp$ the gradient with respect to its orthogonal space $(\mathcal{S}^N)^\perp$, we can decompose the gradient on $\mathbb{R}^{dN}$

$$\nabla_{\mathbb{R}^{dN}} \widetilde{h} = \nabla_\perp \widetilde{h} + \nabla_{\mathcal{S}^N} \widetilde{h} = (\nabla_\perp \rho\eta)\, h + \rho\eta\, \nabla_{\mathcal{S}^N} h = (\nabla_\perp \log(\rho\eta))\, \widetilde{h} + \rho\eta\, \nabla_{\mathcal{S}^N} h. \tag{3.121}$$

For $\widetilde{h}$ we can compute $\nabla_i \widetilde{h} \in \mathbb{R}^d$, for $1 \le i \le N$, as

$$\nabla_i \widetilde{h} = \left(\partial_{v_{i,\alpha}} \widetilde{h} \right)_{1 \le \alpha \le d} = \left(\nabla_{\mathbb{R}^{dN}} \widetilde{h} \cdot e_{i,\alpha} \right)_{1 \le \alpha \le d},$$

where $(e_{i,\alpha})_{j,\beta} = \delta_{ij} \delta_{\alpha\beta} \in \mathbb{R}^{dN}$. Hence by (3.121), for all $1 \le i \le N$ and all $1 \le \alpha \le d$,

$$\partial_{v_{i,\alpha}} \widetilde{h} = (\nabla_\perp \log(\rho\eta) \cdot e_{i,\alpha})\, \widetilde{h} + \rho\eta\, (\nabla_{\mathcal{S}^N} h \cdot e_{i,\alpha}).$$

Now, observing that $(\nabla_\perp \log(\rho\eta)\cdot(e_{i,\alpha}-e_{j,\alpha}))_{1\le\alpha\le d}$ is proportional to (v_i-v_j) and using that $a(z)z=0$ for all $z\in\mathbb{R}^d$, we can evaluate the expression

$$a(v_i-v_j)\left(\nabla_i\widetilde{h}-\nabla_j\widetilde{h}\right)\cdot\left(\nabla_i\widetilde{h}-\nabla_j\widetilde{h}\right)=(\rho\eta)^2\,a(v_i-v_j)(\nabla_{\mathcal{S}_i^N}h-\nabla_{\mathcal{S}_j^N}h)\cdot(\nabla_{\mathcal{S}_i^N}h-\nabla_{\mathcal{S}_j^N}h),$$

where we define

$$\nabla_{\mathcal{S}_i^N}h=(\nabla_{\mathcal{S}^N}h\cdot e_{i,\alpha})_{1\le\alpha\le d}\,. \tag{3.122}$$

Since we have the following Fubini-like theorem for Boltzmann's spheres (see [35, 19])

$$\begin{aligned}&\int_{\mathbb{R}^{dN}}\rho(\mathcal{E}(V))\,\eta(\mathcal{M}(V))\,A\left(h\left(\mathcal{E}\,\frac{V-\mathcal{M}(V)}{\mathcal{E}(V)}\right)\right)\,dV\\&\qquad=\left(\int_{\mathbb{R}^+\times\mathbb{R}^d}B\left(\rho(\mathcal{E}),\eta(\mathcal{M})\right)\,d\mathcal{E}\,d\mathcal{M}\right)\left(\int_{\mathcal{S}^N(\mathcal{E})}A(h)\,d\gamma^N\right),\end{aligned}\tag{3.123}$$

for some functions A and B, thanks to (3.118) with $h=h^N$ and $\widetilde{h}=g^N$, we obtain the equation for the relative entropy $H(f^N|\gamma^N)$,

$$\begin{aligned}&\frac{d}{dt}\int_{\mathcal{S}^N(\mathcal{E})}h^N\log h^N\,d\gamma^N\\&\quad=-\frac{1}{2N}\sum_{i,j}\int_{\mathcal{S}^N(\mathcal{E})}a(v_i-v_j)\left(\frac{\nabla_{\mathcal{S}_i^N}h^N}{h^N}-\frac{\nabla_{\mathcal{S}_j^N}h^N}{h^N}\right)\cdot\left(\frac{\nabla_{\mathcal{S}_i^N}h^N}{h^N}-\frac{\nabla_{\mathcal{S}_j^N}h^N}{h^N}\right)h^N\,d\gamma^N\\&\quad=:-D^N(F^N)\le 0,\end{aligned}\tag{3.124}$$

and D^N is called the entropy-production functional. This implies

$$\frac{1}{N}\,H(F_t^N|\gamma^N)+\int_0^t\frac{1}{N}\,D^N(F_s^N)\,ds=\frac{1}{N}\,H(F_0^N|\gamma^N). \tag{3.125}$$

Moreover for the limit equation we have [74]

$$\frac{d}{dt}H(f):=\frac{d}{dt}\int f\log f\,dv=-\frac12\int f f_*\,a(v-v_*)\left(\frac{\nabla f}{f}-\frac{\nabla_* f_*}{f_*}\right)\cdot\left(\frac{\nabla f}{f}-\frac{\nabla_* f_*}{f_*}\right)dv\,dv_*$$

and then for the relative entropy $H(f|\gamma)=\int(f/\gamma)\log(f/\gamma)\,\gamma(dv)$, we obtain

$$H(f_t|\gamma)+\int_0^t D(f_s)\,ds=H(f_0|\gamma). \tag{3.126}$$

We are able now to prove the following result, which will be useful in the sequel.

Lemma 3.29. *If F^N is f-chaotic, then*

$$H(f|\gamma)\le\liminf_{N\to\infty}\frac{1}{N}\,H(F^N|\gamma^N)\quad\textit{and}\quad D(f)\le\liminf_{N\to\infty}\frac{1}{N}\,D^N(F^N).$$

Proof of Lemma 3.29. The lower semicontinuity property of the relative entropy is proved in [19, Theorem 21], thus we prove only the second inequality.

Let us denote $\nabla_{12} = \nabla_1 - \nabla_2$, $\nabla_{\mathcal{S}_{12}^N} = \nabla_{\mathcal{S}_1^N} - \nabla_{\mathcal{S}_2^N}$, and for all $x, y, z \in \mathbb{R}^d$ we denote $a(z)xy = (a(z)x) \cdot y$. Since a is nonnegative, considering a function $\varphi : \mathbb{R}^{2d} \to \mathbb{R}^d$, we have

$$a(v_1 - v_2) \left(\nabla_{12} \log f_1 f_2 - \frac{\varphi}{2} \right) \left(\nabla_{12} \log f_1 f_2 - \frac{\varphi}{2} \right) \geq 0,$$

which gives the following representation for $D(f)$,

$$\begin{aligned} D(f) &= \frac{1}{2} \sup_{\varphi:\mathbb{R}^{2d}\to\mathbb{R}^d} \iint a(v_1 - v_2) \left[(\nabla_{12} \log f_1 f_2)\varphi - \frac{\varphi\varphi}{4} \right] f_1 f_2 \, dv_1 \, dv_2 \\ &= \frac{1}{2} \sup_{\varphi:\mathbb{R}^{2d}\to\mathbb{R}^d} \iint \left\{ -\nabla_{12} \cdot (a(v_1 - v_2)\varphi) - a(v_1 - v_2) \frac{\varphi\varphi}{4} \right\} f_1 f_2 \, dv_1 \, dv_2 \end{aligned}$$

where $f_1 = f(v_1)$ and $f_2 = f(v_2)$. Let $\varepsilon > 0$ and choose $\varphi = \varphi(v_1, v_2) : \mathbb{R}^{2d} \to \mathbb{R}^d$ such that

$$D(f) - \varepsilon \leq \frac{1}{2} \iint \left\{ -\nabla_{12}(a(v_1 - v_2)\varphi) - a(v_1 - v_2) \frac{\varphi\varphi}{4} \right\} f_1 f_2 \, dv_1 \, dv_2.$$

For the N-particle entropy-production D^N defined in (3.124), we have by symmetry

$$\begin{aligned} \frac{1}{N} D^N(F^N) &= \frac{N(N-1)}{N^2} \frac{1}{2} \int_{\mathcal{S}^N} a(v_1 - v_2) \left(\frac{\nabla_{\mathcal{S}_1^N} h^N}{h^N} - \frac{\nabla_{\mathcal{S}_2^N} h^N}{h^N} \right) \cdot \left(\frac{\nabla_{\mathcal{S}_1^N} h^N}{h^N} - \frac{\nabla_{\mathcal{S}_2^N} h^N}{h^N} \right) h^N \, d\gamma^N \\ &=: \frac{N(N-1)}{N^2} D_{12}^N(F^N), \end{aligned}$$

and then $\liminf_{N\to\infty} N^{-1} D^N(F^N) \geq \liminf_{N\to\infty} D_{12}^N(F^N)$. For $\Phi : \mathbb{R}^{dN} \to \mathbb{R}^d$, $\Phi \in C_b^1$, we have, with $F^N = h^N \gamma^N$,

$$\begin{aligned} D_{12}^N(F^N) &= \frac{1}{2} \int_{\mathcal{S}^N} a(v_1 - v_2) \nabla_{\mathcal{S}_{12}^N} \log h^N \cdot \nabla_{12} \log h^N \, h^N \, d\gamma^N \\ &= \frac{1}{2} \sup_{\Phi:\mathbb{R}^{dN}\to\mathbb{R}^d} \int_{\mathcal{S}^N} a(v_1 - v_2) \left(\nabla_{\mathcal{S}_{12}^N} \log h^N \Phi - \Phi\Phi/4 \right) h^N \, d\gamma^N \\ &= \frac{1}{2} \sup_{\Phi} \left\{ \int_{\mathcal{S}^N} a(v_1 - v_2) \nabla_{\mathcal{S}_{12}^N} h^N \Phi \, d\gamma^N - \int_{\mathcal{S}^N} a(v_1 - v_2) \frac{\Phi\Phi}{4} h^N \, d\gamma^N \right\}. \end{aligned}$$

Choosing $\Phi(V) = \varphi(v_1, v_2)$ we obtain, using (3.122),

$$\begin{aligned} D_{12}^N(F^N) &\geq \frac{1}{2} \int_{\mathcal{S}^N} a(v_1 - v_2) \nabla_{\mathcal{S}_{12}^N} h^N \, \varphi \, d\gamma^N - \frac{1}{2} \int_{\mathcal{S}^N} a(v_1 - v_2) \frac{\varphi\varphi}{4} h^N \, d\gamma^N \\ &\geq \frac{1}{2} \sum_{\alpha,\beta=1}^{d} \int_{\mathcal{S}^N} \nabla_{\mathcal{S}^N} h^N \cdot [(e_{1,\alpha} - e_{2,\alpha}) \, a_{\alpha\beta}(v_1 - v_2) \, \varphi_\beta] \, d\gamma^N - \frac{1}{2} \int_{\mathcal{S}^N} a(v_1 - v_2) \frac{\varphi\varphi}{4} h^N \, d\gamma^N. \end{aligned} \tag{3.127}$$

We need an integration by parts formula for the first term on the right-hand side, thanks to [19, Lemma 22], for a function A and a vector field Ψ, we have

$$\int_{\mathcal{S}^N} \left\{ \nabla_{\mathcal{S}^N} A(V) \cdot \Psi(V) + A(V)\,\mathrm{div}_{\mathcal{S}^N}\,\Psi(V) - \frac{d(N-1)-1}{dN} A(V)\,\Psi(V)\cdot V \right\} d\gamma^N(V) = 0,$$

with

$$\mathrm{div}_{\mathcal{S}^N}\,\Psi(V) = \mathrm{div}\,\Psi(V) - \frac{1}{N}\sum_{i,j=1}^{N}\sum_{\beta=1}^{d} \partial_{v_{i,\beta}}\Psi_{j,\beta}(V) - \sum_{j=1}^{N}\sum_{\beta=1}^{d} V\cdot\nabla\Psi_{j,\beta}\,\frac{v_{j,\beta}}{|V|^2}. \qquad (3.128)$$

Taking $\Psi(V) = (e_{1,\alpha} - e_{2,\alpha})\, a_{\alpha\beta}(v_1 - v_2)\,\varphi_\beta$ we obtain

$$\begin{aligned}
&\int_{\mathcal{S}^N} \nabla_{\mathcal{S}^N} h^N \cdot (e_{1,\alpha} - e_{2,\alpha})\, a_{\alpha\beta}(v_1 - v_2)\,\varphi_\beta\, d\gamma^N \\
&\quad = -\int_{\mathcal{S}^N} h^N\,\mathrm{div}_{\mathcal{S}^N}\left[(e_{1,\alpha} - e_{2,\alpha})\, a_{\alpha\beta}\,\varphi_\beta\right] d\gamma^N + \frac{d(N-1)-1}{dN}\int_{\mathcal{S}^N} h^N\, a_{\alpha\beta}\,\varphi_\beta\,(e_{1,\alpha} - e_{2,\alpha})\cdot V\, d\gamma^N.
\end{aligned}$$

Since $(e_{1,\alpha} - e_{2,\alpha})\cdot V = (v_{1,\alpha} - v_{2,\alpha})$, when performing the summation $\alpha,\beta = 1$ to d in the second term of the right-hand side of last equation, we obtain

$$\int_{\mathcal{S}^N} h^N a(v_1 - v_2)(v_1 - v_2)\,\varphi\, d\gamma^N = 0.$$

For the first term, thanks to (3.128),

$$\begin{aligned}
&\sum_{\alpha,\beta=1}^{d} \int_{\mathcal{S}^N} h^N\,\mathrm{div}_{\mathcal{S}^N}\left[(e_{1,\alpha} - e_{2,\alpha})\, a_{\alpha\beta}\,\varphi_\beta\right] d\gamma^N \\
&\qquad = \int_{\mathcal{S}^N} \nabla_{12}\cdot(a(v_1 - v_2)\,\varphi)\, h^N\, d\gamma^N \\
&\qquad - \sum_{\alpha,\beta=1}^{d} \frac{1}{|V|^2}\int_{\mathcal{S}^N} \left\{ v_1\cdot\nabla_1(a_{\alpha\beta}\,\varphi_\beta) + v_2\cdot\nabla_2(a_{\alpha\beta}\,\varphi_\beta)\right\}(v_{1,\alpha} - v_{2,\alpha})\, h^N\, d\gamma^N
\end{aligned}$$

Getting back to (3.127) with last expression, we split the integral over (v_1, v_2) and $\mathcal{S}^N(v_1,v_2) := \{(v_3,\ldots,v_N) \in \mathbb{R}^{d(N-2)};\ V \in \mathcal{S}^N\}$, use that $|V|^2 = \mathcal{E}N$ and $\int_{\mathcal{S}^N(v_1,v_2)} h^N\, d\gamma^N = F_2^N$, which yields

$$\begin{aligned}
D_{12}^N(F^N) \geq &-\frac{1}{2}\iint \nabla_{12}\cdot(a(v_1 - v_2)\,\varphi)\, F_2^N(v_1,v_2)\, dv_1\, dv_2 - \frac{1}{2}\iint a(v_1 - v_2)\,\frac{\varphi\varphi}{4}\, F_2^N(v_1,v_2)\, dv_1\, dv_2 \\
&+ O\left(\frac{1}{N}\right)\sum_{\alpha,\beta=1}^{d}\iint \left\{ v_1\cdot\nabla_1(a_{\alpha\beta}\,\varphi_\beta) + v_2\cdot\nabla_2(a_{\alpha\beta}\,\varphi_\beta)\right\}(v_{1,\alpha} - v_{2,\alpha})\, F_2^N(v_1,v_2)\, dv_1\, dv_2.
\end{aligned} \qquad (3.129)$$

Passing to the limit $N\to\infty$, since $F_2^N \rightharpoonup f^{\otimes 2}$ we obtain

$$\liminf_{N\to\infty} D_{12}^N(F^N) \geq \frac{1}{2}\int \left\{ -\nabla_{12}\cdot(a(v_1 - v_2)\varphi) - a(v_1 - v_2)\frac{\varphi\varphi}{4}\right\} f_1 f_2\, dv_1\, dv_2 \geq D(f) - \varepsilon$$

and we conclude letting ε go to 0.

$\square$

We define the Fisher information of $G \in \mathbf{P}(\mathbb{R}^{dN})$ that is absolutely continuous with respect to the Lebesgue measure by

$$I(G) := \int_{\mathbb{R}^{dN}} \frac{|\nabla_{\mathbb{R}^{dN}} G|^2}{G} \, dV.$$

Moreover, for a probability measure $F \in \mathbf{P}(\mathcal{S}^N(\mathcal{E}))$ absolutely continuous with respect to γ^N, we define the relative Fisher's information by

$$I(F|\gamma^N) := \int_{\mathcal{S}^N(\mathcal{E})} \frac{|\nabla_{\mathcal{S}^N} h|^2}{h} \, d\gamma^N, \quad h = \frac{dF}{d\gamma^N}, \tag{3.130}$$

where $\nabla_{\mathcal{S}^N}$ stands for the gradient on $\mathcal{S}^N(\mathcal{E})$.

We can now give the following result.

Lemma 3.30. *Let $F_0^N \in \mathbf{P}_{\mathrm{sym}}(\mathcal{S}^N(\mathcal{E}))$ with finite relative Fisher information $I(F_0^N|\gamma^N)$. For all $t > 0$ consider the solution F_t^N of the Landau master equation* (3.32)*. Then we have*

$$I(F_t^N|\gamma^N) \leq I(F_0^N|\gamma^N).$$

Proof of Lemma 3.30. Denote $h_0^N := dF_0^N/d\gamma^N$ and, for all $t \geq 0$, $h_t^N := dF_t^N/d\gamma^N$. Consider $\widetilde{h}_t^N$ defined on $\mathbb{R}^{dN}$ given by (3.120) and define then $\widetilde{F}_t^N = \widetilde{h}_t^N \mathscr{L}$ a solution of (3.116), where $\mathscr{L}$ is the Lebesgue measure on $\mathbb{R}^{dN}$. Following [62, Lemma 7.4], we claim that is enough to prove that

$$I(\widetilde{F}_t^N) := \int_{\mathbb{R}^{dN}} \frac{|\nabla_{\mathbb{R}^{dN}} \widetilde{h}_t^N|^2}{\widetilde{h}_t^N} \, dV \leq \int_{\mathbb{R}^{dN}} \frac{|\nabla_{\mathbb{R}^{dN}} \widetilde{h}_0^N|^2}{\widetilde{h}_0^N} \, dV =: I(\widetilde{F}_0^N).$$

Indeed, this equation, (3.121), (3.123) and the conservation of momentum and energy yield

$$\begin{aligned} I(\widetilde{F}_t^N) &= |\nabla_\perp \log(\rho\eta)|^2 + \left(\int_{\mathbb{R}^+ \times \mathbb{R}^d} B\left(\rho(\mathcal{E}), \eta(\mathcal{M})\right) \, d\mathcal{E} \, d\mathcal{M} \right) \left(\int_{\mathcal{S}^N(\mathcal{E})} \frac{|\nabla_{\mathcal{S}^N} h_t^N|^2}{h_t^N} \, d\gamma^N \right) \\ &\leq |\nabla_\perp \log(\rho\eta)|^2 + \left(\int_{\mathbb{R}^+ \times \mathbb{R}^d} B\left(\rho(\mathcal{E}), \eta(\mathcal{M})\right) \, d\mathcal{E} \, d\mathcal{M} \right) \left(\int_{\mathcal{S}^N(\mathcal{E})} \frac{|\nabla_{\mathcal{S}^N} h_0^N|^2}{h_0^N} \, d\gamma^N \right) = I(\widetilde{F}_0^N), \end{aligned}$$

which implies, dropping the time independent terms,

$$I(F_t^N|\gamma^N) \leq I(F_0^N|\gamma^N).$$

Now, let $F_{t,\varepsilon}^N \in \mathbf{P}_{\mathrm{sym}}(\mathcal{S}^N(\mathcal{E}))$ be the solution of the Boltzmann master equation (3.11)-(3.12) with collision kernel b_ε satisfying the grazing collisions assumptions (3.21) and initial datum $F_0^N \in \mathbf{P}_{\mathrm{sym}}(\mathcal{S}^N(\mathcal{E}))$. Then we have from [62, Lemma 7.4], for all $t \geq 0$,

$$I(\widetilde{F}_{t,\varepsilon}^N) \leq I(\widetilde{F}_0^N),$$

where $\widetilde{F}_0^N, \widetilde{F}_{t,\varepsilon}^N \in \mathbf{P}_{\mathrm{sym}}(\mathbb{R}^{dN})$ are constructed as before.

Since $\widetilde{F}^N_{t,\varepsilon}$ weakly converges towards $\widetilde{F}^N_t$ when $\varepsilon \to 0$ and the Fisher information functional is weakly lower semicontinuous, we obtain

$$I(\widetilde{F}^N_t) \leq \liminf_{\varepsilon\to 0} I(\widetilde{F}^N_{t,\varepsilon}) \leq I(\widetilde{F}^N_0)$$

and that concludes the proof.

$\square$

Now, with the definitions of relative entropy (3.119), relative Fisher information (3.130) and the notion of entropic chaos, described below, we are able to state our main theorem of this section, concerning the propagation of entropic chaos.

Let F^N be a sequence of probability measures $\mathcal{S}^N(\mathcal{E})$ such that F^N_1 weakly converges to f in measure sense, for some $f \in \mathbf{P}(\mathbb{R}^d)$. We say that F^N is entropically f-chaotic if

$$\frac{H(F^N|\gamma^N)}{N} \xrightarrow[N\to\infty]{} H(f|\gamma). \tag{3.131}$$

For more information on entropic chaos we refer to [12, 46, 19].

Theorem 3.31. *Let $f_0 \in \mathbf{P}(\mathbb{R}^d)$ and $F^N_0 \in \mathbf{P}_{\mathrm{sym}}(\mathcal{S}^N(\mathcal{E}))$ that is f_0-chaotic. Consider then, for all $t > 0$, the solution F^N_t of the Landau master equation* (3.32) *with initial condition F^N_0, and the solution f_t of the limit Landau equation* (3.13)-(3.14) *with initial data f_0.*

Then we have

1. *If F^N_0 is entropically f_0-chaotic, then for all $t > 0$ F^N_t is entropically f_t-chaotic, more precisely*
$$\frac{1}{N} H(F^N_t|\gamma^N) \longrightarrow H(f_t|\gamma) \qquad \text{as} \qquad N \to \infty.$$

2. *Consider $f_0 \in \mathbf{P}_6(\mathbb{R}^d)$ with $I(f_0|\gamma) < \infty$ and $F^N_0 = [f_0^{\otimes N}]_{\mathcal{S}^N(\mathcal{E})} \in \mathbf{P}_{\mathrm{sym}}(\mathcal{S}^N(\mathcal{E}))$. Then, for all $t > 0$, F^N_t is entropically f_t-chaotic, more precisely, for any $0 < \varepsilon < 18[5(7d+6)^2(d+9)]^{-1}$ there exists a constant $C := C(\varepsilon)$ such that*
$$\sup_{t\geq 0} \left| \frac{1}{N} H(F^N_t|\gamma^N) - H(f_t|\gamma) \right| \leq C N^{-\varepsilon}.$$

3. *Consider $f_0 \in \mathbf{P}_6(\mathbb{R}^d)$ with $I(f_0|\gamma) < \infty$ and $F^N_0 = [f_0^{\otimes N}]_{\mathcal{S}^N(\mathcal{E})} \in \mathbf{P}_{\mathrm{sym}}(\mathcal{S}^N(\mathcal{E}))$. Then for all N it holds*
$$\frac{1}{N} H(F^N_t|\gamma^N) \leq p(t),$$
for some polynomial function $p(t) \to 0$ as $t \to \infty$.

Proof of Theorem 3.31 (1). The idea is from [62]. Using (3.125), (3.126) and the entropic chaoticity at initial time, one has

$$\frac{1}{N}\,H(f_t^N|\gamma^N) + \int_0^t \frac{1}{N}\,D^N(F_s^N)\,ds = \frac{1}{N}\,H(f_0^N|\gamma^N)$$
$$\xrightarrow[N\to\infty]{} H(f_0|\gamma) = H(f_t|\gamma) + \int_0^t D(f_s)\,ds.$$

By Lemma 3.29 one also has

$$\liminf_{N\to\infty}\left(H(f_t^N|\gamma^N) + \int_0^t \frac{1}{N}\,D^N(F_s^N)\,ds\right) \geq H(f_t|\gamma) + \int_0^t D(f_s)\,ds,$$

and we can conclude with these two last equations togheter with Lemma 3.29.

□

Proof of Theorem 3.31 (2). From Lemma 3.30 we know that, for all $t \geq 0$, $N^{-1}I(F_t^N|\gamma^N) \leq N^{-1}I(F_0^N|\gamma^N)$ and the later one is bounded by construction, we deduce then that the normalized relative Fisher's information $N^{-1}I(F_t^N|\gamma^N)$ is bounded. Since the limit Landau equation for maxwellian molecules propagates moments and the Fisher's information's bound [74, 75], we have, for all $t > 0$, $M_6(f_t)$ and $I(f_t|\gamma)$ bounded.

We can then apply [19, Theorem 31] to F_t^N and we obtain that for any $\beta < (7d+6)^{-1}$ there exists $C' = C'(\beta)$ such that

$$\left|\frac{1}{N}\,H(F_t^N|\gamma^N) - H(f_t|\gamma)\right| \leq C'\left(\frac{W_2(F_t^N, f_t^{\otimes N})}{\sqrt{N}} + N^{-\beta}\right).$$

We have then to estimate the first term of the right-hand side. From [46], the following estimation holds,

$$\frac{W_2(F_t^N, f_t^{\otimes N})}{\sqrt{N}} \leq C\left(\frac{M_6(F_t^N, f_t^{\otimes N})}{N}\right)^{1/10}\left(\frac{W_1(F_t^N, f_t^{\otimes N})}{N}\right)^{2/5} \tag{3.132}$$

where $M_6(F_t^N, f_t^{\otimes N}) = M_6(F_t^N) + M_6(f_t^{\otimes N})$. We observe that $N^{-1}M_6(F_t^N, f_t^{\otimes N})$ is bounded since $N^{-1}M_6(f_t^{\otimes N}) = M_6(f_t)$, $N^{-1}M_6(F_t^N) \leq C\,N^{-1}M_6(F_0^N)$ by Lemma 3.20 and $N^{-1}M_6(F_0^N)$ is bounded by construction, thanks to the assumption $M_6(f_0)$ finite.

Finally, Theorem 3.15 and last equation (3.132) imply that for any $\epsilon < 9[(7d+6)^2(d+9)]^{-1}$ and any $\beta < (7d+6)^{-1}$ there exists a positive constant $C = C(\epsilon, \beta)$ such that

$$\left|\frac{1}{N}\,H(F_t^N|\gamma^N) - H(f_t|\gamma)\right| \leq C\left(N^{-2\epsilon/5} + N^{-\beta}\right), \tag{3.133}$$

which concludes the proof.

□

Proof of Theorem 3.31 (3). By the HWI inequality [78, Theorem 30.21], for all $t \geq 0$, we have

$$\frac{H(F_t^N|\gamma^N)}{N} \leq \frac{\pi}{2}\sqrt{\frac{I(F_t^N|\gamma^N)}{N}}\,\frac{W_2(F_t^N,\gamma^N)}{\sqrt{N}}.$$

From Lemma 3.30 we have $N^{-1}I(F_t^N|\gamma^N) \leq N^{-1}I(F_0^N|\gamma^N) \leq C$ for some constant $C > 0$ independent of N, by construction. Moreover, thanks to Lemma 3.20 and (3.132) we deduce

$$\frac{W_2(F_t^N,\gamma^N)}{\sqrt{N}} \leq C\left(\frac{W_1(F_t^N,\gamma^N)}{N}\right)^{2/5}.$$

Gathering these two estimates with point (2) in Theorem 3.15 it follows

$$\frac{H(F_t^N|\gamma^N)}{N} \leq C\left(\frac{W_1(F_t^N,\gamma^N)}{N}\right)^{2/5} \leq p(t),$$

for a polynomial function $p(t) \to 0$ as $t \to \infty$.

$\square$

Deuxième partie

Retour vers l'équilibre

Chapitre 4

Exponential convergence to equilibrium for the homogeneous Landau equation

Abstract. This paper deals with the long time behaviour of solutions to the spatially homogeneous Landau equation with hard potentials . We prove an exponential in time convergence towards the equilibrium, improving results of Desvillettes and Villani. Our approach is based on new spectral gap estimates for the linearized Landau operator in weighted (polynomial or stretched exponential) L^p-spaces.

4.1 Introduction and main results

This work deals with the asymptotic behaviour of solutions to the spatially homogeneous Landau equation for hard potentials. It is well known that these solutions converge towards the Maxwellian equilibrium when time goes to infinity and we are interested in quantitative rate of convergence.

On the one hand, in the case of Maxwellian molecules, Villani [74] and Desvillettes-Villani [29] prove a linear functional inequality betwenn the entropy and entropy production by constructive methods, from which one deduces an exponential convergence (with quantitave rate) of the solution to the Landau equation towards the Maxwellian equilibrium in relative entropy, which in turn implies an exponential convergence in L^1-distance (thanks to the Csiszár-Kullback-Pinsker inequality). This kind of linear functional inequality entropy-entropy production is known as Cercignani's Conjecture in Boltzmann and Landau theory, for more details and a review of results we refer to [27].

On the other hand, in the case of hard potentials, Desvillettes-Villani [29] proves a functional inequality for entropy-entropy production that is not linear, from which one obtains a polynomial convergence of solutions towards the equilibrium, again in relative entropy, which implies the same type of convergence in L^1-distance.

Before going further on details of existing results and on the contributions of the present work, we shall introduce in a precise manner the problem addressed here. In kinetic theory, the Landau equation is a model in plasma physics that describes the evolution of the density in the phase space of all positions and velocities of particles. Assuming that the density function does not depend on the position, we obtain the *spatially homogeneous Landau equation* in the form

$$\left\{ \begin{array}{rcl} \partial_t F & = & Q(F,F) \\ F_{|t=0} & = & F_0 \end{array} \right. \tag{4.1}$$

where $F = F(t,v) \geq 0$ is the density of particles with velocity v at time t, $v \in \mathbb{R}^3$ and $t \in \mathbb{R}^+$. The Landau operator Q is a bilinear operator given by

$$Q(G,F) = \partial_i \int_{\mathbb{R}^3} a_{ij}(v-v_*)\left[G_* \partial_j F - F \partial_j G_*\right] dv_*, \tag{4.2}$$

where here and below we shall use the convention of implicit summation over repeated indices and we use the shorthand $G_* = G(v_*)$, $\partial_j G_* = \partial_{v_{*j}} G(v_*)$, $F = F(v)$ and $\partial_j F = \partial_{v_j} F(v)$.

The matrix a is nonnegative, symmetric and depends on the interaction between particles. If two particles interact with a potential proportional to $1/r^s$, where r denotes their distance, a is given by (see for instance [76])

$$a_{ij}(v) = |v|^{\gamma+2}\left(\delta_{ij} - \frac{v_i v_j}{|v|^2}\right), \tag{4.3}$$

with $\gamma = (s-4)/s$. We call hard potentials if $\gamma \in (0,1]$, Maxwellian molecules if $\gamma = 0$, soft potentials if $\gamma \in (-3,0)$ and Coulombian potential if $\gamma = -3$. Through this paper we shall consider the case of hard potentials $\gamma \in (0,1]$.

The Landau equation conserves mass, momentum and energy, indeed, at least formally, for any test function φ we have (see e.g. [73])

$$\int_{\mathbb{R}^3} Q(F,F)\varphi(v)\,dv = \frac{1}{2}\int_{\mathbb{R}^3\times\mathbb{R}^3} a_{ij}(v-v_*) F F_* \left(\frac{\partial_i F}{F} - \frac{\partial_i F_*}{F_*}\right)(\partial_j \varphi - \partial_j \varphi_*)\,dv\,dv_*$$

from which we deduce

$$\int Q(F,F)\varphi(v) = 0 \qquad \text{for} \qquad \varphi(v) = 1, v, |v|^2. \tag{4.4}$$

Moreover, the entropy $H(F) = \int F \log F$ is nonincreasing, indeed, at least formally, since a_{ij} is nonnegative we have the following inequality for $D(F)$ the entropy production,

$$\begin{aligned} D(F) :&= -\frac{d}{dt} H(F) \\ &= \frac{1}{2}\int_{\mathbb{R}^3\times\mathbb{R}^3} F F_*\, a_{ij}(v-v_*)\left(\frac{\partial_i F}{F} - \frac{\partial_{i*} F_*}{F_*}\right)\left(\frac{\partial_j F}{F} - \frac{\partial_{j*} F_*}{F_*}\right) dv\,dv_* \geq 0. \end{aligned} \tag{4.5}$$

It follows that any equilibrium is a Maxwellian distribution

$$\mu_{\rho,u,T}(v) := \frac{\rho}{(2\pi T)^{3/2}} e^{-\frac{|v-u|^2}{2T}},$$

for some $\rho > 0$, $u \in \mathbb{R}^3$ and $T > 0$. This is the Landau version of the famous Boltzmann's H-theorem (for more details we refer to [29, 74] again), from which the solution $F(t, \cdot)$ of the Landau equation is expected to converge towards the Maxwellian μ_{ρ_F, u_F, T_F} when $t \to +\infty$, where ρ_F is the density of the gas, u_F the mean velocity and T_F the temperature, defined by

$$\rho_F = \int F(v), \quad u_F = \frac{1}{\rho} \int v F(v), \quad T_F = \frac{1}{3\rho} \int |v-u|^2 F(v),$$

and these quantities are defined by the initial datum F_0 thanks to the conservation properties of the Landau operator (4.4).

We may only consider the case of initial datum F_0 satisfying

$$\int_{\mathbb{R}^3} F_0(v)\, dv = 1, \quad \int_{\mathbb{R}^3} v F_0(v)\, dv = 0, \quad \int_{\mathbb{R}^3} |v|^2 F_0(v)\, dv = 3, \tag{4.6}$$

the general case being reduced to (4.6) by a simple change of coordinates (see [29]). Then, we shall denote $\mu(v) = (2\pi)^{-3/2} e^{-|v|^2/2}$ the standard Gaussian distribution in $\mathbb{R}^3$, which corresponds to the Maxwellian with $\rho = 1$, $u = 0$ and $T = 1$, i.e. the Maxwellian with same mass, momentum and energy of F_0 (4.6).

We linearize the Landau equation around μ, with the perturbation

$$F = \mu + f,$$

hence the equation satisfied by $f = f(t, v)$ takes the form

$$\partial_t f = \mathcal{L} f + Q(f, f), \tag{4.7}$$

with initial datum f_0 defined by $f_0 = F_0 - \mu$, and where the linearized Landau operator $\mathcal{L}$ is given by

$$\mathcal{L} f = Q(\mu, f) + Q(f, \mu). \tag{4.8}$$

Furthermore, from the conservations properties (4.4), we observe that the null space of $\mathcal{L}$ has dimension 5 and is given by (see e.g. [24, 43, 2, 64, 66])

$$\mathcal{N}(\mathcal{L}) = \mathrm{Span}\{\mu, v_1\mu, v_2\mu, v_3\mu, |v|^2\mu\}. \tag{4.9}$$

4.1.1 Known results

We present here existing results concerning spectral gap estimates for the linearized operator and convergence to equilibrium for the nonlinear equation.

For any weight function $m = m(v)$ ($m : \mathbb{R}^3 \to \mathbb{R}^+$) we define the weighted Lebesgue space $L^p(m)$, for $p \in [1, +\infty]$, associated to the norm

$$\|f\|_{L^p(m)} := \|mf\|_{L^p},$$

and the weighted Sobolev spaces $W^{s,p}(m)$ for $s \in \mathbb{N}$, associated to the norm

$$\|f\|_{W^{s,p}(m)} := \left(\sum_{|\alpha| \le s} \|\partial^\alpha f\|^p_{L^p(m)} \right)^{1/p}, \quad \text{if } p \in [1, +\infty),$$
$$\|f\|_{W^{s,\infty}(m)} := \max_{|\alpha| \le s} \|\partial^\alpha f\|_{L^\infty(m)}.$$

We denote by $\mathcal{D}$ the Dirichlet form associated to $-\mathcal{L}$,

$$\mathcal{D}(f) := \langle -\mathcal{L}f, f \rangle_{L^2(\mu^{-1/2})} := \int (-\mathcal{L}f) f \mu^{-1},$$

and we say that $f \in \mathcal{N}(\mathcal{L})^\perp$, where $\mathcal{N}(\mathcal{L})$ denotes the nullspace of $\mathcal{L}$, if f is of the form $f = f - \Pi f$, where Π denotes the projection onto the null space.

We can now state the existing results on the spectral gap of $\mathcal{L}$. The spectral gap inequality for the linearized Landau operator for hard potentials $\gamma \in (0, 1]$,

$$\mathcal{D}(f) \ge \lambda_0 \|f\|^2_{L^2(\mu^{-1/2})}, \quad \forall f \in \mathcal{N}(\mathcal{L})^\perp, \tag{4.10}$$

for some contructive constant $\lambda_0 > 0$, was proven by Baranger-Mouhot [2].

In the case of hard and soft potentials $\gamma \in (-3, 1]$, Mouhot [64] proved the following result

$$\mathcal{D}(f) \ge \lambda_0 \left\{ \|f\|^2_{H^1(\langle v \rangle^{\gamma/2} \mu^{-1/2})} + \|f\|^2_{L^2(\langle v \rangle^{(\gamma+2)/2} \mu^{-1/2})} \right\}, \quad \forall f \in \mathcal{N}(\mathcal{L})^\perp. \tag{4.11}$$

Furthermore, Guo [43], by nonconstructive arguments, and later Mouhot-Strain [66], by constructive arguments, proved a spectral gap inequality for an anisotropic norm for the linearized Landau operator (in all cases: hard, soft and Coulombien potentials) $\gamma \in [-3, 1]$,

$$\mathcal{D}(f) \ge \lambda_0 \|f\|^2_*, \quad \forall f \in \mathcal{N}(\mathcal{L})^\perp, \tag{4.12}$$

with the anisotropic norm $\|\cdot\|_*$ defined by

$$\|f\|^2_* := \|\langle v \rangle^{\gamma/2} P_v \nabla f\|^2_{L^2(\mu^{-1/2})} + \|\langle v \rangle^{(\gamma+2)/2} (I - P_v) \nabla f\|^2_{L^2(\mu^{-1/2})} + \|\langle v \rangle^{(\gamma+2)/2} f\|^2_{L^2(\mu^{-1/2})}$$

where P_v denotes the projection onto the v-direction, more precisely $P_v g = \left(\frac{v}{|v|} \cdot g \right) \frac{v}{|v|}$. We also have from [43], the reverse inequality

$$\mathcal{D}(f) \le C_2 \|f\|^2_*, \quad \forall f \in \mathcal{N}(\mathcal{L})^\perp, \tag{4.13}$$

which, toghether with (4.12), imply a spectral gap for $\mathcal{L}$ in $L^2(\mu^{-1/2})$ if and only if $\gamma + 2 \ge 0$.

Summarizing the results (4.10), (4.11) and (4.12) , for $\gamma \in [0, 1]$ there is a constructive constant $\lambda_0 > 0$ such that

$$\forall t \ge 0, \ \forall f \in L^2(\mu^{-1/2}), \qquad \|e^{t\mathcal{L}} f - \Pi f\|_{L^2(\mu^{-1/2})} \le e^{-\lambda_0 t} \|f - \Pi f\|_{L^2(\mu^{-1/2})}, \tag{4.14}$$

where Π denotes the projection onto $\mathcal{N}(\mathcal{L})$, the null space of $\mathcal{L}$ given by (4.9).

Another approach is to study directly the nonlinear equation, establishing functional inequalities for the entropy production $D(F)$. The entropy production inequality for the (nonlinear) Landau operator for maxwellian molecules $\gamma = 0$

$$D(F) \geq \delta_0 \, H(F|\mu), \quad \forall F \in L^1_{1,0,1}(\mathbb{R}^3) := \Big\{ G \in L^1(\mathbb{R}^3); \rho_G = 1, u_G = 0, T_G = 1 \Big\}\,; \tag{4.15}$$

for some explicit constant δ_0, was proven by Desvillettes-Villani [29] and Villani [74]. Here $H(F|\mu) := \int F \log(F/\mu)$ denotes the relative entropy of F with respect to μ, and this inequality implies an exponential decay to the equilibrium μ. Taking $F = \mu + \varepsilon f$, they also deduce a degenerated spectral gap inequality for the linearized Landau operator for $\gamma = 0$,

$$\mathcal{D}(f) \geq \bar{\delta}_0 \, \|\nabla f\|^2_{L^2(\mu^{-1/2})} \quad \forall \, f \in \mathcal{N}(\mathcal{L})^\perp. \tag{4.16}$$

In the case of hard potentials $\gamma \in (0, 1]$, Desvillettes-Villani [29] proved the following entropy-entropy production inequality, for some explicit $\delta_1, \delta_2 > 0$,

$$D(F) \geq \min \Big\{ \delta_1 H(F|\mu), \delta_2 H(F|\mu)^{1+\gamma/2} \Big\} \quad \forall F \in L^1_{1,0,1}(\mathbb{R}^3), \tag{4.17}$$

which implies a polynomial decay to equilibrium (see theorem 4.16 for more details).

As we can see above, the result (4.17) tell us that any solution to the Landau equation converges to the equilibrium in polynomial time, and from the spectral gap estimate for the linearized operator (4.14), if the solution lies in some suitable neighborhood of the equilibrium, the convergence is exponential in time. One could then expect to prove an exponential convergence to equilibrium combining these to results : for small times one uses (4.17), then for large times, when the solution enters in the appropriated neighborhood of the equilibrium in $L^2(\mu^{-1/2})$-norm, one uses (4.14). However, the spectral gap for the linearized operator holds in $L^2(\mu^{-1/2})$ and the Cauchy theory [28] for the nonlinear Landau equation is constructed in L^1-spaces with polynomial weight, which means that to apply the strategy above, starting from some initial datum in weighted L^1-space, one would need the appearance of the $L^2(\mu^{-1/2})$-norm of any solution in positive time to be able to use (4.14), and this is not known to be true (one does not know even if the $L^2(\mu^{-1/2})$-norm is propagated). Hence, in order to be able to connect the linearized theory with the nonlinear theory, we need to enlarge the functional space of spectral gap estimates for the linearized operator $\mathcal{L}$.

Our goal in this paper is *to prove an exponential in time convergence of solutions to the Landau equation towards the equilibrium* and our strategy is based on (1) new spectral gap estimates for the linearized Landau operator $\mathcal{L}$ in various L^p-spaces with polynomial and streched exponential weight, (2) the well-known Cauchy theory for the nonlinear equation, using the appearance of L^1- moments and (3) the strategy of connecting the linearized theory with the nonlinear one, presented in the above paragraph.

4.1.2 Statement of the main result

Let us state our main result, which proves an exponential decay to equilibrium for the spatially homogeneaous Landau equation with hard potentials.

Theorem 4.1 (Exponential decay to equilibrium)**.** *Let $\gamma \in (0,1]$ and a nonnegative $F_0 \in L^1(\langle v\rangle^{2+\delta})$ for some $\delta > 0$, satisfying* (4.6)*. Then, for any solution $(F_t)_{t\geq 0}$ to the spatially homogeneous Landau equation* (4.1) *with initial datum F_0, there exists a constant $C > 0$ such that*

$$\forall\, t \geq 0, \qquad \|F_t - \mu\|_{L^1} \leq C e^{-\lambda_0 t},$$

where λ_0 is the spectral gap (4.14) *of the linearized operator $\mathcal{L}$ on $L^2(\mu^{-1/2})$.*

As mentioned above, in the case of hard potentials $\gamma \in (0,1]$, a polynomial decay to equilibrium was proven by Desvillettes and Villani [29] and in the case of maxwellian molecules $\gamma = 0$ an exponential decay to equilibrium was proven by Villani [74] and also by Desvillettes and Villani [29]. The proof of theorem 4.1 relies on coupling the polynomial decay from [29] for small times and the exponential decay for the linearized operator in weighte L^p-spaces from theorem 4.3 for large times, when the linearized dynamics is dominant. This method was first used by Mouhot [65] where is proved the exponential decay to equilibrium for the spatially homogeneous Boltzmann equation for hard potentials. Later, the same approach was used by Gualdani, Mischler and Mouhot [42] to prove the exponential decay to the equilibrium for the inhomogeneous Boltzmann equation for hard spheres on the torus.

4.1.3 Organization of the paper

We start Section 4.2 presenting some properties of the linearized equation and then we state and prove the spectral gap extension theorem (Theorem 4.14), which is a key ingredient of the proof of the main theorem. Finally, in Section 4.3, we prove estimates for the (nonlinear) Landau operator and then prove theorem 4.1.

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4.2 The linearized equation

We define (see e.g. [28, 73, 74]) in 3-dimension the following quantities

$$b_i(z) = \partial_j a_{ij}(z) = -2\,|z|^\gamma\, z_i, \quad c(z) = \partial_{ij} a_{ij}(z) = -2(\gamma+3)\,|z|^\gamma. \tag{4.18}$$

Hence, we can rewrite the Landau operator (4.2) in the following way

$$Q(G,F) = (a_{ij} * G)\partial_{ij} F - (c * G)F = \partial_i[(a_{ij} * G)\partial_j F - (b_i * G)F]. \tag{4.19}$$

We also denote

$$\bar{a}_{ij}(v) = a_{ij} * \mu, \quad \bar{b}_i(v) = b_i * \mu, \quad \bar{c}(v) = c * \mu. \tag{4.20}$$

Furthermore we have the following results concerning $\bar{a}_{ij}(v)$.

Lemma 4.2. *The following properties hold:*

(a) The matrix $\bar{a}(v)$ has a simple eigenvalue $\ell_1(v) > 0$ associated with the eigenvector v and a double eigenvalue $\ell_2(v) > 0$ associated with the eigenspace $v^\perp$. Moreover,

$$\ell_1(v) = \int_{\mathbb{R}^3} \left(1 - \left(\frac{v}{|v|} \cdot \frac{w}{|w|}\right)^2\right) |w|^{\gamma+2} \mu(v-w)\, dw$$

$$\ell_2(v) = \int_{\mathbb{R}^3} \left(1 - \frac{1}{2}\left|\frac{v}{|v|} \times \frac{w}{|w|}\right|^2\right) |w|^{\gamma+2} \mu(v-w)\, dw.$$

When $|v| \to +\infty$ we have

$$\ell_1(v) \sim 2|v|^\gamma$$

$$\ell_2(v) \sim |v|^{\gamma+2}.$$

If $\gamma \in (0,1]$ there exists $\ell_0 > 0$ such that, for all $v \in \mathbb{R}^3$, $\min\{\ell_1(v), \ell_2(v)\} \geq \ell_0$.

(b) The function $\bar{a}_{ij}$ is smooth, for any multi-index $\beta \in \mathbb{N}^3$

$$|\partial^\beta \bar{a}_{ij}(v)| \leq C_\beta \langle v \rangle^{\gamma+2-|\beta|}$$

and

$$\bar{a}_{ij}(v)\xi_i\xi_j = \ell_1(v)|P_v\xi|^2 + \ell_2(v)|(I-P_v)\xi|^2,$$

$$\bar{a}_{ij}(v)v_iv_j = \ell_1(v)|v|^2,$$

where P_v is the projection on v, i.e.

$$P_v\xi = \left(\xi \cdot \frac{v}{|v|}\right)\frac{v}{|v|}.$$

(c) We have

$$\bar{a}_{ii}(v) = 2\int_{\mathbb{R}^3} |v-v_*|^{\gamma+2}\mu(v_*)\, dv_* \qquad \textit{and} \qquad \bar{b}_i(v) = -\ell_1(v)\, v_i.$$

Proof. We just give the proof of item (c) since (a) comes from [24, Propositions 2.3 and 2.4, Corollary 2.5] and (b) is [43, Lemma 3].

Hence, for item (c) we write

$$\bar{a}_{ii}(v) = \sum_{i=1}^{3} \int_{\mathbb{R}^3} a_{ii}(v-v_*)\mu(v_*)\, dv_*.$$

Using (4.3) we obtain that

$$a_{ii}(z) = \sum_{i=1}^{3} |z|^{\gamma+2}\left(1 - \frac{z_i^2}{|z|^2}\right) = 2|z|^{\gamma+2}$$

and then

$$\bar{a}_{ii}(v) = 2 \int_{\mathbb{R}^3} |v - v_*|^{\gamma+2} \mu(v_*)\, dv_*.$$

Moreover, we compute

$$\bar{b}_i(v) = (\partial_j a_{ij} * \mu)(v) = (a_{ij} * \partial_j \mu)(v) = -\int_{\mathbb{R}^3} a_{ij}(v - v_*) v_{*j}\, \mu(v_*)\, dv_*,$$

and using that $a_{ij}(z) z_j = 0$ we obtain

$$\begin{aligned}
\bar{b}_i(v) &= -\int_{\mathbb{R}^3} a_{ij}(v - v_*) v_{*j}\, \mu(v_*)\, dv_* \\
&= -\int_{\mathbb{R}^3} a_{ij}(v_*)(v_j - v_{*j})\, \mu(v - v_*)\, dv_* \\
&= -\left(\int_{\mathbb{R}^3} a_{ij}(v_*) \mu(v - v_*)\, dv_* \right) v_j = -\bar{a}_{ij}(v) v_j = -\ell_1(v)\, v_i.
\end{aligned}$$

□

Using the form (4.19) of the operator Q, we decompose the linearized Landau operator $\mathcal{L}$ defined in (4.8) as $\mathcal{L} = \mathcal{A}_0 + \mathcal{B}_0$, where we define

$$\begin{aligned}
\mathcal{A}_0 f &:= Q(f, \mu) = (a_{ij} * f)\partial_{ij}\mu - (c * f)\mu, \\
\mathcal{B}_0 f &:= Q(\mu, f) = (a_{ij} * \mu)\partial_{ij} f - (c * \mu) f.
\end{aligned} \tag{4.21}$$

Consider a smooth nonnegative function $\chi \in C_c^\infty(\mathbb{R}^3)$ such that $0 \le \chi(v) \le 1$, $\chi(v) \equiv 1$ for $|v| \le 1$ and $\chi(v) \equiv 0$ for $|v| > 2$. For any $R \ge 1$ we define $\chi_R(v) := \chi(R^{-1} v)$ and in the sequel we shall consider the function $M\chi_R$, for some constant $M > 0$. Then, we make the final decomposition of the operator $\mathcal{L}$ as $\mathcal{L} = \mathcal{A} + \mathcal{B}$ with

$$\mathcal{A} := \mathcal{A}_0 + M\chi_R, \qquad \mathcal{B} := \mathcal{B}_0 - M\chi_R, \tag{4.22}$$

where M and R will be chosen later (see lemma 4.7).

Let us now make our assumptions on the weight functions $m = m(v)$. We define the polynomial weight, for all $p \in [1, +\infty)$,

$$m = \langle v \rangle^k, \quad \text{with } k > \max\{\gamma(1 - 1/p), \gamma + 2\} + 3(1 - 1/p) \tag{4.23}$$

and the abcissa

$$\begin{aligned}
a_{m,p} &:= 2[(\gamma + 3)(1 - 1/p) - k], && \text{if } \gamma = 0, \\
a_{m,p} &:= -\infty, && \text{if } \gamma \in (0, 1].
\end{aligned} \tag{4.24}$$

Moreover, we define the exponential weight, for $p \in [1, +\infty)$,

$$m = \exp\left(r \langle v \rangle^s\right), \quad \text{with } \begin{cases} r > 0, & \text{if } s \in (0, 2), \\ 0 < r < \frac{1}{2p}, & \text{if } s = 2, \end{cases} \tag{4.25}$$

and we define the abcissa, for all cases,

$$a_{m,p} := -\infty. \tag{4.26}$$

We are able know to state the following result that extends to various weighted L^p-spaces the spectral gap estimate known to hold on $L^2(\mu^{-1/2})$ as presented in (4.14).

Theorem 4.3 (Spectral gap estimates)**.** *Let $\gamma \in [0,1]$, $p \in [1,2]$, a weight function $m = m(v)$ satisfying* (4.23) *or* (4.25) *and their respectives abscissa $a_{m,p}$ given by* (4.24) *or* (4.26)*. Consider the linearized Landau operator* (4.8)*, then for any positive $\lambda \leq \min\{\lambda_0, |a_{m,p}| + 0\}$ there exists $C_\lambda > 0$ such that*

$$\forall\, t \geq 0,\ \forall\, f \in L^p(m), \qquad \|e^{t\mathcal{L}} f - \Pi f\|_{L^p(m)} \leq C_\lambda\, e^{-\lambda t}\, \|f - \Pi f\|_{L^p(m)}, \tag{4.27}$$

where Π is the projection onto the null space of $\mathcal{L}$ (defined in (4.9)*), and $\lambda_0 > 0$ is the spectral gap of $\mathcal{L}$ in $L^2(\mu^{-1/2})$ given by* (4.14)*.*

Remark 4.4. As we can see in the defition of $a_{m,p}$ (4.24) and (4.26), we conclude that:

1. If $\gamma \in [0,1]$ and m is the streched exponential weight (4.25) or if $\gamma \in (0,1]$ and m is the polynomial weight (4.23), then $\lambda = \lambda_0$ since $a_{m,p} := -\infty$.
2. If $\gamma = 0$ and $m = \langle v \rangle^k$ is the polynomial weight (4.23), then $\lambda = \lambda_0$ if k is big enough such that $a_{m,p} := 2[(\gamma+3)(1-1/p) - k] < -\lambda_0$, otherwise $\lambda = 2[k - (\gamma+3)(1-1/p)] + 0$.

This theorem extends the spectral gap to weighted L^p spaces using a method developped by Gualdani, Mischler and Mouhot [42] (see theorem 4.14 below) for Boltzmann and Fokker-Planck equations (see also Mischler and Mouhot [61] for other results on Fokker-Planck equations).

4.2.1 Hypodissipativity properties

In this subsection we shall investigate the hypodissipativity in $L^p(m)$ spaces of the operator $\mathcal{B}$ defined in (4.22) in order to prove assumption (i) of theorem 4.14. Before proving the desired result in lemma 4.7, we give the following lemma that will be useful in the sequel.

Lemma 4.5. *Let $J_\alpha(v) := \int_{\mathbb{R}^3} |v-w|^\alpha \mu(w)\, dw$, for $0 \leq \alpha \leq 3$, and denote $M_\alpha(\mu) := \int |v|^\alpha \mu$. Then it holds:*

(a) $J_0(v) = 1$.

(b) $J_\alpha(v) \leq |v|^\alpha + M_\alpha(\mu)$*, for* $0 < \alpha \leq 1$.

(c) $J_\alpha(v) \leq |v|^\alpha + M_2(\mu)^{\alpha/2}$*, for* $1 < \alpha < 2$.

(d) $J_2(v) = |v|^2 + M_2(\mu)$.

(e) $J_\alpha(v) \leq |v|^\alpha + 10^{\alpha/4}|v|^{\alpha/2} + M_4(\mu)^{\alpha/4}$*, for* $2 < \alpha \leq 3$.

Remark 4.6. As we will see in the proof of lemma 4.7, the important point here is that, for all $0 \leq \alpha \leq 3$, the dominant part of the upper bound of J_α has coefficient 1.

Proof. Items (a) and (d) are evident. For (b) we see that $|v-w|^\alpha \leq |v|^\alpha + |w|^\alpha$ and it implies $J_\alpha(v) \leq |v|^\alpha + M_\alpha(\mu)$. To prove item (c) we use $\alpha/2 < 1$ and Jensen's inequality to write

$$J_\alpha(v) \leq \left(\int_{\mathbb{R}^3} |v-w|^2 \mu(dw) \right)^{\alpha/2} = \Big(|v|^2 + M_2(\mu) \Big)^{\alpha/2} \leq |v|^\alpha + M_2(\mu)^{\alpha/2}.$$

Finally, item (e) can be proven in the same way as (d). Firstly, for $\alpha = 4$ explicit computation gives $J_4(v) = |v|^4 + 10|v|^2 + M_4(\mu)$. Then, from $\alpha/4 < 1$ and Jensen's inequality we obtain

$$\begin{aligned} J_\alpha(v) \le \left(\int_{\mathbb{R}^3} |v-w|^4 \mu(dw) \right)^{\alpha/4} &= \left(|v|^4 + 10|v|^2 + M_4(\mu) \right)^{\alpha/4} \\ &\le |v|^\alpha + 10^{\alpha/4} |v|^{\alpha/2} + M_4(\mu)^{\alpha/4}. \end{aligned}$$

□

With the help of the result above, we are able to state the hypodissipativity result for $\mathcal{B}$.

Lemma 4.7. *Let $\gamma \in [0,1]$, $p \in [1,+\infty)$ and consider a weight function $m = m(v)$ satisfying* (4.23) *or* (4.25) *with the corresponding definitions of the abcissa* (4.24) *or* (4.26)*, respectively. Then, for any $a > a_{m,p}$ we can choose M and R large enough such that the operator $\mathcal{B} - a$ is dissipative in $L^p(m)$.*

Proof. We split the proof into four steps.

Step 1. Let us denote $\Phi'(z) = |z|^{p-1}\mathrm{sign}(z)$ and consider the equation

$$\partial_t f = \mathcal{B} f = \mathcal{B}_0 f - M\chi_R f.$$

For all $1 \le p < +\infty$, we have

$$\begin{aligned} \frac{d}{dt}\|f\|_{L^p(m)} &= \|f\|_{L^p(m)}^{1-p} \left\{ \int (\mathcal{B} f)\Phi'(f) m^p \right\} \\ &= \|f\|_{L^p(m)}^{1-p} \left\{ \int (\mathcal{B}_0 f)\Phi'(f) m^p - \int (M\chi_R f)\Phi'(f) m^p \right\} \end{aligned} \tag{4.28}$$

with, from (4.21) and (4.19),

$$\int (\mathcal{B}_0 f)\Phi'(f)\, m^p = \int \bar{a}_{ij}\partial_{ij} f\, \Phi'(f)\, m^p - \int \bar{c}\, m^p\, |f|^p$$

Let us denote $h = m^\theta f$, for some θ to be chosen later. For the first term, using $\Phi'(f) = \Phi'(h)\, m^{-\theta(p-1)}$, we have

$$\begin{aligned} T_1 &= \int \bar{a}_{ij}\partial_{ij}(hm^{-\theta})\, \Phi'(h)\, m^{p+\theta(1-p)} \\ &= -\int \partial_j(hm^{-\theta})\partial_i \left(\bar{a}_{ij}\Phi'(h) m^{p+\theta(1-p)} \right) \\ &= -\int \partial_j(hm^{-\theta})\bar{a}_{ij}\partial_i \left(\Phi'(h) m^{p+\theta(1-p)} \right) - \int \partial_j(hm^{-\theta})\bar{b}_j\, \Phi'(h) m^{p+\theta(1-p)} \\ &=: T_{11} + T_{12}. \end{aligned}$$

We also have

$$\begin{aligned}
&\partial_j(hm^{-\theta})\partial_i\left(\Phi'(h)m^{p+\theta(1-p)}\right)\\
&\quad= (p-1)\partial_i h\partial_j h\, m^{p(1-\theta)}\,|h|^{p-2} + \frac{[p+\theta(1-p)]}{p}\partial_i m\partial_j(|h|^p)\, m^{p(1-\theta)-1}\\
&\quad- \frac{\theta(p-1)}{p}\partial_i(|h|^p)\partial_j m\, m^{p(1-\theta)-1} - \theta[p-\theta(p-1)]\partial_i m\partial_j m\, m^{p(1-\theta)-2}\,|h|^p,
\end{aligned}$$

then, since $\bar a_{ij}$ is simmetric , it follows

$$\begin{aligned}
T_{11} &= -(p-1)\int \bar a_{ij}\partial_i h\partial_j h\, m^{p(1-\theta)}\,|h|^{p-2}\\
&\quad+\left[2\theta\frac{(p-1)}{p}-1\right]\int \bar a_{ij}\partial_i m\partial_j(|h|^p)\, m^{p(1-\theta)-1}\\
&\quad+\theta[p-\theta(p-1)]\int \bar a_{ij}\partial_i m\partial_j m\, m^{p(1-\theta)-2}\,|h|^p.
\end{aligned}$$

Performing an integration by parts, we obtain

$$\begin{aligned}
T_{11} &= -(p-1)\int \bar a_{ij}\partial_i h\partial_j h\, m^{p(1-\theta)}\,|h|^{p-2}\\
&\quad+\delta_1(p,\theta)\int \bar b_i\partial_i m\, m^{p(1-\theta)-1}\,|h|^p\\
&\quad+\delta_1(p,\theta)\int \bar a_{ij}\partial_{ij} m\, m^{p(1-\theta)-1}\,|h|^p\\
&\quad+\delta_2(p,\theta)\int \bar a_{ij}\partial_i m\partial_j m\, m^{p(1-\theta)-2}\,|h|^p
\end{aligned}\tag{4.29}$$

where

$$\delta_1(p,\theta) := 1-2\theta(1-1/p), \qquad \delta_2(p,\theta) := \delta_1(p,\theta)[p(1-\theta)-1]+\theta[p-\theta(p-1)]. \tag{4.30}$$

For th term T_{12} we have

$$\begin{aligned}
T_{12} &= -\int \partial_j(hm^{-\theta})\bar b_j\,\Phi'(h)m^{p+\theta(1-p)}\\
&= -\int \partial_j h\Phi'(h)\bar b_j\, m^{p(1-\theta)} + \theta\int h\Phi'(h)\bar b_j\partial_j m\, m^{p(1-\theta)-1}\\
&= -\frac{1}{p}\int \partial_j(|h|^p)\bar b_j\, m^{p(1-\theta)} + \theta\int \bar b_j\partial_j m\, m^{p(1-\theta)-1}\,|h|^p\\
&= \frac{1}{p}\int \bar c\, m^{p(1-\theta)}\,|h|^p + \int \bar b_j\partial_j m\, m^{p(1-\theta)-1}\,|h|^p.
\end{aligned}\tag{4.31}$$

Gathering (4.29) and (4.31) one obtains

$$\int(\mathcal{B}_0 f)\Phi'(g)m^p = -(p-1)\int \bar a_{ij}\partial_i(m^\theta f)\partial_j(m^\theta f)\, m^{p-2\theta}\,|f|^{p-2} + \int \varphi_{m,p,\theta}(v)\, m^p\,|f|^p, \tag{4.32}$$

with

$$\begin{aligned}\varphi_{m,p,\theta} := \delta_1(p,\theta)\left(\bar{a}_{ij}\,\frac{\partial_{ij}m}{m}\right) + \delta_2(p,\theta)\left(\bar{a}_{ij}\,\frac{\partial_i m}{m}\,\frac{\partial_j m}{m}\right) \\ + (1+\delta_1(p,\theta))\left(\bar{b}_i\,\frac{\partial_i m}{m}\right) + \left(\frac{1}{p}-1\right)\bar{c},\end{aligned} \tag{4.33}$$

where δ_1 and δ_2 are defined by (4.30).

Let us now split the proof into two steps, corresponding to the polynomial weight (4.23) and to the streched exponential weight (4.25).

Step 2 : Polynomial weight. Consider $m = \langle v\rangle^k$ defined in (4.23). On the one hand, we have

$$\frac{\partial_i m}{m} = k v_i \langle v\rangle^{-2}, \qquad \frac{\partial_i m}{m}\,\frac{\partial_j m}{m} = k^2 v_i v_j \langle v\rangle^{-4},$$

$$\frac{\partial_{ij} m}{m} = \delta_{ij}\, k\langle v\rangle^{-2} + k(k-2) v_i v_j \langle v\rangle^{-4}.$$

Hence, from the definitions (4.18)-(4.20) and lemma 4.2 we obtain

$$\begin{aligned}\bar{a}_{ij}\,\frac{\partial_{ij}m}{m} &= (\delta_{ij}\bar{a}_{ij})\, k\langle v\rangle^{-2} + (\bar{a}_{ij}v_iv_j)\, k(k-2)\langle v\rangle^{-4} \\ &= \bar{a}_{ii}\, k\langle v\rangle^{-2} + \ell_1(v)\, k(k-2)|v|^2\langle v\rangle^{-4},\end{aligned} \tag{4.34}$$

where we recall that the eigenvalue $\ell_1(v) > 0$ is defined in lemma 4.2. Moreover, arguing exactly as above we obtain

$$\bar{a}_{ij}\,\frac{\partial_i m}{m}\,\frac{\partial_j m}{m} = (\bar{a}_{ij}v_iv_j)\, k^2\langle v\rangle^{-4} = \ell_1(v)\, k^2|v|^2\langle v\rangle^{-4} \tag{4.35}$$

and also, using the fact that $\bar{b}_i(v) = -\ell_1(v)v_i$ from lemma 4.2,

$$\bar{b}_i\,\frac{\partial_i m}{m} = -\ell_1(v)v_i\, kv_i\langle v\rangle^{-2} = -\ell_1(v)\, k|v|^2\langle v\rangle^{-2}. \tag{4.36}$$

On the other hand, from item (c) of lemma 4.2 and definitions (4.18)-(4.20) we obtain that

$$\bar{a}_{ii} = 2J_{\gamma+2}(v) \quad \text{and} \quad \bar{c} = -2(\gamma+3)J_\gamma(v), \tag{4.37}$$

where J_α is defined in lemma 4.5. It follows from (4.33)–(4.37) that

$$\begin{aligned}\varphi_{m,p,\theta}(v) = \delta_1(p,\theta)\, 2kJ_{\gamma+2}(v)\langle v\rangle^{-2} + \delta_1(p,\theta)\, k(k-2)\,\ell_1(v)\,|v|^2\langle v\rangle^{-4} \\ + \delta_2(p,\theta)\, k^2\,\ell_1(v)\,|v|^2\langle v\rangle^{-4} - [1+\delta_1(p,\theta)]\, k\,\ell_1(v)\,|v|^2\langle v\rangle^{-2} \\ + 2(\gamma+3)\left(1-\frac{1}{p}\right) J_\gamma(v).\end{aligned} \tag{4.38}$$

Since $\ell_1(v) \sim 2\langle v\rangle^\gamma$ and $J_\alpha(v) \sim \langle v\rangle^\alpha$ when $|v| \to +\infty$ by lemmas 4.2 and 4.5, the dominant terms in (4.38) are the first, fourth and fifth one, all of order $\langle v\rangle^\gamma$.

For $p \in (1, +\infty)$ we choose $\theta = p/[2(p-1)]$, then $\delta_1(p,\theta) = 0$, $\delta_2(p,\theta) = p^2/[4(p-1)]$ and

$$\varphi_{m,p,\theta}(v) = \frac{p^2}{4(p-1)}\, k^2\, \ell_1(v)\, |v|^2 \langle v\rangle^{-4} - k\, \ell_1(v)\, |v|^2 \langle v\rangle^{-2} + 2(\gamma+3)\left(1 - \frac{1}{p}\right) J_\gamma(v).$$

Using lemma 4.5 to bound J_γ, we obtain that $\varphi_{m,p,\theta}(v) \le \bar\varphi_{m,p,\theta}(v)$ where

$$\bar\varphi_{m,p,\theta}(v) \underset{|v|\to+\infty}{\sim} -2k\langle v\rangle^\gamma + 2(\gamma+3)(1-1/p)\langle v\rangle^\gamma.$$

Hence it yields, since $k > (\gamma+3)(1-1/p)$ from (4.23),

$$\begin{cases} \bar\varphi_{m,p,\theta}(v) \underset{|v|\to+\infty}{\sim} -2\left[k - (\gamma+3)(1-1/p)\right] \xrightarrow[|v|\to+\infty]{} k_c < 0, & \text{if } \gamma = 0, \\ \bar\varphi_{m,p,\theta}(v) \underset{|v|\to+\infty}{\sim} -2\left[k - (\gamma+3)(1-1/p)\right]\langle v\rangle^\gamma \xrightarrow[|v|\to+\infty]{} -\infty, & \text{if } \gamma \in (0,1]. \end{cases} \tag{4.39}$$

If $p = 1$, for all θ, we have $\delta_1(1,\theta) = 1$ and $\delta_2(1,\theta) = 0$ which gives

$$\varphi_{m,1,\theta}(v) = 2kJ_{\gamma+2}(v)\langle v\rangle^{-2} + k(k-2)\lambda(v)|v|^2\langle v\rangle^{-4} - 2k\ell_1(v)|v|^2\langle v\rangle^{-2},$$

and the dominant terms are the first and last one, both of order $\langle v\rangle^\gamma$. Using lemma 4.5 to bound $J_{\gamma+2}$, we have $\varphi_{m,1,\theta}(v) \le \bar\varphi_{m,1,\theta}(v)$ where

$$\bar\varphi_{m,1,\theta}(v) \underset{|v|\to+\infty}{\sim} 2k\langle v\rangle^\gamma - 4k\langle v\rangle^\gamma = -2k\langle v\rangle^\gamma.$$

Then we obtain, since $k > 0$ from (4.23),

$$\begin{cases} \bar\varphi_{m,1,\theta}(v) \underset{|v|\to+\infty}{\sim} -2k \xrightarrow[|v|\to+\infty]{} -2k, & \text{if } \gamma = 0, \\ \bar\varphi_{m,1,\theta}(v) \underset{|v|\to+\infty}{\sim} -2k\langle v\rangle^\gamma \xrightarrow[|v|\to+\infty]{} -\infty, & \text{if } \gamma \in (0,1]. \end{cases} \tag{4.40}$$

Step 3 : Exponential weight. We consider now $m = \exp(r\langle v\rangle^s)$ given by (4.25). In this case we have

$$\frac{\partial_i m}{m} = rsv_i\langle v\rangle^{s-2}, \qquad \frac{\partial_i m}{m}\frac{\partial_j m}{m} = r^2s^2v_iv_j\langle v\rangle^{2s-4},$$
$$\frac{\partial_{ij} m}{m} = rs\langle v\rangle^{s-2}\delta_{ij} + rs(s-2)v_iv_j\langle v\rangle^{s-4} + r^2s^2v_iv_j\langle v\rangle^{2s-4}.$$

It follows from last equation that

$$\begin{aligned} \bar a_{ij}\,\frac{\partial_{ij} m}{m} &= (\delta_{ij}\bar a_{ij})\, rs\langle v\rangle^{s-2} + (\bar a_{ij}v_iv_j)\, rs(s-2)\langle v\rangle^{s-4} + (\bar a_{ij}v_iv_j)\, r^2s^2\langle v\rangle^{2s-4} \\ &= \bar a_{ii}\, rs\langle v\rangle^{s-2} + \ell_1(v)\, rs(s-2)|v|^2\langle v\rangle^{s-4} + \ell_1(v)\, r^2s^2|v|^2\langle v\rangle^{2s-4}, \end{aligned} \tag{4.41}$$

where we used lemma 4.2,

$$\bar a_{ij}\,\frac{\partial_i m}{m}\,\frac{\partial_j m}{m} = (\bar a_{ij}v_iv_j)\, r^2s^2\langle v\rangle^{2s-4} = \ell_1(v)\, r^2s^2|v|^2\langle v\rangle^{2s-4} \tag{4.42}$$

and also, using the fact that $\bar{b}_i(v) = -\ell_1(v)v_i$,

$$\bar{b}_i \, \frac{\partial_i m}{m} = -\ell_1(v)v_i \, rsv_i\langle v\rangle^{s-2} = -\ell_1(v)\, rs|v|^2\langle v\rangle^{s-2}. \tag{4.43}$$

Gathering togheter (4.33), (4.41), (4.42) and (4.43), and thanks to lemma 4.2, it yields

$$\begin{aligned}\varphi_{m,p,\theta}(v) &= \delta_1(p,\theta)\, 2rsJ_{\gamma+2}(v)\langle v\rangle^{s-2} + \delta_1(p,\theta)\, rs(s-2)\ell_1(v)|v|^2\langle v\rangle^{s-4} \\ &\quad + \delta_1(p,\theta)\, r^2s^2\ell_1(v)|v|^2\langle v\rangle^{2s-4} + \delta_2(p,\theta)\, r^2s^2\ell_1(v)|v|^2\langle v\rangle^{2s-4} \\ &\quad - [1+\delta_1(p,\theta)]\, rs\ell_1(v)|v|^2\langle v\rangle^{s-2} + 2(\gamma+3)\left(1-\frac{1}{p}\right) J_\gamma(v)\end{aligned} \tag{4.44}$$

where we recall that J_α is given in lemma 4.5.

Let us choose $\theta = 0$ for all cases $p \in [1,+\infty)$. Then $\delta_1(p,0) = 1$, $\delta_2(p,0) = p-1$ and

$$\begin{aligned}\varphi_{m,p,0}(v) &= 2rsJ_{\gamma+2}(v)\langle v\rangle^{s-2} + rs(s-2)\ell_1(v)|v|^2\langle v\rangle^{s-4} + pr^2s^2\ell_1(v)|v|^2\langle v\rangle^{2s-4} \\ &\quad - 2rs\ell_1(v)|v|^2\langle v\rangle^{s-2} + 2(\gamma+3)\left(1-\frac{1}{p}\right) J_\gamma(v),\end{aligned} \tag{4.45}$$

and we recall that $\ell_1(v) \sim 2\langle v\rangle^\gamma$ and $J_\alpha(v) \sim \langle v\rangle^\alpha$ when $|v| \to +\infty$ by lemmas 4.2 and 4.5.

If $0 < s < 2$, the dominant terms in (4.45) is the fourth one, of order $\langle v\rangle^{\gamma+s}$. Then we obtain the asymptotic behaviour

$$\varphi_{m,p,0}(v) \underset{|v|\to+\infty}{\approx} -4rs\langle v\rangle^{s+\gamma} \xrightarrow[|v|\to+\infty]{} -\infty \tag{4.46}$$

since $s+\gamma > 0$. If $s = 2$, the dominant terms in (4.45) are the first, third and fourth one, all of order $\langle v\rangle^{\gamma+2}$. Hence, using lemma 4.5 to bound $J_{\gamma+2}$ and lemma 4.2, we have $\varphi_{m,p,0}(v) \le \bar{\varphi}_{m,p,0}(v)$ where the asymptotic behaviour of $\bar{\varphi}$ is given by

$$\bar{\varphi}_{m,p,0}(v) \underset{|v|\to+\infty}{\approx} 4r\,(2pr-1)\,\langle v\rangle^{\gamma+2} \xrightarrow[|v|\to+\infty]{} -\infty \tag{4.47}$$

since $r < 1/(2p)$ from (4.25).

Remark 4.8. We could also, for $p \in (1,+\infty)$, chose $\theta = p/[2(p-1)]$ as we did for the polynomial weight. This would not change anything for $0 < s < 2$, however for the case $s = 2$ we would obtain

$$\varphi_{m,p,\theta}(v) \underset{|v|\to+\infty}{\approx} \left(\frac{2p^2r^2}{p-1} - 4r\right)\langle v\rangle^{\gamma+2}$$

which goes to $-\infty$ when $|v| \to +\infty$ if $r < 2(p-1)/p^2$, modifiyng then the conditions on r defined in (4.25). Using these two computations, a more general condition on r defined in (4.25) in the case $s = 2$ would be $r < \max\left\{\frac{1}{2p}, \frac{2(p-1)}{p^2}\right\}$.

Step 4. Finally, gathering steps 1, 2 and 3, for any $p \in [1,+\infty)$, for any $a > a_{m,p}$, thanks to the asymptotic behaviour (4.39)-(4.40)-(4.46)-(4.47), we can choose M and R large enough such that $\varphi_{m,p,\theta}(v) - M\chi_R(v) \le a$ for all $v \in \mathbb{R}^3$. It follows that the operator $\mathcal{B} - a = \mathcal{B}_0 - M\chi_R - a$ is dissipative in $L^p(m)$ for all $a > a_{m,p}$, and we have, for all $f \in L^p(m)$,

$$\|\mathcal{S}_\mathcal{B}(t) f\|_{L^p(m)} \le e^{at} \|f\|_{L^p(m)}. \tag{4.48}$$

Indeed, from (4.28) and (4.32) we obtain

$$\begin{aligned}\frac{1}{p}\frac{d}{dt}\|f\|^p_{L^p(m)} &\le -(p-1)\int \bar a_{ij}\partial_i(m^\theta f)\partial_j(m^\theta f) m^{p-2\theta}|f|^{p-2} + \int(\varphi_{m,p,\theta} - M\chi_R) m^p |f|^p \\ &\le \int(\varphi_{m,p,\theta} - M\chi_R) m^p |f|^p \\ &\le a\int m^p|f|^p\end{aligned}$$

which yields (4.48).

□

4.2.2 Regularization properties

We are now interested in regularization properties of the operator $\mathcal{A}$ and the iterated convolutions of $\mathcal{A}\mathcal{S}_\mathcal{B}$ to prove assumptions (ii) and (iii) of theorem 4.14. Let us recall the operator $\mathcal{A}$ defined in (4.22),

$$\mathcal{A}g = \mathcal{A}_0 g + M\chi_R g = (a_{ij} * g)\partial_{ij}\mu - (c * g)\mu + M\chi_R g,$$

for M and R large enough chosen before.

Thanks to the function χ_R, for any $q \in [1,+\infty)$, $p \ge q$ and any weight function m_0, we have

$$\|M\chi_R g\|_{L^q(m_0)} \le C\|\chi_R m_0 m^{-1}\|_{L^{pq/(p-q)}}\|g\|_{L^p(m)} \le C\|g\|_{L^p(m)}, \tag{4.49}$$

from which we deduce that $M\chi_R \in \mathscr{B}(L^p(m), L^q(m_0))$.

Let us now focus on regularization estimates for the operator $\mathcal{A}_0$. First of all we give the following result, which will be useful in the sequel.

Lemma 4.9. *Let $\gamma \in [0,1]$ and $\beta \in \mathbb{N}^3$ be a multindex such that $|\beta| \le 2$. Then*

$$|\partial_\beta(a_{ij} * g)(v)| \lesssim \langle v\rangle^{\gamma+2}\|\partial_\beta g\|_{L^1(\langle v\rangle^{\gamma+2})} \quad \textit{and} \quad |\partial_\beta(a_{ij} * g)(v)| \lesssim \langle v\rangle^{\gamma+2-|\beta|}\|g\|_{L^1(\langle v\rangle^{\gamma+2-|\beta|})}$$

Proof. First of all, we write $\partial_\beta(a_{ij} * g) = a_{ij} * \partial_\beta g$ and then

$$|(a_{ij} * \partial_\beta g)(v)| \le \int |a_{ij}(v - v_*)||\partial_\beta g_*|\, dv_*.$$

For $\gamma \in [0,1]$ we have $|a_{ij}(v - v_*)| \le |v - v_*|^{\gamma+2} \le C\langle v\rangle^{\gamma+2}\langle v_*\rangle^{\gamma+2}$, which yields

$$|(a_{ij} * \partial_\beta g)(v)| \lesssim \langle v\rangle^{\gamma+2}\|\partial_\beta g\|_{L^1(\langle v\rangle^{\gamma+2})}.$$

Finally, writing $\partial_\beta(a_{ij} * g) = \partial_\beta a_{ij} * g$ and using $|\partial_\beta a_{ij}(v - v_*)| \lesssim |v - v_*|^{\gamma+2-|\beta|} \lesssim \langle v\rangle^{\gamma+2-|\beta|}\langle v_*\rangle^{\gamma+2-|\beta|}$ from lemma 4.2 and because $\gamma + 2 - |\beta| \geq 0$, it follows

$$|(\partial_\beta a_{ij} * g)(v)| \lesssim \int \langle v\rangle^{\gamma+2-|\beta|}\langle v_*\rangle^{\gamma+2-|\beta|}|g_*|\, dv_* \lesssim \langle v\rangle^{\gamma+2-|\beta|}\|g\|_{L^1(\langle v\rangle^{\gamma+2-|\beta|})},$$

which finishes the proof. □

Lemma 4.10. *Let $\gamma \in [0, 1]$ and $p \in [1, +\infty]$. Then we have*

$$\|\mathcal{A}_0 g\|_{L^p(m)} \leq C_\mu \left(\|g\|_{L^1(\langle v\rangle^{\gamma+2})} + \|g\|_{L^1(\langle v\rangle^{\gamma})}\right). \tag{4.50}$$

As a consequence, $\mathcal{A}_0 \in \mathscr{B}(L^p(m))$ and

$$\|\mathcal{A}_0 g\|_{L^p(m)} \leq C_\mu \|g\|_{L^p(m)}.$$

Proof. For the first inequality, we write

$$\|\mathcal{A}_0 g\|_{L^p(m)} \leq \|(a_{ij} * g)\partial_{ij}\mu\|_{L^p(m)} + \|(c * g)\mu\|_{L^p(m)}.$$

For the first term, using lemma 4.9, we compute

$$\begin{aligned}\|(a_{ij} * g)\partial_{ij}\mu\|^p_{L^p(m)} &\leq C\,\|g\|^p_{L^1(\langle v\rangle^{\gamma+2})} \int \langle v\rangle^{(\gamma+2)p}|\partial_{ij}\mu(v)|^p\, m^p(v)\, dv \\ &\leq C_\mu\,\|g\|^p_{L^1(\langle v\rangle^{\gamma+2})}.\end{aligned}$$

Arguing in the same way, we also obtain

$$\begin{aligned}\|(c * g)\mu\|^p_{L^p(m)} &\leq C\,\|g\|^p_{L^1(\langle v\rangle^{\gamma})} \int \langle v\rangle^{\gamma p}|\mu(v)|^p\, m^p(v)\, dv \\ &\leq C_\mu\,\|g\|^p_{L^1(\langle v\rangle^{\gamma})},\end{aligned}$$

which completes the proof of the first inequality of the lemma.

Then we compute, for some $\sigma > 0$ and using Hölder's inequality,

$$\begin{aligned}\|g\|_{L^1(\langle v\rangle^{\gamma+2})} &\leq \left(\int \langle v\rangle^{-\sigma p/(p-1)}\right)^{(p-1)/p} \|g\|_{L^p(\langle v\rangle^{\gamma+2+\sigma})} \\ &\leq C\|g\|_{L^p(\langle v\rangle^{\gamma+2+\sigma})},\end{aligned}$$

if $\sigma > 3(1-1/p)$. This implies that $\|\mathcal{A}_0 g\|_{L^p(m)} \leq C_\mu\|g\|_{L^p(m)}$ since $k > \gamma+2+3(1-1/p)$ when $m = \langle v\rangle^k$ satisfies (4.23) or $m = e^{r\langle v\rangle^s}$ satisfies (4.25). □

Corollary 4.11. *Let $p \in [2, +\infty]$. Then $\mathcal{A} \in \mathscr{B}(L^p(m), L^2(\mu^{-1/2}))$ and for any $a > a_{m,p}$ we have*

$$\|\mathcal{A}\mathcal{S}_\mathcal{B}(t)\|_{\mathscr{B}(L^p(m),L^2(\mu^{-1/2}))} \leq C_a e^{at}.$$

Proof. From lemma 4.10 and equation (4.49) it follows that $\mathcal{A} \in \mathscr{B}(L^p(m), L^2(\mu^{-1/2}))$ for all $p \in [2, +\infty]$. Then we compute using lemma 4.7,

$$\|\mathcal{A}\mathcal{S}_{\mathcal{B}}(t)f\|_{L^2(\mu^{-1/2})} \leq \|\mathcal{A}\|_{\mathscr{B}(L^p(m),L^2(\mu^{-1/2}))} \|\mathcal{S}_{\mathcal{B}}(t)f\|_{L^p(m)} \leq Ce^{at}\|f\|_{L^p(m)},$$

which concludes the proof. □

Let us denote $m_0 = e^{r\langle v\rangle^2}$ with $r \in (0, 1/4)$, then $L^2(\mu^{-1/2}) \subset L^q(m_0)$ for any $1 \leq q \leq 2$.

Lemma 4.12. *There exists $C > 0$ such that for all $1 \leq p < 2$,*

$$\|\mathcal{S}_{\mathcal{B}}(t)f\|_{L^2(m_0)} \leq C\, t^{-\frac{3}{2}\left(\frac{1}{p}-\frac{1}{2}\right)}\, e^{at}\, \|f\|_{L^p(m_0)}, \qquad \forall\, t \geq 0. \tag{4.51}$$

As a consequence, for all $1 \leq p < 2$ and m satisfying (4.23) or (4.25), for any $a' > a$ we have

$$\|(\mathcal{A}\mathcal{S}_{\mathcal{B}})^{*2}(t)f\|_{L^2(\mu^{-1/2})} \leq C\, e^{a't}\, \|f\|_{L^p(m)}, \qquad \forall\, t \geq 0. \tag{4.52}$$

Proof of Lemma 4.12. Consider the equation $\partial_t f = \mathcal{B}f$. Then from (4.28) and (4.32) we have

$$\frac{1}{2}\frac{d}{dt}\|f\|^2_{L^2(m_0)} = -\int \bar{a}_{ij}\partial_i(m_0 f)\partial_j(m_0 f) + \int (\varphi_{m_0,2,1} - M\chi_R)m_0^2 f^2$$

From lemma 4.2 there exists $\ell_0 > 0$ such that $\bar{a}_{ij}\xi_i\xi_j \geq \ell_0|\xi|^2$. We obtain

$$\frac{1}{2}\frac{d}{dt}\|f\|^2_{L^2(m_0)} \leq -\ell_0 \int |\nabla(m_0 f)|^2 + \int (\varphi_{m_0,2,1} - M\chi_R)m_0^2 f^2 \tag{4.53}$$

The weight function m_0 satisfies (4.25), then Lemma 4.7 holds, more precisely

$$\|\mathcal{S}_{\mathcal{B}}(t)f\|_{L^p(m_0)} \leq e^{at}\|f\|_{L^p(m_0)}, \qquad \forall\, t \geq 0. \tag{4.54}$$

Applying Nash's inequality in 3-dimension: $\|g\|^2_{L^2} \leq c_1\|\nabla g\|^{6/5}_{L^2}\|g\|^{4/5}_{L^1}$ with $g = m_0 f$ we obtain

$$c_1^{-1}\|m_0 f\|^{10/3}_{L^2}\|m_0 f\|^{-4/3}_{L^1} \leq \int |\nabla(m_0 f)|^2.$$

Putting togheter last inequality with (4.55), it follows

$$\frac{1}{2}\frac{d}{dt}\|f\|^2_{L^2(m_0)} \leq -C\,\|f\|^{10/3}_{L^2(m_0)}\|f\|^{-4/3}_{L^1(m_0)} + a\|f\|^2_{L^2(m_0)}. \tag{4.55}$$

Les us denote $x(t) := \|f(t)\|^2_{L^2(m_0)}$ and $y(t) := \|f(t)\|_{L^1(m_0)}$, then we have the following inequality $\dot{x}(t) \leq -C_1 x(t)^{5/3}y(t)^{-4/3} + 2ax(t)$. From (4.54) we have $y(t) \leq y_0$ and then

$$\dot{x}(t) \leq -C_1 x(t)^{5/3}y_0^{-4/3} + 2ax(t).$$

If $x_0 \le C y_0$ by (4.54) we have $x(t) \le C e^{at} y_0$. If x_0 is such that $x_0 > [C_1/4a] y_0$ then $x(t) \le C(y_0^{-4/3} t)^{-3/2}$, and we obtain

$$\|\mathcal{S}_{\mathcal{B}}(t) f\|_{L^2(m_0)} \le C\, t^{-\frac{3}{4}} e^{at}\, \|f\|_{L^1(m_0)}.$$

Using Riesz-Thorin interpolation theorem to $\mathcal{S}_{\mathcal{B}}(t)$ which acts from $L^2 \to L^2$ with estimate (4.54) and from $L^1 \to L^2$ with the estimate above, we obtain (4.51).

Let us prove now (4.52). From lemma 4.10 and equation (4.49) we have the following estimates, for any $p \in [1, +\infty]$,

$$\|\mathcal{A} g\|_{L^2(\mu^{-1/2})} \lesssim \|g\|_{L^2(m_0)}, \qquad \|\mathcal{A} g\|_{L^p(m_0)} \lesssim \|g\|_{L^p(m)}. \tag{4.56}$$

Hence, by (4.56) and (4.51), for $1 \le p \le 2$, it follows

$$\|\mathcal{A}\mathcal{S}_{\mathcal{B}}(t) f\|_{L^2(\mu^{-1/2})} \lesssim \|\mathcal{S}_{\mathcal{B}}(t) f\|_{L^2(m_0)} \lesssim t^{-\frac{3}{2}\left(\frac{1}{p}-\frac{1}{2}\right)}\, e^{at}\, \|f\|_{L^p(m_0)}. \tag{4.57}$$

Computing the convolution of $\mathcal{A}\mathcal{S}_{\mathcal{B}}(t)$ we have

$$\begin{aligned}
\|(\mathcal{A}\mathcal{S}_{\mathcal{B}})^{*2}(t) f\|_{L^2(\mu^{-1/2})} &\lesssim \int_0^t \|\mathcal{A}\mathcal{S}_{\mathcal{B}}(t-s)\mathcal{A}\mathcal{S}_{\mathcal{B}}(s) f\|_{L^2(\mu^{-1/2})}\, ds \\
&\lesssim \int_0^t \|\mathcal{S}_{\mathcal{B}}(t-s)\mathcal{A}\mathcal{S}_{\mathcal{B}}(s) f\|_{L^2(m_0)}\, ds \\
&\lesssim \int_0^t (t-s)^{-\frac{3}{2}\left(\frac{1}{p}-\frac{1}{2}\right)} e^{a(t-s)}\, \|\mathcal{A}\mathcal{S}_{\mathcal{B}}(s) f\|_{L^p(m_0)}\, ds \\
&\lesssim \int_0^t (t-s)^{-\frac{3}{2}\left(\frac{1}{p}-\frac{1}{2}\right)} e^{a(t-s)}\, \|\mathcal{S}_{\mathcal{B}}(s) f\|_{L^p(m)}\, ds \\
&\lesssim \int_0^t (t-s)^{-\frac{3}{2}\left(\frac{1}{p}-\frac{1}{2}\right)} e^{a(t-s)}\, e^{as}\, \|f\|_{L^p(m)}\, ds \\
&\lesssim t^{\frac{1}{2}\left(\frac{7}{2}-\frac{3}{p}\right)} e^{at}\, \|f\|_{L^p(m)} \\
&\lesssim e^{a' t}\, \|f\|_{L^p(m)},
\end{aligned}$$

where we have used sucessively (4.56), (4.51), (4.56), lemma 4.7 and the fact that $(\frac{7}{2} - \frac{3}{p}) > 0$ for $1 \le p < 2$. Hence for all $t \ge 0$ we have $\|(\mathcal{A}\mathcal{S}_{\mathcal{B}})^{*2}(t)\|_{\mathscr{B}(L^p(m), L^2(\mu^{-1/2}))} \lesssim e^{a't}$, for any $a' > a$ (where $a > a_{m,p}$ is fixed in lemma 4.7).

$\square$

4.2.3 Abstract theorem

We shall present in this subsection an abstract theorem from [42, 61], which will be used to prove theorem 4.3.

Let us introduce some notation before state the theorem. Consider two Banach spaces $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$. We denote by $\mathscr{B}(X, Y)$ the space of bounded linear operators from X to Y and by $\|\cdot\|_{\mathscr{B}(X,Y)}$ its operator norm. Moreover we write $\mathscr{C}(X, Y)$

the space of closed unbounded linear operators from X to Y with dense domain. When $X = Y$ we simply denote $\mathscr{B}(X) = \mathscr{B}(X,X)$ and $\mathscr{C}(X) = \mathscr{C}(X,X)$.

Given a Banach space X and a operator $\Lambda : X \to X$, we denote $\mathcal{S}_\Lambda(t)$ or $e^{t\Lambda}$ the semigroup generated by Λ. We also denote $\mathcal{N}(\Lambda)$ its null space, $\mathrm{dom}(\Lambda)$ its domain, $\Sigma(\Lambda)$ its spectrum and $\mathrm{R}(\Lambda)$ its range. Recall that for any z in the resolvent set $\rho(\Lambda) := \mathbb{C} \setminus \Sigma(\Lambda)$, the operator $\Lambda - z$ is invertible, moreover the resolvent operator $(\Lambda - z)^{-1} \in \mathscr{B}(X)$ and its range equals $\mathrm{dom}(\Lambda)$. An eigenvalue $\xi \in \Sigma(\Lambda)$ is isolated if

$$\Sigma(\Lambda) \cap \{z \in \mathbb{C};\ |z - \xi| \leq r\} = \{\xi\} \quad \text{for some } r > 0.$$

Then for an isolated eigenvalue ξ we define the associated spectral projector $\Pi_{\Lambda,\xi} \in \mathscr{B}(X)$ by

$$\Pi_{\Lambda,\xi} := -\frac{1}{2i\pi} \int_{|z-\xi|=r'} (\Lambda - z)^{-1}\, dz \quad \text{with } 0 < r' < r. \tag{4.58}$$

If moreover the algebraic eigenspace $\mathrm{R}(\Pi_{\mathcal{L},\xi})$ is finite dimensional, we say that ξ is a discrete eigenvalue and write $\xi \in \Sigma_d(\Lambda)$.

Definition 4.13. Let X_1, X_2 and X_3 be Banach spaces and $\mathcal{S}_1 \in L^1(\mathbb{R}_+, \mathscr{B}(X_1, X_2))$, $\mathcal{S}_2 \in L^1(\mathbb{R}_+, \mathscr{B}(X_2, X_3))$. We define the convolution $\mathcal{S}_2 * \mathcal{S}_1 \in L^1(\mathbb{R}_+, \mathscr{B}(X_1, X_3))$ by

$$\forall\, t \geq 0, \qquad \mathcal{S}_2 * \mathcal{S}_1(t) := \int_0^t \mathcal{S}_2(s)\mathcal{S}_1(t-s)\, ds.$$

If $X_1 = X_2 = X_3$ and $\mathcal{S} = \mathcal{S}_1 = \mathcal{S}_2$, we define $\mathcal{S}^1 = \mathcal{S}$ and $\mathcal{S}^{*n} = \mathcal{S} * \mathcal{S}^{*(n-1)}$ for all $n \geq 2$.

We can now state a the following theorem from [42, Theorem 2.13].

Theorem 4.14. *Let E and $\mathcal{E}$ be Banach spaces such that $E \subset \mathcal{E}$ is dense with continuous embedding. Consider the operators $L \in \mathscr{C}(E)$, $\mathcal{L} \in \mathscr{C}(\mathcal{E})$ with $L = \mathcal{L}_{|E}$ and $a \in \mathbb{R}$. Assume that :*

(1) L generates a semigroup e^{tL} on E, $L - a$ is hypodissipative on $\mathrm{R}(\mathrm{Id} - \Pi_{L,a})$ and

$$\Sigma(L) \cap \Delta_a := \{\xi_1, \dots, \xi_k\} \subset \Sigma_d(L) \quad \textit{(dinstinct discrete eigenvalues)}$$

(2) There are $\mathcal{A}, \mathcal{B} \in \mathscr{C}(\mathcal{E})$ such that $\mathcal{L} = \mathcal{A} + \mathcal{B}$, with the corresponding restrictions A and B on E, some $n \in \mathbb{N}^*$ and some constant $C_a > 0$ such that

(i) $\mathcal{B} - a$ is hypodissipative on $\mathcal{E}$;

(ii) $A \in \mathscr{B}(E)$ and $\mathcal{A} \in \mathscr{B}(\mathcal{E})$;

(iii) we have

$$\|(\mathcal{A}\mathcal{S}_{\mathcal{B}})^{*n}(t)\|_{\mathscr{B}(\mathcal{E},E)} \leq C_a e^{at}.$$

Then $\mathcal{L}$ is hypodissipative on $\mathcal{E}$ and we have

$$\forall\, t \geq 0, \qquad \left\| \mathcal{S}_{\mathcal{L}}(t) - \sum_{j=1}^{k} \mathcal{S}_{\mathcal{L}}(t)\Pi_{\mathcal{L},\xi_j} \right\|_{\mathscr{B}(\mathcal{E})} \leq C'_a\, t^n\, e^{at}, \tag{4.59}$$

where $C'_a > 0$ is an explicit constant depending on the constants from the assumptions.

This theorem permets us to *enlarge the space of the spectral estimates* of a given operator. More precisely, the knowledge of the spectral information in the "small space" (1) permet us to extend this information to a "big space" (4.59). Hence, the strategy to prove theorem 4.3 is to consider $E = L^2(\mu^{-1/2})$, in which space the spectral gap is known to hold (assumption (1)), and prove assumptions (2i), (2ii) and (2iii) with $\mathcal{E} = L^p(m)$ to obtain spectral gap estimates in $\mathcal{E}$ applying theorem 4.14.

4.2.4 Proof of theorem 4.3

With the results of subsections 4.2.1 and 4.2.2 and theorem 4.14 we are able to prove the spectral gap for the linearized Landau operator.

Let $E = L^2(\mu^{-1/2})$, in which space we already know the spectral gap (4.14), and $\mathcal{E} = L^p(m)$, for any $p \in [1, 2]$ and m satisfying (4.23) or (4.25). We consider the decomposition $\mathcal{L} = \mathcal{A} + \mathcal{B}$ as in (4.22). For any $a > a_{m,p}$, the operator $\mathcal{B} - a$ is hypodissipative in $\mathcal{E}$ from lemma 4.7 (assumption (i) of theorem 4.14); $\mathcal{A} \in \mathscr{B}(\mathcal{E})$ and $A \in \mathscr{B}(E)$ from lemma 4.10 and equation (4.49) (assumption (ii) of theorem 4.14). Hence we only need to prove assumption (iii) to conclude applying theorem 4.14.

We split the proof in different cases.

Case $p = 2$. First of all, in this case we have $E \subset \mathcal{E}$. Moreover, $\mathcal{A}\mathcal{S}_{\mathcal{B}}(t) \in \mathscr{B}(\mathcal{E}, E)$ with exponential decay rate from corollary 4.11, which proves assumption (iii) with $n = 1$.

Case $p \in [1, 2)$. Here $E \subset \mathcal{E}$ and from lemma 4.12 we have $(\mathcal{A}\mathcal{S}_{\mathcal{B}})^{*2}(t) \in \mathscr{B}(\mathcal{E}, E)$ with exponential decay rate, which gives assumption (iii) with $n = 2$.

4.3 Proof of the main result

Let us consider the Landau operator (4.19)

$$Q(g, h) = (a_{ij} * g)\partial_{ij}h - (c * g)h.$$

We shall prove some estimates for the nonlinear operator Q before proving the theorem 4.1.

Proposition 4.15. *Let* $\gamma \in [0, 1]$ *and* $p \in [1, +\infty]$. *Then*

$$\|Q(g, h)\|_{L^p(m)} \lesssim \|g\|_{L^1(\langle v\rangle^{\gamma+2})} \|\partial_{ij}h\|_{L^p(m\langle v\rangle^{\gamma+2})} + \|g\|_{L^1(\langle v\rangle^{\gamma})} \|h\|_{L^p(m\langle v\rangle^{\gamma})}$$

Proof. We write

$$\|Q(g, h)\|_{L^p(m)} \le \|(a_{ij} * g)\partial_{ij}h\|_{L^p(m)} + \|(c * g)h\|_{L^p(m)}.$$

Thanks to lemma 4.9

$$\|(a_{ij} * g)\partial_{ij}h\|_{L^p(m)} \lesssim \|g\|_{L^1(\langle v\rangle^{\gamma+2})} \|\partial_{ij}h\|_{L^p(m\langle v\rangle^{\gamma+2})}$$

Moreover, by lemma 4.9 one obtains, since $c = \partial_{ij} a_{ij}$ and $|(c * g)(v)| \le C\langle v\rangle^\gamma \|g\|_{L^1(\langle v\rangle^\gamma)}$,

$$\|(c * g)h\|_{L^p(m)} \lesssim \|g\|_{L^1(\langle v\rangle^\gamma)} \|h\|_{L^p(m\langle v\rangle^\gamma)},$$

and the proof is complete.

□

The proof of theorem 4.1 relies on known results by Desvillettes and Villani [28, 29] corcerning the polynomial decay rate to equilibrium, togheter with the spectral estimates from theorem 4.3 and some estimates on the nonlinear operator from 4.15. We follow the strategy of [65].

Let us first summarize the results on the Cauchy theory for the Landau equation with hard potentials.

Theorem 4.16 ([28] (Theorems 3, 6 and 7) and [29] (Theorem 8))**.** *Let $\gamma \in (0, 1]$.*

1. *if $F_0 \in L^1(\langle v\rangle^{2+\delta})$ for some $\delta > 0$, then there exists a weak solution F to* (4.1) *such that :*
   (a) for all $t_0 > 0$, $F \in \mathcal{C}^\infty([t_0, +\infty); \mathcal{S}(\mathbb{R}^3_v))$.
   (b) for all $t_0 > 0$, all integer $k > 0$ and all $\theta > 0$, there exists $C_{t_0} > 0$ such that
   $$\sup_{t\ge t_0} \|F(t,\cdot)\|_{H^k(\langle v\rangle^\theta)} \le C_{t_0}.$$
2. *let F be any weak solution of* (4.1) *with initial datum $F_0 \in L^1(\langle v\rangle^2)$ satisfying the decay of energy, then for all $t_0 > 0$ and all $\theta > 0$, there is a constant $C_{t_0} > 0$ such that*
   $$\sup_{t\ge t_0} \|F(t,\cdot)\|_{L^1(\langle v\rangle^\theta)} \le C_{t_0}.$$
3. *if F is a smooth solution of* (4.1) *(in the sense of (1a)), then for all $t \ge 0$ there is $C > 0$ such that*
   $$H(F_t|\mu) := \int_{\mathbb{R}^3} F_t \log \frac{F_t}{\mu}\, dv \le C(1+t)^{-2/\gamma}$$

Corollary 4.17. *For all $t_0 > 0$ and all $\ell > 0$, there exists $C_{t_0} > 0$ such that*

$$\forall\, t \ge t_0, \qquad \|F_t - \mu\|_{L^1(\langle v\rangle^\ell)} \le C_{t_0}(1+t)^{-\frac{1}{2\gamma}}.$$

Proof. Let us fixe some $t_0 > 0$. First of all, from theorem 4.16 and the Csiszár-Kullback-Pinsker inequality (see e.g. [78, Remark 22.12])

$$\|F - \mu\|_{L^1} \le C\sqrt{H(F|\mu)},$$

we obtain

$$\forall\, t \ge 0, \qquad \|F_t - \mu\|_{L^1} \le C(1+t)^{-1/\gamma}.$$

Then, using the bounds of theorem 4.16 and Hölder's inequality we obtain

$$\forall\, t \ge t_0, \qquad \|F_t - \mu\|_{L^1(\langle v\rangle^\ell)} \le \|F_t - \mu\|^{1/2}_{L^1(\langle v\rangle^{2\ell})} \|F_t - \mu\|^{1/2}_{L^1} \le C_{t_0}(1+t)^{-\frac{1}{2\gamma}}.$$

□

Proof of Theorem 4.1. Let $F_0 = \mu + f_0$ and consider the equation

$$\left\{ \begin{array}{rcl} \partial_t f & = & \mathcal{L} f + Q(f,f) \\ f_{|t=0} & = & f_0. \end{array} \right. \tag{4.60}$$

Since F_0 has same mass, momentum and energy than μ, we have $\Pi f_0 = 0$ and for all $t \geq 0$, thanks to the conservation of these quantities, we also have $\Pi f_t = 0$.

Lemma 4.18. *Consider $m = \langle v \rangle^k$ satisfying* (4.23)*. There exists $\epsilon > 0$ such that, if the solution f of* (4.60) *satisfies*

$$\forall\, t \geq 0, \qquad \|f_t\|_{L^1(\langle v \rangle^\ell)} \leq \epsilon,$$

with $\ell := \gamma + 4 + k$, and if

$$\forall\, t \geq 0, \qquad \|f_t\|_{H^4(\langle v \rangle^\ell)} \leq C,$$

then there is $C' > 0$ such that

$$\forall\, t \geq 0, \qquad \|f_t\|_{L^1(\langle v \rangle^k)} \leq C' e^{-\lambda t}\, \|f_0\|_{L^1(\langle v \rangle^k)},$$

where $\lambda > 0$ is given by theorem 4.3.

Proof of lemma 4.18. By Duhamel's formula for the solution of (4.60), we write,

$$f_t = e^{t\mathcal{L}} f_0 + \int_{t_0}^t e^{(t-s)\mathcal{L}} Q(f_s, f_s)\, ds.$$

Using theorem 4.3 and proposition 4.15, one deduces

$$\begin{aligned} \|f_t\|_{L^p(m)} &\leq \|e^{t\mathcal{L}} f_0\|_{L^p(m)} + \int_0^t \|e^{(t-s)\mathcal{L}} Q(f_s, f_s)\|_{L^p(m)}\, ds \\ &\leq e^{-\lambda t} \|f_0\|_{L^p(m)} + \int_0^t e^{-\lambda(t-s)} \|Q(f_s, f_s)\|_{L^p(m)}\, ds \\ &\leq e^{-\lambda t} \|f_0\|_{L^p(m)} + C \int_0^t e^{-\lambda(t-s)} \Big(\|f_s\|_{L^1(\langle v \rangle^\gamma)} \|f_s\|_{L^p(m\langle v \rangle^\gamma)} \\ &\qquad\qquad + \|f_s\|_{L^1(\langle v \rangle^{\gamma+2})} \|\partial_{ij} f_s\|_{L^p(m\langle v \rangle^{\gamma+2})} \Big)\, ds. \end{aligned}$$

For $L^p(m) = L^1(\langle v \rangle^k)$ we obtain (recall that $k > \gamma + 2$ from (4.23))

$$\begin{aligned} \|f_t\|_{L^1(\langle v \rangle^k)} \leq e^{-\lambda t} \|f_0\|_{L^1(\langle v \rangle^k)} + C \int_0^t e^{-\lambda(t-s)} \Big(\|f_s\|_{L^1(\langle v \rangle^\gamma)} \|f_s\|_{L^1(\langle v \rangle^{\gamma+k})} \\ + \|f_s\|_{L^1(\langle v \rangle^{\gamma+2})} \|\partial_{ij} f_s\|_{L^1(\langle v \rangle^{\gamma+2+k})} \Big)\, ds. \end{aligned} \tag{4.61}$$

Let us recall the following interpolation inequalities

$$\begin{aligned} \|g\|_{L^2}^2 &\lesssim \|g\|_{L^1} \|g\|_{L^\infty} \lesssim \|g\|_{L^1} \|g\|_{H^{3/2+1}}, \\ \|g\|_{H^2}^2 &\lesssim \|g\|_{L^2} \|g\|_{H^4}, \\ \|g\|_{L^1(\langle v \rangle^\alpha)} &\lesssim \|g\|_{L^2(\langle v \rangle^{\alpha+r})}, \quad \text{if } r > 3/2. \end{aligned} \tag{4.62}$$

Using the third equation of (4.62) with $g = \partial^\alpha f_s$ and $r = 2$, and then first and second equations of (4.62) with $g = \langle v\rangle^\ell f_s$, it yields

$$\|\partial_{ij} f_s\|_{L^1(\langle v\rangle^{\gamma+2+k})} \lesssim \|f_s\|^{1/4}_{L^1(\langle v\rangle^\ell)} \|f_s\|^{1/4}_{H^{3/2+1}(\langle v\rangle^\ell)} \|f_s\|^{1/2}_{H^4(\langle v\rangle^\ell)}.$$

Gathering last inequality with (4.61) it follows

$$\begin{aligned}\|f_t\|_{L^1(\langle v\rangle^k)} \le e^{-\lambda t}\|f_0\|_{L^1(\langle v\rangle^k)} &+ C\int_0^t e^{-\lambda(t-s)}\|f_s\|_{L^1(\langle v\rangle^\gamma)}\|f_s\|_{L^1(\langle v\rangle^{\gamma+k})}\,ds \\ &+ C\int_0^t e^{-\lambda(t-s)}\|f_s\|_{L^1(\langle v\rangle^{\gamma+2})}\|f_s\|^{1/4}_{L^1(\langle v\rangle^\ell)} \|f_s\|^{1/4}_{H^{3/2+1}(\langle v\rangle^\ell)} \|f_s\|^{1/2}_{H^4(\langle v\rangle^\ell)}\,ds \\ \le e^{-\lambda t}\|f_0\|_{L^1(\langle v\rangle^k)} &+ C(\epsilon+\epsilon^{1/4})\int_0^t e^{-\lambda(t-s)}\|f_s\|_{L^1(\langle v\rangle^k)}\,ds\end{aligned}$$

using the hypotheses of the lemma and $k > \gamma + 2$.

Denoting $x(t) := \|f_t\|_{L^1(\langle v\rangle^k)}$ and $\epsilon_1 := C(\epsilon+\epsilon^{1/4})$ we obtain the following differential inequality

$$x(t) \le e^{-\lambda t}x(0) + \epsilon_1\int_0^t e^{-\lambda(t-s)}x(s)\,ds,$$

and arguing as in [65, Lemma 4.5], if $x(0)$ and ϵ_1 are small enough we obtain, for all $t \ge 0$, $x(t) \le C' e^{-\lambda t}x(0)$, i.e.

$$\|f_t\|_{L^1(\langle v\rangle^k)} \le C' e^{-\lambda t}\|f_0\|_{L^1(\langle v\rangle^k)}.$$

□

We can now complete the proof of theorem 4.1. From corollary 4.17 we pick $t_0 > 0$ such that

$$\forall\, t \ge t_0, \qquad \|F_t - \mu\|_{L^1(\langle v\rangle^\ell)} = \|f_t\|_{L^1(\langle v\rangle^\ell)} \le \epsilon,$$

where ϵ is chosen in lemma 4.18. From theorem 4.16-(2) we have that, for all $t \ge t_0$, $\|f_t\|_{H^4(\langle v\rangle^\ell)} \le \|F_t\|_{H^4(\langle v\rangle^\ell)} + \|\mu\|_{H^4(\langle v\rangle^\ell)} \le C$. We can then apply lemma 4.18 to f starting from t_0, then

$$\forall\, t \ge t_0, \qquad \|F_t - \mu\|_{L^1(\langle v\rangle^k)} = \|f_t\|_{L^1(\langle v\rangle^k)} \le C' e^{-\lambda t}\|f_{t_0}\|_{L^1(\langle v\rangle^k)} \le C'' e^{-\lambda t},$$

which completes the proof.

□

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