

# CORRIGENDUM TO “HYDRODYNAMIC LIMIT FOR THE NON-CUTOFF BOLTZMANN EQUATION”

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We correct an error in “Hydrodynamic limit for the non-cutoff Boltzmann equation”, *Ann. Inst. Henri Poincaré C, Anal. Non Linéaire* 43, no. 2, 417–482, 2026.

The first mistake appears in Section 3, more precisely in Proposition 3.1 in the case  $\ell > 0$ . The second mistake appears at one estimate in Section 6, concerning the hydrodynamic limit, in the case of soft potentials  $\gamma + 2s < 0$ . All the main results of the paper (Theorem 2.1, Theorem 2.2 and Theorem 2.3) remain unchanged.

## A. CHANGES IN SECTION 3

We correct Proposition 3.1 which, as originally stated, is not valid for all  $0 < \varepsilon \leq 1$  and all  $\ell \geq 0$ , but only for all  $0 < \varepsilon \leq 1$  and  $\ell = 0$ , or for  $\varepsilon = 1$  and all  $\ell > 0$ .

In order to obtain a statement that holds for all  $0 < \varepsilon \leq 1$  and all  $\ell \geq 0$  at once, we need to slightly change the definition of the new inner products in (3.1) and (3.2). Namely, instead of considering the term  $\varepsilon B[f, g](\xi)$ , we shall consider a rescaled version  $B^\varepsilon$  of  $B$ . More precisely, we first define, for every  $0 < \varepsilon \leq 1$  and every  $\xi \in \mathbf{R}^3$ ,

$$\begin{aligned}
 (A.1) \quad B^\varepsilon[f, g](\xi) &:= \delta_1 i \theta[f] \frac{\varepsilon \xi}{\langle \varepsilon \xi \rangle^2} \cdot M[\mathbf{P}^\perp g] + \delta_1 i \theta[g] \frac{\varepsilon \xi}{\langle \varepsilon \xi \rangle^2} \cdot M[\mathbf{P}^\perp f] \\
 &+ \delta_2 i \left( \frac{\varepsilon \xi}{\langle \varepsilon \xi \rangle^2} \otimes u[f] \right)^{\text{sym}} : \left\{ \Theta[\mathbf{P}^\perp \widehat{g}(\xi)] + \theta[g] I \right\} \\
 &+ \delta_2 i \left( \frac{\varepsilon \xi}{\langle \varepsilon \xi \rangle^2} \otimes u[g] \right)^{\text{sym}} : \left\{ \Theta[\mathbf{P}^\perp f] + \theta[f] I \right\} \\
 &+ \delta_3 i \rho[f] \frac{\varepsilon \xi}{\langle \varepsilon \xi \rangle^2} \cdot u[\widehat{g}(\xi)] + \delta_3 i \rho[g] \frac{\varepsilon \xi}{\langle \varepsilon \xi \rangle^2} \cdot u[\widehat{f}(\xi)],
 \end{aligned}$$

with constants  $0 < \delta_3 \ll \delta_2 \ll \delta_1 \ll 1$ . We then define the inner product  $\langle \cdot, \cdot \rangle_{L_v^2}$  on  $L_v^2$  (depending on  $\varepsilon$  and  $\xi$ ) by

$$(A.2) \quad \langle \widehat{f}(\xi), \widehat{g}(\xi) \rangle_{L_v^2} := \langle \widehat{f}(\xi), \widehat{g}(\xi) \rangle_{L_v^2} + B^\varepsilon [\widehat{f}(\xi), \widehat{g}(\xi)] (\xi).$$

Similarly, for any  $\ell > 0$  we define the inner product  $\langle \cdot, \cdot \rangle_{L_v^2(\langle v \rangle^\ell)}$  on  $L_v^2(\langle v \rangle^\ell)$  (depending on  $\varepsilon$  and  $\xi$ ) by

$$\begin{aligned}
 (A.3) \quad \langle \widehat{f}(\xi), \widehat{g}(\xi) \rangle_{L_v^2(\langle v \rangle^\ell)} &:= \langle \widehat{f}(\xi), \widehat{g}(\xi) \rangle_{L_v^2} + B^\varepsilon [\widehat{f}(\xi), \widehat{g}(\xi)] (\xi) \\
 &+ \delta \mathbf{1}_{|\varepsilon \xi| \leq 1} \langle \mathbf{P}^\perp \widehat{f}(\xi), \mathbf{P}^\perp \widehat{g}(\xi) \rangle_{L_v^2(\langle v \rangle^\ell)} + \delta \mathbf{1}_{|\varepsilon \xi| > 1} \langle \widehat{f}(\xi), \widehat{g}(\xi) \rangle_{L_v^2(\langle v \rangle^\ell)},
 \end{aligned}$$

with  $0 < \delta \ll \delta_3 \ll \delta_2 \ll \delta_1 \ll 1$ . We denote by

$$\|\widehat{f}(\xi)\|_{L_v^2(\langle v \rangle^\ell)}^2 := \langle \widehat{f}(\xi), \widehat{f}(\xi) \rangle_{L_v^2(\langle v \rangle^\ell)}$$

the associated norm.

Following the proof of [69, Theorem 2.3] (see also [Dua11] for more precise computations) we then obtain a dissipativity result for the rescaled operator  $\Lambda^\varepsilon(\xi)$ .

**Proposition A.1.** *We can choose  $0 < \delta \ll \delta_3 \ll \delta_2 \ll \delta_1 \ll 1$  appropriately such that, for any  $0 < \varepsilon \leq 1$  and  $\ell \geq 0$ :*

(1) *The new norm  $\|\cdot\|_{L_v^2(\langle v \rangle^\ell)}$  is equivalent to the usual norm  $\|\cdot\|_{L_v^2(\langle v \rangle^\ell)}$  on  $L_v^2(\langle v \rangle^\ell)$  with bounds that are independent of  $\varepsilon$  and  $\xi$ .*

(2) For every  $f$  we have, for all  $\xi \in \mathbf{R}^3$ ,

$$\operatorname{Re} \left\langle \Lambda^\varepsilon(\xi) \widehat{f}(\xi), \widehat{f}(\xi) \right\rangle_{L_v^2(\langle v \rangle^\ell)} \leq -\lambda_0 \left( \frac{1}{\varepsilon^2} \|\mathbf{P}^\perp \widehat{f}(\xi)\|_{H_v^{s,*}(\langle v \rangle^\ell)}^2 + \frac{|\varepsilon \xi|^2}{\varepsilon^2 \langle \varepsilon \xi \rangle^2} \|\mathbf{P} \widehat{f}(\xi)\|_{L_v^2}^2 \right),$$

for some constant  $\lambda_0 > 0$ .

Observing that

$$\frac{|\varepsilon \xi|^2}{\varepsilon^2 \langle \varepsilon \xi \rangle^2} \gtrsim \frac{|\xi|^2}{\langle \xi \rangle^2} + \frac{1}{\varepsilon^2} \mathbf{1}_{|\varepsilon \xi| \leq 1} |\varepsilon \xi|^2 + \frac{1}{\varepsilon^2} \mathbf{1}_{|\varepsilon \xi| > 1}$$

we see that Proposition A.1 is slightly stronger than Proposition 3.1. Therefore all the other propositions in Section 3 remain however unchanged. We only need to change the estimate of the term  $\langle \widehat{S}(\xi), \widehat{g}_S(\xi) \rangle_{L_v^2(\langle v \rangle^\ell)}$  in the proof of Proposition 3.3. Using the inner product (A.3), a straightforward computation gives now

(A.4)

$$\left| \langle \widehat{S}(\xi), \widehat{g}_S(\xi) \rangle_{L_v^2(\langle v \rangle^\ell)} \right| \lesssim \varepsilon \|\langle v \rangle^\ell \widehat{S}(\xi)\|_{(H_v^{s,*})'} \left\{ \frac{1}{\varepsilon} \|\mathbf{P}^\perp \widehat{g}_S(\xi)\|_{H_v^{s,*}(\langle v \rangle^\ell)} + \frac{|\varepsilon \xi|}{\varepsilon \langle \varepsilon \xi \rangle} \|\mathbf{P} \widehat{g}_S(\xi)\|_{L_v^2} \right\},$$

and the rest of the proof remains exactly the same.

## B. CHANGES IN SECTION 6

The first modification concerns Lemma 6.1–(4), which should be changed into:

**Lemma B.1.** *In the soft potential case  $\gamma + 2s < 0$ , for all  $t \geq 0$  and all  $\xi \in \mathbf{R}^3$  there holds, for any  $k, \ell \geq 0$ ,*

$$(B.1) \quad \begin{aligned} & \|\widehat{U}^{\varepsilon\#}(t, \xi) \widehat{f}(\xi)\|_{L_v^2(\langle v \rangle^k)} \\ & \lesssim \left(1 + \frac{t}{\varepsilon^2}\right)^{-\frac{\ell}{|\gamma+2s|}} \left[ \|\widehat{f}(\xi)\|_{L_v^2(\langle v \rangle^{k+\ell})} + \chi\left(\frac{\varepsilon|\xi|}{\kappa}\right) \varrho_{\frac{\ell}{|\gamma+2s|} + \frac{1}{2}}(\varepsilon|\xi|) \|\widehat{f}(\xi)\|_{L_v^2(\langle v \rangle^k)} \right], \end{aligned}$$

as well as

$$(B.2) \quad \begin{aligned} & \|\widehat{U}^{\varepsilon\#}(t, \xi) \mathbf{P}^\perp \widehat{f}(\xi)\|_{L_v^2(\langle v \rangle^k)} \\ & \lesssim \left(1 + \frac{t}{\varepsilon^2}\right)^{-\frac{\ell}{|\gamma+2s|}} \left[ \|\mathbf{P}^\perp \widehat{f}(\xi)\|_{L_v^2(\langle v \rangle^{k+\ell})} + \chi\left(\frac{\varepsilon|\xi|}{\kappa}\right) \varrho_{\frac{\ell}{|\gamma+2s|}}(\varepsilon|\xi|) \|\mathbf{P}^\perp \widehat{f}(\xi)\|_{L_v^2(\langle v \rangle^k)} \right], \end{aligned}$$

where  $\varrho_m : \mathbf{R}_+ \rightarrow \mathbf{R}_+$  with  $m \geq 0$  is defined by  $\varrho_m(0) = 0$  and for  $z > 0$

$$\varrho_m(z) = \begin{cases} 1 & \text{if } m < 1, \\ \log\left(2 + \frac{1}{z}\right) & \text{if } m = 1, \\ z^{-2(m-1)} & \text{if } m > 1. \end{cases}$$

Estimates (B.1) and (B.2) are straightforward adaptations of [76, Theorem 1.1]. For the sake of completeness, we provide the proofs for both estimates below, although only (B.2) is required for our subsequent arguments.

Using this new bound (B.2) of Lemma B.1, we need to change the estimate of the term  $\widehat{\Psi}^{\varepsilon\#}[g, g]$  in Step 3 of the proof of Lemma 6.5 in the soft potential case  $\gamma + 2s < 0$ . More precisely, Lemma 6.5 in the case of soft potentials should be changed into:

**Lemma B.2.** *Consider the case of soft potentials  $\gamma + 2s < 0$ . Let  $g_0 = g_0(x, v) \in \operatorname{Ker} L$  and  $g = g(t, x, v) \in \operatorname{Ker} L$  be as in Lemma 6.5.*

(1) *Torus case  $\Omega_x = \mathbf{T}^3$ : There holds, for any  $\eta \in (0, 1)$ ,*

$$\|\widehat{\Psi}^\varepsilon[g, g] - \widehat{\Psi}[g, g]\|_{L_\xi^1 L_t^\infty L_v^2} \lesssim \varepsilon^\eta \left( \|\widehat{g}_0\|_{L_\xi^1 L_v^2}^2 + \|\widehat{g}_0\|_{L_\xi^1 L_v^2}^3 \right)$$

(2) *Whole space case  $\Omega_x = \mathbf{R}^3$ : For any  $p \in (3/2, \infty]$  there holds, for any  $\eta \in (0, 1)$ ,*

$$\|\widehat{\Psi}^\varepsilon[g, g] - \widehat{\Psi}[g, g]\|_{L_\xi^1 L_t^\infty L_v^2} \lesssim \varepsilon^\eta \left( \|\widehat{g}_0\|_{L_\xi^1 L_v^2}^2 + \|\widehat{g}_0\|_{L_\xi^1 L_v^2}^3 + \|\widehat{g}_0\|_{L_\xi^p L_v^2}^2 + \|\widehat{g}_0\|_{L_\xi^p L_v^2}^3 \right).$$

Thanks to the estimates of Lemma B.2, the statement and proof of the main hydrodynamic limit result in Theorem 2.3 remain unchanged.

**B.1. Proof of Lemma B.1.** We consider the case of soft potentials  $\gamma + 2s < 0$ . First of all we recall that

$$\widehat{U}_j^\varepsilon(t, \xi) := \widehat{U}_j\left(\frac{t}{\varepsilon^2}, \varepsilon\xi\right) \quad \text{and} \quad \widehat{U}^{\varepsilon\#}(t, \xi) = \widehat{U}^\# \left(\frac{t}{\varepsilon^2}, \varepsilon\xi\right),$$

so that hereafter we may consider  $\varepsilon = 1$ . Moreover, the case of high frequencies  $|\xi| > \kappa$ , where  $\kappa > 0$  is fixed in Lemma 6.1, is already proven in [76, Theorem 1.1], thus we shall focus only on the case of low frequencies  $|\xi| \leq \kappa$ .

We adopt the same notations as [76], that is, we denote

$$B(\xi) := L - i\xi \cdot v \quad \text{and} \quad B_0(\xi) := L - P - i\xi \cdot v.$$

The eigenvalue problem, for  $j = 1, \dots, 4$ ,

$$B(\xi)\psi_j(\xi) = \lambda_j(\xi)\psi_j(\xi)$$

is solved in [76, Proof of Lemma 2.20], which gives the asymptotic behavior of  $\lambda_j(\xi)$  stated in Lemma 6.1–(1). In the soft potentials case these eigenvalues are not isolated, since  $L$  does not possess a spectral gap, therefore the spectral projectors are not defined. One can however define the algebraic eigenprojections  $P_j(\xi)$  using the associated eigenfunctions  $\psi_j(\xi)$ , for which one also has an asymptotic expansion in [76, Proof of Lemma 2.20]. These operators

$$P_j(\xi) = P_j^0\left(\frac{\xi}{|\xi|}\right) + |\xi|P_j^1\left(\frac{\xi}{|\xi|}\right) + |\xi|^2P_j^2(\xi)$$

satisfy the properties stated in Lemma 6.1–(1) and Lemma 6.1–(2), and moreover we also have

$$P_j^0\left(\frac{\xi}{|\xi|}\right)\mathbf{P} = \mathbf{P}P_j^0\left(\frac{\xi}{|\xi|}\right) = P_j^0\left(\frac{\xi}{|\xi|}\right),$$

and using [75, Equation (3.36)], we have

$$P_j^1\left(\frac{\xi}{|\xi|}\right) = P_j^0\left(\frac{\xi}{|\xi|}\right)(-i\frac{\xi}{|\xi|} \cdot v)(-L^{-1}\mathbf{P}^\perp) + (-L^{-1}\mathbf{P}^\perp)(-i\frac{\xi}{|\xi|} \cdot v)P_j^0\left(\frac{\xi}{|\xi|}\right) + \mathbf{P}\mathbb{P}_j^1\left(\frac{\xi}{|\xi|}\right)\mathbf{P},$$

where  $\mathbb{P}_j^1\left(\frac{\xi}{|\xi|}\right)$  is a finite-dimensional matrix whose explicit form will not be important here, so that

$$P_j^1\left(\frac{\xi}{|\xi|}\right)\mathbf{P}^\perp = P_j^0\left(\frac{\xi}{|\xi|}\right)(-i\frac{\xi}{|\xi|} \cdot v)(-L^{-1}\mathbf{P}^\perp).$$

In [76, Theorem 1.1 and Theorem 2.21] the authors obtain a time-decay estimate for the operator

$$(B.3) \quad e^{tB(\xi)} - \chi\left(\frac{|\xi|}{\kappa}\right) \sum_{j=1}^4 e^{t\lambda_j(\xi)} \mathcal{P}_j(0, 0)$$

where the operators  $\mathcal{P}_j(\lambda, \xi)$ , with  $\lambda = \sigma + i\tau \in \mathbf{C}$  and  $\xi \in \mathbf{R}^3$ , corresponds to the eigenprojections associated to the eigenvalues  $\mu_j(\lambda, |\xi|)$  of the finite-dimensional operator

$$H(\lambda, \xi) := \mathbf{P}(\lambda - B_0(\xi))^{-1}\mathbf{P}$$

defined in [76, Equation (2.50)–(2.51)]. In particular  $\mathcal{P}_j(0, 0)g := \langle g, e_j \rangle e_j$  where  $\{e_j\}_{0 \leq j \leq 4}$  is an orthonormal basis of  $\text{Ker } L$  (see [76, Equation (2.63)]), so that  $\mathcal{P}_j(0, 0)\mathbf{P}^\perp = 0$ .

We are however interested in estimating the operator

$$\widehat{U}^\#(t, \xi) = e^{tB(\xi)} - \chi\left(\frac{|\xi|}{\kappa}\right) \sum_{j=1}^4 e^{t\lambda_j(\xi)} P_j(\xi),$$

which is slightly different than the operator in (B.3) above, and more importantly the operator

$$\widehat{U}^\#(t, \xi)\mathbf{P}^\perp = \left\{ e^{tB(\xi)} - \chi\left(\frac{|\xi|}{\kappa}\right) \sum_{j=1}^4 e^{t\lambda_j(\xi)} P_j(\xi) \right\} \mathbf{P}^\perp.$$

*Proof of (B.1).* We follow [76, Proof of Theorem 2.21]. As in [76, Equations (2.61)–(2.63)], we fix some integer  $l \in \mathbf{N}^*$  and define the operator

$$\mathcal{U}_j(\lambda, \xi) := (\lambda - B_0(\xi))^{-1} \mathbf{P} \mathcal{P}_j(\lambda, \xi) \mathbf{P} (\lambda - B_0(\xi))^{-1}$$

as well as  $\mathcal{U}_{j,m}^{(l)}(\lambda, \xi)$ , with  $m \in \mathbf{N}$  such that  $0 \leq m \leq l$ , which is given by a linear combination of products of  $\mu_j(\lambda, \xi)$ ,  $\mathcal{U}_j(\lambda, k)$  and their derivatives. In particular  $\mathcal{U}_{j,l}^{(l)}(\lambda, \xi) = i^{-l} (\partial_\tau \mu_j(\lambda, \xi))^l \mathcal{U}_j(\lambda, \xi)$  and  $\mathcal{U}_j(0, 0) = \mathcal{P}_j(0, 0) = P_j(0)$ .

We then replace [76, Equation (2.77)] by

$$\begin{aligned} & \|\mathcal{U}_{j,l}^{(l)}(i\tau, \xi) - P_j(\xi)\|_{L_v^2(\langle v \rangle^k) \rightarrow L_v^2(\langle v \rangle^k)} \\ & \lesssim \|\mathcal{U}_{j,m}^{(l)}(i\tau, \xi) - P_j(0, 0)\|_{L_v^2(\langle v \rangle^k) \rightarrow L_v^2(\langle v \rangle^k)} + \|P_j(0, 0) - P_j(\xi)\|_{L_v^2(\langle v \rangle^k) \rightarrow L_v^2(\langle v \rangle^k)} \\ & \lesssim O(|\xi|) + O(|i\tau - \lambda_j(\xi)|), \end{aligned}$$

and the rest of the proof of [76, Theorem 2.21] remains exactly the same, from which it follows

$$\begin{aligned} & \left\| \left\{ e^{tB(\xi)} - \chi\left(\frac{|\xi|}{\kappa}\right) \sum_{j=1}^4 e^{t\lambda_j(\xi)} P_j(\xi) \right\} g \right\|_{L_v^2(\langle v \rangle^k)} \\ & \lesssim (1+t)^{-l} \left[ \|g\|_{L_v^2(\langle v \rangle^{k+l|\gamma+2s|})} + \varrho_{l+1/2}(|\xi|) \|g\|_{L_v^2(\langle v \rangle^k)} \right]. \end{aligned}$$

□

*Proof of (B.2).* Again we follow [76, Proof of Theorem 2.21]. We claim that

$$(B.4) \quad \|\mathcal{U}_{j,l}^{(l)}(i\tau, \xi) \mathbf{P}^\perp - P_j(\xi) \mathbf{P}^\perp\|_{L_v^2(\langle v \rangle^k) \rightarrow L_v^2(\langle v \rangle^k)} \lesssim O(|\xi|^2) + O(|i\tau - \lambda_j(\xi)|^2).$$

This estimate then replaces [76, Equation (2.77)] and the proof of [76, Theorem 2.21] remains the same (using  $|\xi|^2 \varrho_{m+1}(|\xi|) = \varrho_m(|\xi|)$ ), which gives

$$\begin{aligned} & \left\| \left\{ e^{tB(\xi)} - \chi\left(\frac{|\xi|}{\kappa}\right) \sum_{j=1}^4 e^{t\lambda_j(\xi)} P_j(\xi) \right\} \mathbf{P}^\perp g \right\|_{L_v^2(\langle v \rangle^k)} \\ & \lesssim (1+t)^{-l} \left[ \|g\|_{L_v^2(\langle v \rangle^{k+l|\gamma+2s|})} + \varrho_l(|\xi|) \|g\|_{L_v^2(\langle v \rangle^k)} \right]. \end{aligned}$$

We therefore prove now (B.4). We write

$$\begin{aligned} & \mathcal{U}_{j,l}^{(l)}(i\tau, \xi) \mathbf{P}^\perp - P_j(\xi) \mathbf{P}^\perp \\ & = i^{-l} (\partial_\tau \mu_j(i\tau, \xi))^l \mathcal{U}(i\tau, \xi) \mathbf{P}^\perp - P_j(\xi) \mathbf{P}^\perp \\ & = \mathcal{U}(i\tau, \xi) \mathbf{P}^\perp - P_j(\xi) \mathbf{P}^\perp + [O(|\xi|) + O(|i\tau - \lambda_j(\xi)|)] \mathcal{U}(i\tau, \xi) \mathbf{P}^\perp. \end{aligned}$$

Writing  $\mathcal{U}(i\tau, \xi) = \mathcal{P}_j(0, 0) + O(|\xi|) + O(|i\tau - \lambda_j(\xi)|)$  and recalling that  $\mathcal{P}_j(0, 0) \mathbf{P}^\perp = 0$ , for the second term above we already obtain

$$\|[O(|\xi|) + O(|i\tau - \lambda_j(\xi)|)] \mathcal{U}(i\tau, \xi) \mathbf{P}^\perp\|_{L_v^2(\langle v \rangle^k) \rightarrow L_v^2(\langle v \rangle^k)} \lesssim O(|\xi|^2) + O(|i\tau - \lambda_j(\xi)|^2).$$

Thanks to the definition of  $\mathcal{U}_j$  and the resolvent identity for  $B_0(\xi)$ , see [76, Equation (2.49)], we get

$$\mathcal{U}_j(i\tau, \xi) = T_1 + T_2 + T_3 + T_4$$

with

$$\begin{aligned} T_1 &= (i\tau - L + \mathbf{P})^{-1} \mathbf{P} \mathcal{P}_j(i\tau, \xi) \mathbf{P} (i\tau - L + \mathbf{P})^{-1}, \\ T_2 &= (i\tau - L + \mathbf{P})^{-1} \mathbf{P} \mathcal{P}_j(i\tau, \xi) \mathbf{P} (i\tau - B_0(\xi))^{-1} (-i\xi \cdot v) (i\tau - L + \mathbf{P})^{-1}, \\ T_3 &= (i\tau - B_0(\xi))^{-1} (-i\xi \cdot v) (i\tau - L + \mathbf{P})^{-1} \mathbf{P} \mathcal{P}_j(i\tau, \xi) \mathbf{P} (i\tau - L + \mathbf{P})^{-1}, \\ T_4 &= (i\tau - B_0(\xi))^{-1} (-i\xi \cdot v) (i\tau - L + \mathbf{P})^{-1} \mathbf{P} \mathcal{P}_j(i\tau, \xi) \mathbf{P} (i\tau - B_0(\xi))^{-1} (-i\xi \cdot v) (i\tau - L + \mathbf{P})^{-1}. \end{aligned}$$

Since

$$\mathbf{P} (i\tau - L + \mathbf{P})^{-1} = (i\tau - L + \mathbf{P})^{-1} \mathbf{P} = \frac{\mathbf{P}}{i\tau + 1}$$

we obtain

$$T_1 \mathbf{P}^\perp = T_3 \mathbf{P}^\perp = 0.$$

Moreover we easily get

$$T_4 = O(|\xi|^2).$$

We now investigate the term  $T_2$ . Using the resolvent identity for  $B_0(\xi)$  again, we first obtain  $T_2 = T_{2,1} + T_{2,2}$  with

$$T_{2,1} = \frac{\mathbf{P}}{i\tau + 1} \mathcal{P}_j(i\tau, \xi) \frac{\mathbf{P}}{i\tau + 1} (-i\xi \cdot v) (i\tau - L + \mathbf{P})^{-1}$$

and

$$T_{2,2} = \frac{\mathbf{P}}{i\tau + 1} \mathcal{P}_j(i\tau, \xi) \mathbf{P} (i\tau - B_0(\xi))^{-1} (-i\xi \cdot v) (i\tau - L + \mathbf{P})^{-1} (-i\xi \cdot v) (i\tau - L + \mathbf{P})^{-1}.$$

We already observe that

$$T_{2,2} = O(|\xi|^2).$$

We now write

$$\frac{\mathbf{P}}{i\tau + 1} = [1 + O(|i\tau - \lambda_j(\xi)|) + O(|\xi|)] \mathbf{P},$$

and

$$\mathcal{P}_j(i\tau, \xi) = \mathcal{P}_j(0, 0) + O(|i\tau - \lambda_j(\xi)|) + O(|\xi|)$$

and also

$$(i\tau - L + \mathbf{P})^{-1} = (-L + \mathbf{P})^{-1} + O(|i\tau - \lambda_j(\xi)|) + O(|\xi|).$$

Then we remark that  $(-L + \mathbf{P})^{-1} \mathbf{P} = \mathbf{P}$  and  $(-L + \mathbf{P})^{-1} \mathbf{P}^\perp = -L^{-1} \mathbf{P}^\perp$ , which implies

$$T_{2,1} = \mathbf{P} \mathcal{P}_j(0, 0) \mathbf{P} (-i\xi \cdot v) (-L + \mathbf{P})^{-1} + O(|\xi|^2) + O(|i\tau - \lambda_j(\xi)|^2)$$

and thus

$$T_{2,1} \mathbf{P}^\perp = \mathbf{P} \mathcal{P}_j(0, 0) \mathbf{P} (-i\xi \cdot v) (-L^{-1} \mathbf{P}^\perp) + O(|\xi|^2) + O(|i\tau - \lambda_j(\xi)|^2).$$

Gathering the previous estimates and recalling that

$$\begin{aligned} P_j(\xi) \mathbf{P}^\perp &= P_j^0 \left( \frac{\xi}{|\xi|} \right) \mathbf{P}^\perp + |\xi| P_j^1 \left( \frac{\xi}{|\xi|} \right) \mathbf{P}^\perp + O(|\xi|^2) \\ &= P_j^0 \left( \frac{\xi}{|\xi|} \right) (-i\xi \cdot v) (-L^{-1} \mathbf{P}^\perp) + O(|\xi|^2) \end{aligned}$$

yields

$$\begin{aligned} \mathcal{U}_{j,l}^{(l)}(i\tau, \xi) \mathbf{P}^\perp - P_j(\xi) \mathbf{P}^\perp &= \left[ \mathbf{P} \mathcal{P}_j(0, 0) \mathbf{P} - P_j^0 \left( \frac{\xi}{|\xi|} \right) \right] (-i\xi \cdot v) (-L^{-1} \mathbf{P}^\perp) \\ &\quad + O(|\xi|^2) + O(|i\tau - \lambda_j(\xi)|^2), \end{aligned}$$

from which we deduce (B.4).  $\square$

**B.2. Proof of Lemma B.2.** Consider the case of soft potentials  $\gamma + 2s < 0$ . We only need to modify the estimate of the term  $\widehat{\Psi}^{\varepsilon\sharp}[g, g]$  in Step 3 of the proof of Lemma 6.5.

For high frequencies  $\varepsilon|\xi| > \kappa$ , the argument of Step 3 of the proof of Lemma 6.5 works, therefore we only consider the case of low frequencies  $\varepsilon|\xi| \leq \kappa$ .

We fix  $\ell > 0$  such that  $1 < \frac{\ell}{|\gamma+2s|} < \frac{3}{2}$  and then use estimate (B.2) of Lemma B.1 to obtain, for all  $t \geq 0$  and  $\xi \in \mathbf{R}^3 \setminus \{0\}$ ,

$$\begin{aligned} \|\widehat{\Psi}^{\varepsilon\sharp}[g, g](t, \xi)\|_{L_v^2} &\lesssim \frac{1}{\varepsilon} \int_0^t \left(1 + \frac{t-\tau}{\varepsilon^2}\right)^{-\frac{\ell}{|\gamma+2s|}} \left(1 + (\varepsilon|\xi|)^{\frac{-2\ell}{|\gamma+2s|}+2}\right) \|\widehat{\Gamma}(g, g)(\tau, \xi)\|_{L_v^2((v)^\ell)} d\tau \\ &\lesssim \frac{1}{\varepsilon} \left(1 + (\varepsilon|\xi|)^{\frac{-2\ell}{|\gamma+2s|}+2}\right) \|\widehat{\Gamma}(g, g)\|_{L_t^\infty L_v^2((v)^\ell)} \int_0^t \left(1 + \frac{t-\tau}{\varepsilon^2}\right)^{-\frac{\ell}{|\gamma+2s|}} d\tau \\ &\lesssim \left(\varepsilon + \varepsilon^{3-\frac{2\ell}{|\gamma+2s|}} \mathbf{1}_{|\xi| \geq 1} + \varepsilon^{3-\frac{2\ell}{|\gamma+2s|}} |\xi|^{2-\frac{2\ell}{|\gamma+2s|}} \mathbf{1}_{0 < |\xi| \leq 1}\right) \|\widehat{\Gamma}(g, g)\|_{L_t^\infty L_v^2((v)^\ell)}. \end{aligned}$$

In the case of the torus  $\Omega_x = \mathbf{T}^3$ , we take the  $L_\xi^1 L_t^\infty$  norm of the above estimate to conclude

$$\|\widehat{\Psi}^{\varepsilon\sharp}[g, g]\|_{L_\xi^1 L_t^\infty L_v^2} \lesssim \left(\varepsilon + \varepsilon^{3-\frac{2\ell}{|\gamma+2s|}}\right) \|\widehat{\Gamma}(g, g)\|_{L_\xi^1 L_t^\infty L_v^2((v)^\ell)}.$$

Using the bound (6.13) together with estimates on  $g$  from Theorem 2.2 completes the proof of point (1) in Lemma B.2.

In the case of the whole space  $\Omega_x = \mathbf{R}^3$ , we take the  $L_\xi^1 L_t^\infty$  norm and apply Hölder's inequality to obtain

$$\begin{aligned} \|\widehat{\Psi}^{\varepsilon\sharp}[g, g]\|_{L_\xi^1 L_t^\infty L_v^2} &\lesssim \left( \varepsilon + \varepsilon^{3 - \frac{2\ell}{|\gamma+2s|}} \right) \|\widehat{\Gamma}(g, g)\|_{L_\xi^1 L_t^\infty L_v^2(\langle v \rangle^\ell)} \\ &\quad + \varepsilon^{3 - \frac{2\ell}{|\gamma+2s|}} \|\xi\|^{2 - \frac{2\ell}{|\gamma+2s|}} \mathbf{1}_{0 < |\xi| \leq 1} \|_{L_\xi^{p'}} \|\widehat{\Gamma}(g, g)\|_{L_\xi^p L_t^\infty L_v^2(\langle v \rangle^\ell)}. \end{aligned}$$

Observing that  $\|\xi\|^{2 - \frac{2\ell}{|\gamma+2s|}} \mathbf{1}_{0 < |\xi| \leq 1} \in L_\xi^{p'}$  since  $p \in (3/2, \infty]$  and  $1 < \frac{\ell}{|\gamma+2s|} < \frac{3}{2}$ , we deduce point (2) in Lemma B.2 by using again the bound (6.12) together with estimates on  $g$  from Theorem 2.2.  $\square$

## REFERENCES

- [Dua11] R. Duan, Hypocoercivity of linear degenerately dissipative kinetic equations, *Nonlinearity*, 24 (2011), 2165-2189.

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